ON CONTROL OF 2D SWITCHED SYSTEMS BY MEANS OF GEOMETRIC ALGEBRA FOR CONICS

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Summary of the Abstract

The paper deals with an algorithm for control of a linear switched system by means of Geometric Algebra. More precisely, we develop a switching path searching algorithm for a two–dimensional linear dynamical switched system with non–singular matrix. Then it is natural to represent them as elements of Geometric Algebra for Conics (GAC) and construct the switching path by calculating the switching points, i.e. intersections and contact points. For this, we use symbolic algebra operations, more precisely the wedge and inner products, that are realisable by sums of products in the coordinate form. This choice guarantees optimality of the switching path with respect to the number of switches. On two examples we demonstrate the search for conics' intersections and, consequently, we describe a construction of a switching path in both cases.

Introduction

Switched systems are a special case of hybrid dynamical systems with discrete and continuous dynamics. They are widely used when a real system cannot be described by a single model. There are many examples in engineering systems such as electronics, power systems, traffic control and others. Since the 1990s, research on the stability of switched systems has become very popular, see e.g. [8, 6]. The particular case of linear switched systems can be found in [5]. More recent literature about different types of switched systems is represented by the works of Patrizio Colaneri [5], Yuan Lin, Yuan Sun-Ge Wang, and Jiang-Wang [13], Zhong-Ping, Yuan Wang [12]. To control a switched system it is enough to find the switching points, ie., the points where the integral curves of respective systems of ODEs intersect. In the paper [2], we presented a switching path searching algorithm for a 2D linear dynamical switched system with non-singular matrix whose integral curves are formed by two sets of centralised ellipses. Then it is natural to represent them as elements of Geometric Algebra for Conics (GAC) and construct the switching path by calculating the switching points, ie., intersections and contact points. The purpose of the current paper is to provide the classification of controllable switched systems, ie., those where the switching path may be found between two arbitrary states, and to demonstrate the functionality of our algorithm for different types of non-singular controllable 2x2 systems. Another goal is to discuss the optimality of proposed solution.

1 Geometric Algebra for Conics

Section deals with the descriptipn of Geometric Algebra for Conics $Cl(5,3)$, also referred to as GAC, which is a generalisation of 2D conformal geometric algebra $Cl(3,1)$, [7]. Analogously to the notation in [10], the corresponding basis elements are denoted as

$$\bar{n}_+, \bar{n}_-, \bar{n}_\times, e_1, e_2, n_+, n_-, n_\times. \tag{1}$$

This notation suggests that the basis elements e_1, e_2 play the usual role of standard basis of the plane while the null vectors $\bar{n}$, n represent the origin and infinity, respectively. Note that there are three orthogonal 'origins' $\bar{n}$ and three corresponding orthogonal 'infinities' n, [10]. In terms of this basis, a point of the plane $\mathbf{x} \in \mathbb{R}^2$ defined by $\mathbf{x} = xe_1 + ye_2$ is embedded using the operator $C : \mathbb{R}^2 \to Cone \subset \mathbb{R}^{5,3}$, which is defined by

$$C(x,y) = \bar{n}_+ + xe_1 + ye_2 + \frac{1}{2}(x^2 + y^2)n_+ + \frac{1}{2}(x^2 - y^2)n_- + xyn_\times. \tag{2}$$

For our purposes, we stress that the operations in GAC involve sums and products only. This minimises calculation errors. In fact, the wedge product is computed as the outer product of vectors on each vector space of the same grade, while the inner product acts on these spaces as the scalar product. The extension of both operations to general multivectors does not add any computational complexity due to the linearity of both operations. Let us also recall that if a conic C is considered as a wedge of five different points (which determine a conic uniquely), the appropriate 5-vector E^* is called an outer product null space representation (OPNS) and its dual E, indeed a 1-vector, is called the inner product null space (IPNS) representation. The reason is that if a point P lies on a conic C then

$$P \cdot E = 0 \quad \text{and} \quad P \wedge E^* = 0.$$

Consequently, intersections of two geometric primitives are given as the wedge product of their IPNS representations, ie.,

$$C_1 \cap C_2 = E_1 \wedge E_2$$

for two conics C_1, C_2 and their IPNS representations E_1 and E_2, respectively, see [10].

Further we provide a procedure for intersecting two conics, particularly ellipses with common centre in the coordinate origin but in a general mutual position otherwise. Moreover, we consider a system of circumscribed ellipses and show a procedure for detecting the first order contact points, ie., points where the ellipses touch with identical first order derivative. Again, the contribution of GAC lies in avoiding the use of a solver which leads to accuracy improvement.

Let us first describe some differences to CRA or its 3D version CGA (Conformal Geometric Algebra). Crucial difference lies in the type of objects that are intrinsic to respective structures. For CRA (CGA), spheres (circles) are the geometric primitives that may be represented by specific elements. Taking into account that lines and planes are spheres with infinite radii and a point pair is a 1D sphere, we receive all geometric primitives for analytic geometry. Moreover, intersection still remain such objects, indeed, an intersection of two spheres or two circles are circles or point pairs, respectively. Therefore, intersections that are realised by wedge of IPNS representations remain representatives of Euclidean primitives intrinsic to CRA (CGA). On contrary, in GAC the situation is different. Even if we restrict to the case of co–centric ellipses, their intersection is a ”four point” which has no meaning in the sense of conic–sections. Indeed, a planar conic is generated by five points at least. This leads to an algorithm that may be used for co–centric conics (all types). On the other hand, the algorithm is still geometric–based and may be realised by a sequence of simple operations in GAC, ie., there is no numerical solver involved.

2 Switched Systems

First, consider the oscillation problem of a spring pendulum under the condition of absence of external and friction forces

$$\ddot{x} = -kx, \quad k \in \mathbb{R}$$

with a switchable stiffness coefficient $k > 0$, that changes value between k_1 and k_2 by joining and removing an additional spring with a stiffness coefficient k_2. Two cases can be considered. If the springs are connected in parallel, the parameter k of the system switches between $k = k_1$ and $k = k_1 + k_2$. If the connection is in series, the parameter k of the system switches between $k = k_1$ and $k = \frac{k_1 k_2}{k_1 + k_2}$.

Let us rewrite the differential equation of the pendulum oscillations as a switched system, using the standard procedure:

$$\boldsymbol{x}_1 = x, \quad \boldsymbol{x}_2 = \dot{x}.$$

Therefore

$$\dot{\boldsymbol{x}}(t) = A_i \boldsymbol{x}(t), \quad A_i \in \mathcal{M}at_2(\mathbb{R}), \quad i = 1, 2. \tag{3}$$

Definition 2.1 *We say that the switched system*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_{\sigma(t)}(\boldsymbol{x})$$

is controllable if for any two points $A, B \in \mathbb{R}^m$ from the state space there exists a switching signal generating a trajectory from A to B.

Let us consider a linear homogeneous system with constant coefficients in the form

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}, \quad A \in \mathcal{M}at_2. \tag{4}$$

In the system (4), three types of phase trajectories are possible: point, closed curve, unclosed curve. A point on the phase plane corresponds to the equilibrium of the system (4), while the closed curve corresponds to the periodic solution, and unclosed to non-periodic solutions of the system (4), respectively. The equilibrium points of the system (4) can be found by solving the homogeneous system: $A\boldsymbol{x} = 0$.

Classification of equilibrium points in the case when $\det A \neq 0$ is shown in the Table 1.

3 Control of the switched system by means of GAC

In this section the case of 2×2 matrices with both subsystems having pure imaginary eigenvalues is studied. This case has already been considered in [9], and the main difference lies in using GAC as a suitable space for geometric operations with the ellipses.

Table 1: Classification of equilibrium points in the case det$A \neq 0$

Roots of characteristic Equation	Point Type
λ_1, λ_2 are real numbers of the same sign $\lambda_1\lambda_2 > 0$	Node
λ_1, λ_2 are real numbers of the opposite sign $\lambda_1\lambda_2 < 0$	Saddle
λ_1, λ_2 are complex numbers Reλ_1 = Re$\lambda_2 \neq 0$	Focus
λ_1, λ_2 are complex numbers Reλ_1 = Re$\lambda_2 = 0$	Center

Let us describe the case of a spring pendulum (3) with respect to the equilibrium type as center-center, see Table 1 for full classification. Without the loss of generality, let us assume that we start and end with the first system $i = 1$. Suppose that two nonzero points (initial $A(x_1, y_1) \in \mathbb{R}^2$ and final $B(x_2, y_2) \in \mathbb{R}^2$) are given.
Let us consider the particular type of the switched system (3)

$$\dot{\boldsymbol{x}}(t) = A_i(\boldsymbol{x}(t)),$$

where

$$A_i = \begin{bmatrix} 0 & 1 \\ -\alpha_i & 0 \end{bmatrix}, \quad \alpha_i \in \mathbb{R}^+ \quad \text{for} \quad i = 1, 2.$$

In the following, we describe the generalised algorithm for finding a control of a switched system [2], ie., finding a path composed of the systems' integral curves from the initial point A to the endpoint B such that the number of switches is minimal.

4 Algorithm for a switching path construction

Consider the case $n = 2$, ie., only two systems are included, and both starting and final conic belong to the same family. To apply the GAC based calculations, it is necessary to get the exact GAC form of the representatives of both families of ellipses. Thus the system of ODEs is solved numerically (e.g. by Runge-Kutta method) with the initial condition at the starting point A. This will give us a set of points representing the initial conic (ellipse or hyperbola). After applying the GAC conic fitting algorithm, [10], we get the conic in IPNS representation. Note that according to [11], the algorithm may be further specified by prescribing the resulting ellipse to be axis-aligned and with its centre placed in the origin. This makes the initial trajectories very precise.
As a result, the above algorithm provides a sequence of switching points as well as a sequence of trajectories in GAC [2]. Consider the following examples, which generalize the system from [1, p. 6].
In the next step, we will consider a switched system, where one of the matrices has real eigenvalues of different signs (a singular point of the Saddle type), and the other has purely imaginary eigenvalues.
Let us consider switched system

$$\dot{x}(t) = A_i(x(t)), \tag{5}$$

where the matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ \frac{1}{\alpha} & 0 \end{bmatrix}, \quad \alpha > 0.$$

In the case of a switched system of the Center-Saddle type, the algorithm for finding is also applicable, but it is greatly simplified. The fact is that in this case, only two switches are enough to find the path.

Theorem 4.1 *The switched system 5 is controlled, and, moreover, if the movement from the starting point corresponds to the system of the type of Saddle, then it is possible to get to an arbitrary end point using two switches.*

Upon further investigation, it turned out that among the switched systems with non-singular 2x2 matrices, except considered above systems, controlled can be also switched systems, corresponding to switch between matrices with equilibrium point of the type stable and unstable focuses (λ_1, λ_2 are complex numbers $Re\lambda_1 = Re\lambda_2 \neq 0$) and partially controlled systems are systems, which have switches between saddle(λ_1, λ_2 are real numbers of the opposite sign $\lambda_1\lambda_2 < 0$) and focus(λ_1, λ_2 are complex numbers $Re\lambda_1 = Re\lambda_2 \neq 0$).
In order to compare the result, received by use of GAC with numerical solution, we consider the same system, but instead of GAC conic fitting and searching for intersections, we use Runge-Kutta method for the next ellipse construction and for the last ellipse we get the system of two quadratic equations.
It was demonstrated that the only controllable systems are systems of the type Center-Center (both matrices have pure complex eigenvalues), Center-Saddle and the combination of Stable and Unstable focuses.
The complete classification of integral curves and its intersections for the nonsingular 2×2 systems can be presented in the Table 2.

Table 2: Controllability of the switched 2x2 systems

$A_1 \mid A_2$		**Center**	**Saddle**	**Node** **Stable**	**Unstable**	**Focus** **Stable**	**Unstable**
Center		+	+	-	-	-	-
Saddle		+	-	-	-	Controlled under condition	Controlled under condition
Node	**Stable**	-	-	-	-	-	-
	Unstable	-	-	-	-	-	-
Focus	**Stable**	-	Controlled under condition	-	-	-	+
	Unstable	-	Controlled under condition	-	-	+	-

5 Conclusion

Paper dealed with the controllability of the 2×2 switched systems with regular matrices of each subsystem. For the systems, whose phase portraits contain conics (which are the elements of GAC), the Geometric algebra approach was used. It was demonstrated that the use of GAC for construction of switching points of 2D switched systems leads to the solution that is optimal with respect to the number of switches. From the geometric nature of our approach we can easily see that the number of switches can only differ by 1 from the numerical solution but also the numerical error for particular switches must be taken into account. We provided examples with axes aligned ellipses but from the description of GAC it is clear, that their approach will handle rotated conics of any type as well, in which case numerical solution will carry even larger error. Moreover, we provided full classification of 2D switched systems with respect to their controllability. From the geometry of the controlled systems it is clear that the only controllable systems are systems of the type Center-Center, Center-Saddle and the combination of Stable and Unstable focuses. The switched systems with matrices's curves of the types Saddle and Focus are "controlled under condition", ie., dependent on the quadrant in which starting and ending points are placed.

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