PROJECTIVE AND CONFORMAL FORMULATIONS OF ELECTROMAGNETISM

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Summary of the Abstract

It is well known that the source-free Maxwell equations exhibit conformal invariance. As well as being invariant under the familiar operations of spacetime translations and rotations, they are also invariant under inversions. This symmetry is straightforward to understand in terms of a non-linear transformation in the STA. But it is less clear how to exhibit the symmetry using the linear form of the conformal group in $G(2,4)$*. In this paper we provide explicit constructions of the Maxwell equations in both projective and conformal settings that make the symmetry explicit.*

Conformal Invariance of the Maxwell Equations

The source-free Maxwell equations have a uniquely compact expression in geometric algebra:

$$\nabla F = 0. \tag{1}$$

Here F is a bivector in the Spacetime Algebra (STA), and ∇ is the vector derivative. As well as the obvious symmetries of spacetime translations and rotations, this equation exhibits a surprising additional symmetry under inversion To see this, set

$$x' = -\frac{1}{x} = -\frac{x}{x^2} \tag{2}$$

This has the differential

$$\begin{aligned} a\cdot\nabla x' &= -\frac{a}{x^2} + 2\frac{a\cdot x\, x}{x^4} \\ &= \frac{x}{x^2} a \frac{x}{x^2} \end{aligned} \tag{3}$$

From this we construct the new solution

$$F'(x) = \frac{1}{x^6} x F(x') x, \tag{4}$$

and this new solution also satisfies the vacuum Maxwell equations. By combining inversion in the origin with translations and rotations (and reflections) we generate the entire conformal group.
There is a serious problem with treating inversions in the STA, however. The mapping $x \mapsto -x/x^2$ is not only singular at the origin, it is singular for all points on the null cone from the origin. In Euclidean spaces this is taken care of by adding a single extra point at infinity, creating inversive geometry (the traditional name for conformal geometry). Clearly in spacetime this is not sufficient and a more careful treatment of the various types of infinity is required.
Once the behaviour at infinity is understood, we next face the question of how to linearise transformations such as 4. The answer to this was initially provided by Dirac, who noted that a natural environment for electromagnetism is the 'projective null cone'. This is the familiar construct of vectors X in $G(2,4)$ satisfying

$$X^2 = 0. \tag{5}$$

We also demand that X and λX represent the same point back in spacetime, so any fields in $G(2,4)$ must satisfy a homogeneity property of the form

$$\mathcal{A}(\lambda X) = \lambda^c \mathcal{A}(X) \tag{6}$$

where c is a conformal weight for the field. Here we use calligraphic symbols for fields in $G(2,4)$, and standard font for their equivalents in spacetime.
Following the treatment in [1] we show how to reformulate the Maxwell equations on the projective null cone. The key observation is that we introduce an additional invariance. If we let D denote the vector derivative in $G(2,4)$, and suppose we have a scalar field $\phi(X)$ defined on the projective null cone. Then we can add any amount of X^2 to this field, so that

$$\phi \mapsto \phi + \eta X^2. \tag{7}$$

The derivative of ϕ then maps to

$$D\phi \mapsto D\phi + 2\eta X \tag{8}$$

when evaluated on the projective null cone. We can add in an arbitrary amount of the null vector X to the result. So to define differentiation on the projective null cone we have to work with equivalence classes, where two vectors are in the same equivalence class if their difference is a multiple of X.
With this concept in hand we can define analogues of the Maxwell equations in $G(2,4)$ which exhibit full conformal invariance via linear transformations. The way all this is achieved is quite subtle and deserves to be better known.
There is an additional way of treating electromagnetism in a projective context, which only requires a single extra dimension. Some results of this for a constant curvature background space have been described briefly in [2]. Here we discuss how this relates to the conformal embedding, and generalise to the case of zero curvature using an extension of 'plane-based' geometric algebra relevant to 4d spacetime.

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References

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[2] A.N. Lasenby, Recent Applications of Conformal Geometric Algebra, H. Li, P. J. Olver and G. Sommer (Eds.): IWMM 2004, LNCS 3519 Springer-Verlag (2005).