# Dual Spaces are Real:
## Orientation Types in Geometric Algebras

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**Abstract**

Traditional duality embodies the concept of orthogonal complementation of subspaces, which allows flexible specification of geometrical elements as nullspaces of inner or outer product. But duality also encodes different orientation types for the 'same' geometric element: in $d$-D PGA $\mathbb{R}_{d,0,1}$, an axis line with 'around' orientation is a bivector, and a spear line with 'along' orientation is a $(d-1)$-vector. All geometric elements come in such *extrinsic* and *intrinsic* oriented versions.
This contribution shows how maintaining both an algebra and its dual (thus doubling our representational framework) allows a pseudoscalar-defined dualization operation that is properly Clifford equivariant: independent of basis and of sign conventions, even for degenerate metrics (such as PGA). That double bookkeeping provides mathematical clarity on the orientation types and how they are connected by the *meet* and *join* operations. By the time of AGACSE, we moreover hope to report on how freely one can choose the metrics of the two dual spaces without losing their complementary geometric semantics in modelling tasks.

## 1 Duality, Degeneracy and Orientation

Modeling a particular geometry and its primitive objects in geometric algebras is done as follows. One chooses as vectors the geometric elements that act as reflectors to span its symmetry group (in the case of 3D Euclidean geometry, the planes, leading to PGA $\mathbb{R}_{d,0,1}$). Then invariants of composed reflections can be characterized by blades of the algebra representing the geometric primitive 'object' (for PGA, planes, lines and points). This construction guarantees a consistent equivariant framework, combining operations and objects. Some extra methods are required for constructive relationships between the various primitives, such as *projection* operators (enabling the specification of general linear operators), and the *meet* and *join* for the geometric intersection and union of primitives.

In this paper, we consider the 'duality' operator, which is related to orthogonal complementation, and allows switching between constructions based on intersection and union of subspace elements. The execution of this geometric principle is codified by a pseudoscalar of the representational space (the highest grade blade), which involves a signed scaling factor. The commonly used duality was introduced by Hestenes [6], and produces the dual of an element by dividing by the pseudoscalar (with signs to conform to pre-existing conventions in 3D Euclidean space involving the cross product). We show how to define a pseudoscalar-based form of duality that also works for degenerate algebras like PGA, which do not have an invertible pseudoscalar. It uses an orthogonal basis of blades for its definition, but can be shown to be properly equivariant under orthogonal transformations.

The resulting dual elements transform with an extra minus sign under reflections, relative to the primal elements. We have therefore recently made a case for the *geometric semantics of duality* in terms of orientation types of geometric primitives [1], as a rather neglected aspect of geometry with practical implications. Thus in the encoding of Newtonian mechanics by 3D PGA [3], the meet of two planes encodes an axis line, a bivector that can be used to kinematically characterize a rotation *around* that axis; this orientation of the axis line is called *extrinsic*. For the dynamics characterization, we need forces and momenta, which are 'spear' lines, with an orientation *along* the lines – this was called *intrinsic*. The two are dually related; and if one want a framework that contains both, a dual algebra needs to be maintained, since only then do oriented elements transform property under all versors, including reflections. The consequence of maintaining the two dual algebras is that the meet and join operations come into a clearer focus, with equivariance properties that are more cleanly defined, see [1].

This contribution reports on a project undertaken with mathematician colleague Patrick Forré, who introduced some standard structural mathematical techniques not usually employed by practitioners of geometric algebra to formalize the 'projective duality' present in Gunn [5] and used in [1]. We spell out the exact relationships of duals, in a manner that is general (i.e., not dependent on the degeneracy or sign conventions) and which is demonstrably equivariant under the symmetries encoded in the versors of the algebra. Projective relationships are naturally related between primal and dual elements, but there appears to be some freedom in the metric correspondence. At the time of this submission we are investigating the freedom of choice in relating the quadratic forms of the two dual geometric algebras, which we hope to have resolved by August 2024.

## 2 Duality, Even in Degenerate Algebras

### 2.1 Duality as Projection on the Pseudoscalar

Let us set the scene of treatment by the non-degenerate case, so consider a geometric algebra $\mathrm{Cl}(W, q)$ over a vector space $W$ with quadratic form $q$. Choose an invertible pseudoscalar $\mathbb{I}$. We have for any multivector $A$ (though usually $A$ is taken to be a blade):

$$A = A\,\mathbb{I}^{-1}\,\mathbb{I} = (A \cdot \mathbb{I}^{-1}) \cdot \mathbb{I}.$$

This rewriting to a scalar product is due to the special property of $\mathbb{I}$ that any vector factor $\mathbf{a}$ of a term of $A$ has $\mathbf{a} \wedge \mathbb{I} = 0$. In the final form, this is recognizable as a projection onto the pseudoscalar $\mathbb{I}$, which clearly retrieves $A$. If $A$ is a blade, the element $A \cdot \mathbb{I}^{-1}$ is geometrically like an orthogonal complement of $A$, since $A \cdot (A \cdot \mathbb{I}^{-1}) = 0$, and the grades of $A$ and $A \cdot \mathbb{I}^{-1}$ are complementary.

When considering duality, we focus only on the part $A^* \equiv A \cdot \mathbb{I}^{-1} = A\,\mathbb{I}^{-1}$ (we prefer the geometric product in formulations). One could say that we are performing a '*pseudoscalar split*'. Mathematically, we then maintain $A^*$ and $\mathbb{I}$ separately. This can be formulated by a tensor product construction:

$$\mathrm{Cl}^{(k)}() \to \mathrm{Cl}^{(n-k)}() \otimes \mathrm{Cl}^{(n)}() : \quad A \mapsto (A\,\mathbb{I}^{-1}) \otimes \mathbb{I}.$$

The total expression is linear in $A$, and transforms invariantly under orthogonal transformations $\underline{V}[\,]$. However, due to the appearance of a determinant factor in the two terms:

$$\underline{V}[A] \quad \mapsto \quad (\underline{V}[A]\,\underline{V}[\mathbb{I}^{-1}]) \otimes \underline{V}[\mathbb{I}] = \big(\underline{V}[A]\,\mathbb{I}^{-1}/\det \underline{V}\big) \otimes \big(\mathbb{I}\,\det \underline{V}\big). \tag{1}$$

the dual transforms with a possibly different sign than the original blade $A$ under odd versors.[1]

## 2.2 Duality as an Index-Complementation Function

In the Hestenes approach, dual elements map to the geometric algebra of $W$, yet transform differently (under reflections) than the primal elements they share a basis with. This is confusing, and we should therefore be open to maintaining, besides the base space $W$ for the algebra $\mathrm{Cl}(W, q)$, a separate dual space $W^*$ (the space of duals), and endowing that with its own geometric algebra $\mathrm{Cl}(W^*, q^*)$. We should do so in a way that also works for degenerate algebras.

An easy way to define (and implement!) a duality is by considering an *orthogonal basis* consisting of vectors $e_i$ of $W$ spanning blades $e_J$ (with $J$ this index set of the blade). Choose a pseudoscalar $e_I = \mathbb{I}$ by imposing some preferred order $I$ on the indices, and define a dual element for each basis blade $e_J$ with index $J$ through complementation of the index set $I \backslash J$ to produce the properly ordered index set $I$ of the pseudoscalar $\mathbb{I}$, with a sign for each permuted index. This is the approach that leads to the *Hodge dual* $\star e_J$, which satisfies a defining equation like $e_J \wedge (\star e_J) = e_I$. That dual is then an element of the original space, with a complementary set of indices and a permutation sign $\star e_J = \mathrm{sign}_{J, I\backslash J} e_{J\backslash I}$. But as we have seen by eq.(1), we now obtain elements $e_{I\backslash J}$ that should transform differently depending on whether they were 'intended' as dual or primary.

A more mathematically pure approach is to consider as a dual element the *mapping that produces this connection* to the pseudoscalar. Let us denote that pseudoscalar-based dual by the superscript $^\natural$, or by an upper indexing (as in [1]):

$$e_A^\natural \equiv e^{I\backslash A} \equiv \mathrm{sign}_{I\backslash A, A} \, \langle e_{I\backslash A} \, (\_\_) \rangle_n, . \tag{2}$$

Here $(\_\_)$ denotes where to plug in the function argument to form the geometric product with $e_{I\backslash A}$ before selecting the grade $n$ part (which incidentally extends the dual to non-blades). Note that $e^A[e_A] = e_I = \mathbb{I}$, as desired. The index-based construction of basis elements clearly also works for a degenerate space, with a null pseudoscalar.

## 2.3 A Separate Space of Duals

The $e_A^\natural$ are elements of a space of linear functions that produce the *pseudoscalar* $\mathbb{I}$ when applied to elements of the linear space $W$. Those also form a linear space, which we denote as $W^\natural$. This is different from the traditional way of associating a dual vector space $W^*$ with a vector space $W$ through algebraic forms, which are linear functions that take a blade to produce a *scalar*.

Forré [4] shows that the index-based definition of this 'natural dual' is equivariant: the actual dual does not depend on the actual orthogonal basis chosen to define it, and therefore is truly geometric. This formalizes the 'projective dual' used in [1].

[1] At AGACSE, we may talk about the various *sign conventions* in dualizations, and how the tensor product view could be used to make the duality concept independent of the chosen convention in any implementation. Note that the focus on vectors as hyperplanes of reflection (in PGA and CGA) is already 'dual' in the traditional sense. Even the Hestenes dualization of $D = A/\mathbb{I}$ would now be formulated as $A = D\,\mathbb{I}$, since the originally dual elements are now considered as primal. Still this does not give unique duals in degenerate spaces.

# 3 Complementary Orientation Types

In oriented geometry, subspaces can be given two types of orientation (each of which can be positive or negative): an *intrinsic* orientation related to the ordering of a basis that spans them; and an *extrinsic orientation* related to their orientation relative to the embedding space (like the sidedness of hyperplanes). The two are complementary, and in many applications it is useful to have the capability to switch between the two types. Studying how these act under orthogonal transformations, we find that there is an extra minus sign involved for nett reflections, see Figure.

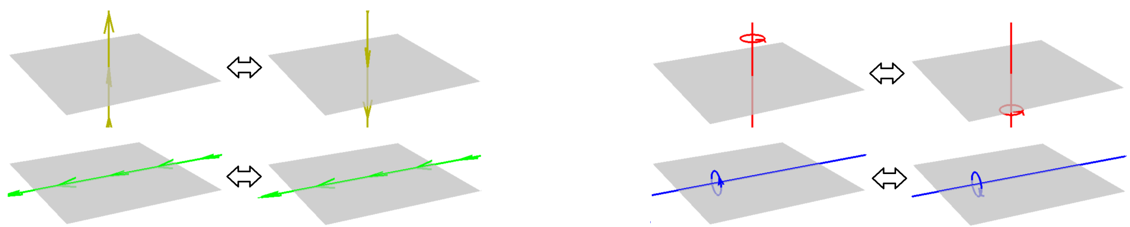

Figure 1: Lines of complementary orientations reflect differently in a mirror.

This precisely corresponds to the factor $\det \underline{V}$ involved in the difference in transformation of an element and its dual [1].[2][3] Using duality, geometric algebra can therefore encode such orientational aspects consistently, with its operators treating them equivariantly under the transformations of its symmetry group. That shows that those aspects are indeed geometrical.

We have thus found a geometric meaning for duality on blades: it switches the orientation type of the subspace represented by the blade: a primal element and its dual represent the 'same' geometric element (i.e., occupying the same part of space), but with a different orientation type (extrinsic vs intrinsic). The precise signed relationship between the two orientations is of course consistently encoded by the specific pseudoscalar $\mathbb{I}$ used. It is then also geometrically reasonable to maintain two distinct copies of each subspace: one for each representation type (each of which may still be weighted by a non-zero scalar to flip handedness and density).

One geometric element, in both its orientations, spans the whole space $W$ (or $W^*$) (i.e., determines their pseudoscalar). For instance, in 3D GA, a hyperplane in extrinsic orientation $e_1$ combines with its intrinsically oriented version $e^{23}$, to produce $e^{23}[e_1] = e_{123} = \mathbb{I}$.

[2]When one is not too specific about orientations of subspaces as represented by blades, such signs may be considered unimportant. This 'projective' or 'homogeneous' view of how reality is encoded into a geometric algebra can therefore be agnostic to the difference between a space repesented directly or dually. In the usual duality, the terms OPNS and IPNS have been used for this distinction, depending on whether a blade $A$ is probed with a GA point probe representative $p$ as $p \wedge A = 0$ (what $p$ is contained in $A$?) or $p \cdot A = 0$ (to what $p$ is $A$ perpendicular?) to determine the set of points it represents. Such equations indeed do not depend on non-zero scalar factors, they do not encode an 'oriented geometry'.

[3]We have of course been using both types in many applications in the past, as the need arose. As long as we were using even versors (and those are the Lie group of 'motions' which may be done a little at a time), there is no distinction between the transformations of the orientation type. We could therefore get away with just ignoring that type, deciding on the actual type either before or after performing the transformation, without any difference in the result. However, this is not how oriented geometry really is, and as soon as we would need to include reflections, the necessity for discrimination arises.

# 4 Constructive Connections: Meet and Join

The practical difference in orientation types manifests itself most clearly in the relative relationships of subspaces.

- We construct blades from vectors of the representational space; these represent the basic reflectors in the Cartan-Dieudonné-based view of orthogonal transformations as multiple reflections in hyperplanes. This is the *extrinsic* view, which is thus our standard approach to GA: in the geometric algebra OGA of planar reflections at an origin, a hyperplane is represented by a normal vector, the direction of the vector allowing distinction between a positive/negative half-space, an inside/outside relative to the hyperplane considered as (local) object boundary.[4] Higher order subspaces with higher grades are then constructed by *intersection* of these representational hyperplanes, which in their geometric algebra corresponds to the wedge product of their vectors. Thus the wedge of extrinsic elements can be considered as their *meet*. In 3D PGA, the wedge of 3 vectors produces a trivector which represents the intersection point of the corresponding planes.

- The 'complementation' view of the dual shows clearly that combining indices of inputs to produce elements with more indices has the effect on the dual of combining their duals by eliminating overlapping indices (with always some possible sign changes). Thus in 3D PGA (which has $n = 4$), an external point 3-vector from the algebra of $W$ is dually represented as an intrinsic 1-vector in the dual space $W^\natural$.

  When we perform a wedge in the algebra of the dual space $W^\natural$ a union of the geometric elements is formed, which is called the *join*, and it produces the composite *intrinsic* elements. In point-based 3D PGA, two points can connect to form an intrinsic line bivector (a spear), which is the dual of the extrinsic bivector (an axis) in the primal plane-based algebra, the meet of two planes.

Thus the operations of meet and join are computable as simple index operations; they are both bilinear (with for non-disjoint indices a zero wedge result), and they are dual to each other, being wedge (i.e., index union) operations in dually related spaces.

In applications, we are also interested in joining extrinsic points (they are, after all, just geometrical points) or intersecting intrinsic planes. This can be constructed by dualization and a wedge; in [1] we introduced that as a 'dual join' $\triangledown$ to complete the set of four sibling operations we desire We now denote this as $\vee$; the arguments of either $\wedge$ or $\triangledown$ should be of the same type, but may be from a space or its dual. For 3D DGA, the algebra of planes through the origin, this looks like:

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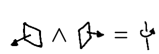
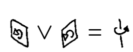
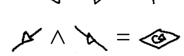
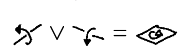
|---|---|---|---|---|---|
| **Meeting Extrinsics:** | $e_3 \wedge e_1 = e_{31}$ | $\longleftrightarrow$ |  | $\longleftrightarrow$ | $\star e^{12} \wedge \star e^{23} = \star e^2$ |
| **Meeting Intrinsics:** | ${e_3}^\natural \vee {e_1}^\natural = e_{31}$ | $\longleftrightarrow$ |  | $\longleftrightarrow$ | $e^{12} \vee e^{23} = \star e^2$ |
| **Joining Intrinsics:** | ${e_{12}}^\natural \wedge {e_{23}}^\natural = {e_2}^\natural$ | $\longleftrightarrow$ |  | $\longleftrightarrow$ | $e^3 \wedge e^1 = e^{31}$ |
| **Joining Extrinsics:** | $e_{12} \vee e_{23} = {e_2}^\natural$ | $\longleftrightarrow$ |  | $\longleftrightarrow$ | $\star e^3 \vee \star e^1 = e^{31}$ |

However we take the meet and join on their legally permissible elements in this manner, they will only involve an *even* number of pseudoscalars, and therefore transform equivariantly under

[4]When taking the extrinsic view, hyperplanes (in the representational space, they may of course represent much less planar realities. In CGA they are spheres, in the projective $\mathbb{R}_{3,3}$ they are lines; it all depends on the embedding function establishing the relationship between reality and the model algebra.

versors (without nett $\det(\underline{V})$ sign). By contrast, the classical join defined as: 'the elements whose dual is the meet of duals' involves three pseudoscalars, and hence flips sign under reflection. The confusing behaviour this causes may be a reason why practitioners commonly neglect orientation signs altogether.

## 5 The Metric of the Dual Space

In 3D PGA $\mathbb{R}_{3,0,1}$, intersecting three independent planes results in the point where they meet being represented as a trivector – which after dualization becomes a vector of $W^\natural$. Hence the dual of plane-based geometric algebra is point-based geometric algebra (i.e., and algebra in which vectors model geometric points). There is a clear metric relationship between the two: a translation versor from the plane-based view (a ratio of two points) leads to a natural 'join distance' along their connecting line in the point-based view. Thus we desire to relate the quadratic forms $q$ and $q^\natural$ for the dually related geometric algebras $\mathrm{Cl}(W,q)$ and $\mathrm{Cl}(W,q)^\natural \simeq \mathrm{Cl}(W^\natural, q^\natural)$. In the particular case of PGA, the Euclidean motion algebra $\mathbb{R}_{d,0,1}$ useful for Newtonian mechanics [3] can be dualized to the algebra of paraxial optics $\mathbb{R}^*_{d,0,1}$, see [2].

A first attempt at a mathematically natural procedure based on preserving the null structures [4] transfers the metric relationships from $\mathrm{Cl}(W,q)$ to the dual space, by isomorphically identifying the metric properties of an element ${e_A}^\natural$ to those of $e_A$. In the case of $\mathbb{R}_{d,0,1}$ that produces the algebra $\mathbb{R}^*_{1,0,d}$ (e.g., ${e_2}^\natural = e^{01}$ is null in 2D PGA becaues $e_{01}$ is). This is not what we desire in the PGA application above, so the mapping of an application field to one's chosen geometric algebra may have to play an essential role in the choice of metrics.

At the moment, we are investigating how much freedom exists in defining metrics in the two spaces, while maintaining the meet and join relationships (which are of a more homogeneous nature, so allow some leeway) and preferably the orthogonality of the two dual bases of eq.(2). In the case of PGA, this may be most naturally described by relating $\mathbb{R}_{d,0,1}$ and $R^*_{d,0,1}$ through they common CGA algebra $\mathbb{R}_{d+1,1,0}$.

We will report on the latest developments at AGACSE, either circumscribing the freedom of choice, or delineating the issues involved.

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