ON GENERALIZED DEGENERATE LIPSCHITZ AND SPIN GROUPS

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Summary of the Abstract

In this talk, we introduce and study generalized Lipschitz and spin groups in degenerate geometric (Clifford) algebras of arbitrary dimension and signature. The generalized degenerate Lipschitz and spin groups contain the corresponding ordinary Lipschitz and spin groups as subgroups and coincide with them in the low-dimensional cases. We prove that an element of the generalized degenerate Lipschitz group can be represented as a product of an element of fixed parity and an element of the Grassmann subalgebra. It is shown that the values of norm functions of elements of the generalized degenerate Lipschitz groups belong to the kernel of the twisted adjoint representation. The introduced groups can be interesting for applications in physics, engineering, and computer science.

Abstract

In this talk, we consider degenerate and non-degenerate real and complex geometric (Clifford) algebras $\mathcal{G}_{p,q,r}$, $p+q+r=n\geq 1$, of arbitrary dimension and signature (in the case of complex geometric algebra, we can take $q=0$). We concentrate on the degenerate $\mathcal{G}_{p,q,r}$, $r\neq 0$, however all the statements are true in the case $r=0$. In particular, we consider the Grassmann (exterior) algebra $\mathcal{G}_{0,0,r}$, which is denoted by Λ_r.

One of the most significant notions in the theory of spin groups is the twisted adjoint representation $\check{\mathrm{ad}}$. It is used to describe two-sheeted coverings of orthogonal groups by spin groups. For the first time, $\check{\mathrm{ad}}$ has been introduced by Atiyah, Bott, and Shapiro in [2] in the case $r=0$. This definition can be straightforwardly generalized to the case of arbitrary $\mathcal{G}_{p,q,r}$ in the following way. The twisted adjoint representation $\check{\mathrm{ad}}$ acts on the Lipschitz group $\Gamma^{\pm\Lambda}_{p,q,r}$ (3) in the way $\check{\mathrm{ad}}:\Gamma^{\pm\Lambda}_{p,q,r}\to\mathrm{Aut}(\mathcal{G}^1_{p,q,r})$ as $T\mapsto\check{\mathrm{ad}}_T$, where $\check{\mathrm{ad}}_T:\mathcal{G}^1_{p,q,r}\to\mathcal{G}^1_{p,q,r}$ is defined for elements of the grade-1 subspace $\mathcal{G}^1_{p,q,r}$ as

$$\check{\mathrm{ad}}_T(U)=\widehat{T}UT^{-1},\qquad U\in\mathcal{G}^1_{p,q,r},\qquad T\in\Gamma^{\pm\Lambda}_{p,q,r},\tag{1}$$

where $\widehat{\ }$ is the grade involution. The kernel of the twisted adjoint representation $\check{\mathrm{ad}}$ coincides with the set of all invertible elements of the Grassmann subalgebra Λ_r:

$$\ker(\check{\mathrm{ad}}) = \{T \in \Gamma^{\pm\Lambda}_{p,q,r} : \quad \widehat{T}UT^{-1} = U, \quad \forall U \in \mathcal{G}^1_{p,q,r}\} = \Lambda^\times_r, \tag{2}$$

where $\times$ denotes the subset of all invertible elements of a set.

In the non-degenerate geometric algebras $\mathcal{G}_{p,q,0}$, the Lipschitz group is defined as the group of all invertible elements preserving the grade-1 subspace under the twisted adjoint representation $\check{\mathrm{ad}}$. Generalizing this definition to the case of the degenerate $\mathcal{G}_{p,q,r}$, we similarly define the Lipschitz group $\Gamma^{\pm\Lambda}_{p,q,r}$ as

$$\Gamma^{\pm\Lambda}_{p,q,r} := \{T \in \mathcal{G}^\times_{p,q,r} : \quad \widehat{T}\mathcal{G}^1_{p,q,r}T^{-1} \subseteq \mathcal{G}^1_{p,q,r}\}. \tag{3}$$

The definition (3) of the degenerate Lipschitz group is used, for example, in the works [3, 4, 6]. The upper index $^{\pm\Lambda}$ in the notation of the Lipschitz group is due to the equivalent definition (4), which we prove using Theorems 2 and 3:

$$\Gamma^{\pm\Lambda}_{p,q,r} = \{T \in (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^\times_r : \quad \widehat{T}\mathcal{G}^1_{p,q,r}T^{-1} \subseteq \mathcal{G}^1_{p,q,r}\}. \tag{4}$$

In the talk, we also discuss other approaches to define the degenerate Lipschitz group $\Gamma^{\pm\Lambda}_{p,q,r}$ considered in the literature. In the particular case of the Grassmann algebra Λ_n, the Lipschitz group $\Gamma^{\pm\Lambda}_{0,0,n}$ coincides with the kernel (2) of the twisted adjoint representation $\check{\mathrm{ad}}$:

$$\Gamma^{\pm\Lambda}_{0,0,n} = \Lambda^\times_n = \ker(\check{\mathrm{ad}}). \tag{5}$$

The structure of the Lipschitz groups in the case of other degenerate geometric algebras $\mathcal{G}_{p,q,r}$, $r \neq n$, is more complicated. We discuss the details in the talk.

We also consider the subgroup $\Gamma^{\pm}_{p,q,r}$ of the Lipschitz group $\Gamma^{\pm\Lambda}_{p,q,r}$ (4):

$$\Gamma^{\pm}_{p,q,r} := \{T \in \mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r} : \quad T\mathcal{G}^1_{p,q,r}T^{-1} \subseteq \mathcal{G}^1_{p,q,r}\} \subseteq \Gamma^{\pm\Lambda}_{p,q,r}. \tag{6}$$

The subgroup $\Gamma^{\pm}_{p,q,r}$ is discussed, for example, in [11]. Let us note Corollary E.27 of this work regarding the form of an arbitrary element of the group $\Gamma^{\pm}_{p,q,r}$ (this statement can be found, for example, in [6] as well). It is of interest to generalize this statement to more general cases.

In the case of the non-degenerate geometric algebra $\mathcal{G}_{p,q,0}$, the groups $\Gamma^{\pm}_{p,q,0}$ (6) and $\Gamma^{\pm\Lambda}_{p,q,0}$ (3) coincide:

$$\Gamma^{\pm}_{p,q,0} = \Gamma^{\pm\Lambda}_{p,q,0}. \tag{7}$$

Theorem 1 *The Lipschitz group $\Gamma^{\pm\Lambda}_{p,q,r}$ (4) can be represented as a product of the groups:*

$$\Gamma^{\pm\Lambda}_{p,q,r} = \Gamma^{\pm}_{p,q,r}\Lambda^\times_r. \tag{8}$$

We consider the Lie group $\check{\mathrm{Q}}^{\overline{1}}_{p,q,r}$:

$$\check{\mathrm{Q}}^{\overline{1}}_{p,q,r} := \{T \in \mathcal{G}^\times_{p,q,r} : \quad \widehat{T}\mathcal{G}^{\overline{1}}_{p,q,r}T^{-1} \subseteq \mathcal{G}^{\overline{1}}_{p,q,r}\}, \tag{9}$$

where $\mathcal{G}^{\overline{1}}_{p,q,r}$ is the subspace determined by the grade involution $\widehat{\ }$ and the reversion $\widetilde{\ }$:

$$\mathcal{G}^{\overline{k}}_{p,q,r} := \{U \in \mathcal{G}_{p,q,r}: \quad \widehat{U} = (-1)^k U, \quad \widetilde{U} = (-1)^{\frac{k(k-1)}{2}} U\} = \bigoplus_{j=k \bmod 4} \mathcal{G}^j_{p,q,r}, \quad k = 0,1,2,3.$$

We call the group $\check{\mathrm{Q}}^{\overline{1}}_{p,q,r}$ (9) *the generalized degenerate Lipschitz group* because of Theorem 2.

Theorem 2 *The group $\check{\mathrm{Q}}^{\overline{1}}_{p,q,r}$ contains the Lipschitz group $\Gamma^{\pm\Lambda}_{p,q,r}$ as a subgroup and coincides with it in the case of the low-dimensional $\mathcal{G}_{p,q,r}$:*

$$\Gamma^{\pm\Lambda}_{p,q,r} \subseteq \check{\mathrm{Q}}^{\overline{1}}_{p,q,r}, \quad \forall n; \qquad \Gamma^{\pm\Lambda}_{p,q,r} = \check{\mathrm{Q}}^{\overline{1}}_{p,q,r}, \quad n \leq 4. \tag{10}$$

We also consider the following Lie group introduced and studied in [8, 9]:

$$\mathrm{P}^{\pm\Lambda}_{p,q,r} := (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^{\times}_r, \tag{11}$$

where $\mathcal{G}^{(0)}_{p,q,r}$ and $\mathcal{G}^{(1)}_{p,q,r}$ are the even and odd subspaces respectively.

Theorem 3 *We have the following inclusion:*

$$\check{\mathrm{Q}}^{\overline{1}}_{p,q,r} \subseteq \mathrm{P}^{\pm\Lambda}_{p,q,r}. \tag{12}$$

Consider the following two norm functions, which are widely used in the theory of spin groups:

$$\psi(T) := \widetilde{T}T, \qquad \chi(T) := \widehat{\widetilde{T}}T, \qquad \forall T \in \mathcal{G}_{p,q,r}. \tag{13}$$

Theorem 4 *The generalized Lipschitz group $\check{\mathrm{Q}}^{\overline{1}}_{p,q,r}$ has the following equivalent definition:*

$$\check{\mathrm{Q}}^{\overline{1}}_{p,q,r} = \{T \in \mathcal{G}^{\times}_{p,q,r}: \quad \widetilde{T}T \in \ker(\check{\mathrm{ad}}), \quad \widehat{\widetilde{T}}T \in \ker(\check{\mathrm{ad}})\}. \tag{14}$$

Therefore, the values of the norm functions (13) of the degenerate Lipschitz groups' elements are in the kernel of $\check{\mathrm{ad}}$ (2):

$$\widetilde{T}T \in \ker(\check{\mathrm{ad}}), \qquad \widehat{\widetilde{T}}T \in \ker(\check{\mathrm{ad}}), \qquad \forall T \in \Gamma^{\pm\Lambda}_{p,q,r}. \tag{15}$$

The degenerate spin groups are discussed in many papers [1, 3, 4, 5, 6, 7, 11, 10, 12]. In the talk, we discuss several approaches how to define these groups and the relation between them. We define the ordinary degenerate spin groups (17)–(20) as normalized subgroups of the Lipschitz group $\Gamma^{\pm\Lambda}_{p,q,r}$ (4) and its even subgroup

$$\Gamma^{+}_{p,q,r} := \{T \in \mathcal{G}^{(0)\times}_{p,q,r}: \quad T\mathcal{G}^1_{p,q,r}T^{-1} \subseteq \mathcal{G}^1_{p,q,r}\} \subseteq \Gamma^{\pm}_{p,q,r} \tag{16}$$

in the following way:

$$\mathrm{Pin}_{\psi}(p,q,r) := \{T \in \Gamma^{\pm\Lambda}_{p,q,r}: \quad \widetilde{T}T = \pm e\}, \quad \mathrm{Pin}_{\chi}(p,q,r) := \{T \in \Gamma^{\pm\Lambda}_{p,q,r}: \quad \widehat{\widetilde{T}}T = \pm e\}, \tag{17}$$

$$\mathrm{Pin}_{+\psi}(p,q,r) := \{T \in \Gamma^{\pm\Lambda}_{p,q,r}: \quad \widetilde{T}T = +e\}, \quad \mathrm{Pin}_{+\chi}(p,q,r) := \{T \in \Gamma^{\pm\Lambda}_{p,q,r}: \quad \widehat{\widetilde{T}}T = +e\}, \tag{18}$$

$$\mathrm{Spin}(p,q,r) := \{T \in \Gamma^{+}_{p,q,r}: \quad \widetilde{T}T = \pm e\} = \{T \in \Gamma^{+}_{p,q,r}: \quad \widehat{\widetilde{T}}T = \pm e\}, \tag{19}$$

$$\mathrm{Spin}_{+}(p,q,r) := \{T \in \Gamma^{+}_{p,q,r}: \quad \widetilde{T}T = +e\} = \{T \in \Gamma^{+}_{p,q,r}: \quad \widehat{\widetilde{T}}T = +e\}. \tag{20}$$

In the particular case of the non-degenerate geometric algebras $\mathcal{G}_{p,q,0}$, the groups $\mathrm{Pin}_{\psi}(p,q,0)$ and $\mathrm{Pin}_{\chi}(p,q,0)$ coincide. However in the general case of arbitrary $\mathcal{G}_{p,q,r}$, the six groups (17)–(20) are different.

In this talk, we introduce and study the *generalized degenerate spin groups*. We define them as normalized subgroups of the generalized degenerate Lipschitz group $\check{\mathrm{Q}}^{\overline{1}}_{p,q,r}$ (9) and its even subgroup $\mathcal{G}^{(0)\times}_{p,q,r}$:

$$\mathrm{Pin}^{\mathrm{Q}}_{\psi}(p,q,r) := \{T \in (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^{\times}_{r} : \quad \widetilde{T}T = \pm e\}, \tag{21}$$

$$\mathrm{Pin}^{\mathrm{Q}}_{\chi}(p,q,r) := \{T \in (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^{\times}_{r} : \quad \widehat{\widetilde{T}}T = \pm e\}, \tag{22}$$

$$\mathrm{Pin}^{\mathrm{Q}}_{+\psi}(p,q,r) := \{T \in (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^{\times}_{r} : \quad \widetilde{T}T = +e\}, \tag{23}$$

$$\mathrm{Pin}^{\mathrm{Q}}_{+\chi}(p,q,r) := \{T \in (\mathcal{G}^{(0)\times}_{p,q,r} \cup \mathcal{G}^{(1)\times}_{p,q,r})\Lambda^{\times}_{r} : \quad \widehat{\widetilde{T}}T = +e\}, \tag{24}$$

$$\mathrm{Spin}^{\mathrm{Q}}(p,q,r) := \{T \in \mathcal{G}^{(0)\times}_{p,q,r} : \quad \widetilde{T}T = \pm e\} = \{T \in \mathcal{G}^{(0)\times}_{p,q,r} : \quad \widehat{\widetilde{T}}T = \pm e\}, \tag{25}$$

$$\mathrm{Spin}^{\mathrm{Q}}_{+}(p,q,r) := \{T \in \mathcal{G}^{(0)\times}_{p,q,r} : \quad \widetilde{T}T = +e\} = \{T \in \mathcal{G}^{(0)\times}_{p,q,r} : \quad \widehat{\widetilde{T}}T = +e\}. \tag{26}$$

The generalized degenerate spin groups (21)–(26) contain the corresponding ordinary degenerate spin groups (17)–(20) as subgroups.

The ordinary degenerate Lipschitz groups and spin groups [1, 3, 4, 6] have applications in construction of Clifford group equivariant neural networks [11] and geometric Clifford algebra networks [12], representation theory of Galilei group in quantum mechanics [4], etc. The generalized degenerate spin groups can be interesting for applications of geometric algebras, for example, in physics, computer vision, and neural networks.

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