Compatible Null Vectors, Heron's Formula and Conformal Duality

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Summary of Abstract

The second author has recently shown that one can define and study Lorentzian vector spaces based on the concept of compatible null vectors. Around the same time the first author used the relation $r = t/s$ between the in-radius r, area t and semi-perimeter s of a triangle to generalize Heron's classical formula to simplices in all dimensions. A few years earlier, Udo Hertrich-Jeromin, Alastair King and Jun O'Hara used the conformal model of Euclidean geometry to show that the vertices of a triangle are related to its ex-centers via reflection w.r.t. a time-like vector exchanging its in-center with the point-at-infinity, an operation they named the conformal dual. In this Abstract, we will explore the interconnections among these three previously independent lines of study of the Lorentzian geometric algebras, focusing on the simple case of $\mathcal{G}_{3,1}$.

Compatible Null Vectors and Conformal Geometric Algebra. In a recent series of papers [4, 5], the second author has characterized Lorentzian metric vector spaces, which have signature $[1, -1, \ldots, -1]$ or $[-1, 1, \ldots, 1]$, in terms of the geometric algebras generated by a set of abstract null vectors. Any such pair of non-zero null vectors $\boldsymbol{a}^2 = \boldsymbol{b}^2 = 0$ is easily seen to satisfy the multiplication table

Null Vector Multiplication Table

$2\,\boldsymbol{a}\cdot\boldsymbol{b} = \gamma$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{ab}$	$\boldsymbol{ba}$
$\boldsymbol{a}$	0	$\boldsymbol{ab}$	0	$\gamma\boldsymbol{a}$
$\boldsymbol{b}$	$\boldsymbol{ba}$	0	$\gamma\boldsymbol{b}$	0
$\boldsymbol{ab}$	$\gamma\boldsymbol{a}$	0	$\gamma\boldsymbol{ab}$	0
$\boldsymbol{ba}$	0	$\gamma\boldsymbol{b}$	0	$\gamma\boldsymbol{ba}$

(1)

wherein $\gamma = (\boldsymbol{ab}+\boldsymbol{ba}) = 2\,\boldsymbol{a}\cdot\boldsymbol{b} \neq 0$, and to generate the geometric algebra $\mathcal{G}_{1,1}$. A pair of null vectors for which $2\,\boldsymbol{a}\cdot\boldsymbol{b} = 1$ is called a *conjugate pair*, in which case $\boldsymbol{ab}$ and $\boldsymbol{ba}$ are mutually annihilating idempotents that partition unity. A set of null vectors $\{\boldsymbol{a}_i \,|\, i = 0, \ldots, n\}$ with $\gamma_{ij} = \gamma_{ji} \coloneqq 2\,\boldsymbol{a}_i \cdot \boldsymbol{a}_j$ is said to *positively correlated* if $\gamma_{ij} > 0$ and *negatively correlated* if $\gamma_{ij} < 0$ for all $0 \leqslant i, j \leqslant n$. More generally, if there exist $\epsilon_i \in \{\pm 1\}$ such that $\{\epsilon_i\,\boldsymbol{a}_i\}$ is positively or negatively correlated, the vectors will be called mutually *compatible*. Such a (mutually) compatible system of null vectors generates an algebra which is isomorphic to $\mathcal{G}_{1,n}$ when it is positively correlated, or $\mathcal{G}_{n,1}$ when it is negatively correlated.

In particular, the geometric algebra $\mathcal{G}_{n+1,1}$ of the well-known "conformal model" of n-dimensional Euclidean (and conformal) geometry is obtained from any negatively correlated system of $n+1$ null vectors with $\boldsymbol{a}_0 \cdot \boldsymbol{a}_i = -1$ for $0 \leqslant i \leqslant n+1$. In this case, the indefinite analogue of Gram-Schmidt ortho-normalization given in Refs [4, 5] will produce an orthonormal basis including $\mathbf{f} = \boldsymbol{a}_1 + \boldsymbol{a}_0/2$, $\mathbf{e}_0 = \boldsymbol{a}_1 - \boldsymbol{a}_0/2$, and an additional n anti-commuting linear combinations of the null vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ with $-\mathbf{f}^2 = \mathbf{e}_1^2 = \cdots \mathbf{e}_n^2 = 1$. It can then be shown that $\boldsymbol{a}_i = \boldsymbol{a}_1 + \mathbf{a}_i + \boldsymbol{a}_0\, \mathbf{a}_i^2/2$, where $\mathbf{a}_i$ is the orthogonal projection of $\boldsymbol{a}_i$ on the subspace $\langle\!\langle \mathbf{e}_1, \ldots, \mathbf{e}_n \rangle\!\rangle$ for $i = 2, \ldots, n+1$, so that $\boldsymbol{a}_0$ serves as the point-at-infinity $\boldsymbol{n}_\infty$ of the conformal model while $\boldsymbol{a}_1$ plays the role of the origin $\boldsymbol{n}_0$. It should be noted however that a *positively* correlated system could be used just as well, and that this is in some respects more natural since then the inner products are half the squared distances rather than the negatives thereof, and $\boldsymbol{n}_\infty = \boldsymbol{a}_0$ is conjugate to $\boldsymbol{n}_0 = \boldsymbol{a}_1$ as well as $\boldsymbol{a}_2, \ldots, \boldsymbol{a}_{n+1}$. Nonetheless, in what follows we will stick with negatively correlated null vectors and conformal geometric algebra as it is commonly defined.

Hence consider a negatively correlated system of four null vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{n}_\infty \in \mathcal{G}_{3,1}$ satisfying $\boldsymbol{a} \cdot \boldsymbol{n}_\infty = \boldsymbol{b} \cdot \boldsymbol{n}_\infty = \boldsymbol{c} \cdot \boldsymbol{n}_\infty = -1$, as in Ref. [2, §5]. These vectors determine a triangle in the Euclidean plane having squared edge lengths $a^2 = -2\,\boldsymbol{b} \cdot \boldsymbol{c}$, $b^2 = -2\,\boldsymbol{a} \cdot \boldsymbol{c}$, $c^2 = -2\,\boldsymbol{a} \cdot \boldsymbol{b}$. Without loss of generality we may take $\boldsymbol{a}$ as the origin and write

$$\boldsymbol{a} \;=\; \boldsymbol{n}_0\,, \quad \boldsymbol{b} \;=\; \boldsymbol{n}_0 + \mathbf{b} + \boldsymbol{n}_\infty\, c^2/2\,, \quad \boldsymbol{c} \;=\; \boldsymbol{n}_0 + \mathbf{c} + \boldsymbol{n}_\infty\, b^2/2 \tag{2}$$

($\mathbf{a} = 0, \mathbf{b}, \mathbf{c} \in \mathcal{G}_2 \subset \mathcal{G}_{3,1}$). Then the squared area of the triangle is given by

$$\begin{aligned} t^2 \;\coloneqq\; \tfrac{1}{4}\,\|\mathbf{b} \wedge \mathbf{c}\|^2 \;&=\; \tfrac{1}{4}\left(\mathbf{b}^2\mathbf{c}^2 - (\mathbf{b} \cdot \mathbf{c})^2\right) \;=\; \tfrac{1}{4}\left(b^2c^2 - (b^2 + c^2 - a^2)^2/4\right) \\ &=\; \tfrac{1}{16}(a+b+c)(a+b-c)(a-b+c)(-a+b+c)\;, \end{aligned} \tag{3}$$

where the law of cosines was used at the end of the first line. The r.h.s. is of course Heron's formula for the (squared) area of a triangle $[\![\mathbf{a}, \mathbf{b}, \mathbf{c}]\!]$.

We can "lift" this $\mathcal{G}_2$ formula into an expression involving the corresponding null vectors together with the point-at-infinity in $\mathcal{G}_{3,1}$ as follows. First, note that $\boldsymbol{a}^2 = \mathbf{a}^2 = 0$, $(\boldsymbol{b} - \boldsymbol{a})^2 = 2\,\boldsymbol{a} \cdot \boldsymbol{b} = c^2$ and $(\boldsymbol{c} - \boldsymbol{a})^2 = 2\,\boldsymbol{a} \cdot \boldsymbol{c} = b^2$ while $(\boldsymbol{b} - \boldsymbol{a}) \cdot (\boldsymbol{c} - \boldsymbol{a}) = \boldsymbol{b} \cdot \boldsymbol{c} - \boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \frac{1}{2}(b^2 + c^2 - a^2) = \mathbf{b} \cdot \mathbf{c}$. It follows that $t^2 = \|(\boldsymbol{b} - \boldsymbol{a}) \wedge (\boldsymbol{c} - \boldsymbol{a})\|^2$ as well. Expanding this outer product gives $\boldsymbol{b} \wedge \boldsymbol{c} - \boldsymbol{a} \wedge \boldsymbol{c} + \boldsymbol{a} \wedge \boldsymbol{b}$, which is the *boundary* $-\boldsymbol{n}_\infty \cdot (\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c})$ of the trivector formed from the vertices of the triangle. From this we can show that t^2 is given by the *Cayley-Menger determinant* $-\|\boldsymbol{n}_\infty \wedge \boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c}\|^2$ of the vertices, namely

$$\begin{aligned} &\big((\boldsymbol{c} \wedge \boldsymbol{b} \wedge \boldsymbol{a}) \cdot \boldsymbol{n}_\infty\big) \cdot \big(\boldsymbol{n}_\infty \cdot (\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c})\big) \;= \\ &\quad (\boldsymbol{c} \wedge \boldsymbol{b} \wedge \boldsymbol{a}) \cdot \big(\boldsymbol{n}_\infty \wedge \big(\boldsymbol{n}_\infty \cdot (\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c})\big)\big) \;= \\ &\quad -\big(\boldsymbol{c} \wedge \boldsymbol{b} \wedge \boldsymbol{a}\big) \cdot \big(\boldsymbol{n}_\infty \cdot \big(\boldsymbol{n}_\infty \wedge (\boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c})\big)\big) \;= \\ &\quad\quad -\big(\boldsymbol{c} \wedge \boldsymbol{b} \wedge \boldsymbol{a} \wedge \boldsymbol{n}_\infty\big) \cdot \big(\boldsymbol{n}_\infty \wedge \boldsymbol{a} \wedge \boldsymbol{b} \wedge \boldsymbol{c}\big)\;. \end{aligned} \tag{4}$$

Expansion of this determinant in the squared edge lengths yields a polynomial which factorizes to give Heron's formula as above.

Heron's Formula and the Heron Parameters. In the course of extending Heron's formula for the area of a Euclidean triangle to the hyper-volumes of simplices in all dimensions n, the first author introduced a set of $n(n-1)/2$ invariant non-negative parameters that jointly determine an n-simplex up to isometry [1, 2]. He called them the *natural parameters* because, in contrast to the multi-variate polynomial equalities and inequalities the inter-vertex distances must fulfill, the consistency relations they satisfy are nearly trivial. In particular, a triangle $[\![\mathbf{a},\mathbf{b},\mathbf{c}]\!] \subset \mathbb{R}^2$ may be specified up to isometry by the three independent parameters $u, v, w \geqslant 0$, which determine its edge lengths simply as $a = v + w$, $b = u + w$, $c = u + v$. The tangency of the triangle's edges to its in-circle together with the Pythagorean theorem shows that u, v, w are the equal lengths of the pairs of line segments connecting its vertices to its *in-touch points* $\mathbf{j}, \mathbf{k}, \mathbf{l}$ (cf. Fig. 1). This simple geometric interpretation played a key role in generalizing Heron's formula to higher dimensions.

The geometric properties of triangles and triangle centers often become algebraically much simpler when expressed in terms of u, v, w. For example, the Pythagorean criterion for the triangle to have a right angle at $\mathbf{a}$ is $su = vw$ where $s \coloneqq u + v + w = (a + b + c)/2$ is its semi-perimeter, while Heron's formula for the squared area of the triangle becomes

$$t^2 \;=\; s\,uvw \;=\; \tfrac{1}{2}\,s\,\Omega(u,v,w) \;\coloneqq\; \tfrac{1}{2}\,(u+v+w)\det\begin{bmatrix} 0 & u & v \\ u & 0 & w \\ v & w & 0 \end{bmatrix}\,. \tag{5}$$

Hence the natural parameters of a triangle are also known as *Heron parameters.* Upon solving the above three linear equations giving a, b, c in terms of u, v, w, we find that $u = (-a + b + c)/2$, $v = (a - b + c)/2$ & $w = (a + b - c)/2$, which yields the traditional version of Heron's formula (3) upon substitution.

The zeros of Heron's formula are degenerate triangles with collinear vertices, and the factor u, v or w thereof that vanishes depends upon which vertex lies between the other two.

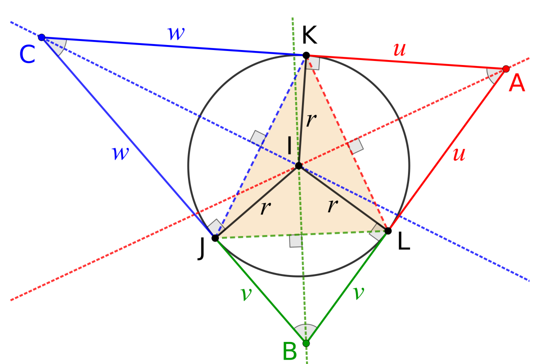

Figure 1: The geometric interpretation of the Heron parameters u, v, w as the equal lengths of the pairs of line segments into which the edges incident each vertex of the triangle ABC are divided by the *in-touch points* J, K, L of its in-circle. These parameters determine the in-radius $r = t/s = \sqrt{uvw/s}$, where $s = u + v + w$ is the semi-perimeter, and hence its area as $t = \sqrt{s\,uvw}$. (Reproduced with permission from Ref. [1].)

Surprisingly, the zeros in higher dimensions are much more subtle, and indeed unprecedented in classical Euclidean geometry. In particular, the generic zeros in three dimensions do *not* correspond to tetrahedra with co-planar vertices, but to collinear tetrahedra with vertices separated by *infinite* distances [1]! The conformal geometric algebra derivation of the formula for tetrahedra [2] however suggests that these strange geometric configurations can be better understood by taking the full conformal symmetry of the equations into account. The final section of this Abstract gives an overview of how that works in two dimensions.

The Conformal Center and Dual of a Triangle. In an absolutely beautiful expository paper that has garnered but a single independent citation in the decade since it was published [3], Udo Hertrich-Jeromin, Alastair King & Jun O'Hare showed that the three ex-centers and in-center of a triangle are "dual" to its vertices and the point-at-infinity. This duality relation is moreover preserved under conformal (or Möbius) transformations, earning it the cognomen of the *conformal dual.* Here we reformulate their main ideas in the conformal geometric algebra $\mathcal{G}_{3,1}$, with emphasis on their interpretations in the Euclidean plane.

This is most elegantly done by renormalizing the null vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathcal{G}_{3,1}$ representing the vertices along with the point-at-infinity $\boldsymbol{n}_\infty$ as follows. First, one computes the reciprocal vector space basis $[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}]$ of the basis $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c},\ \boldsymbol{d} \coloneqq \boldsymbol{n}_\infty]$, which satisfies $\boldsymbol{a} \cdot \boldsymbol{A} = \boldsymbol{b} \cdot \boldsymbol{B} = \boldsymbol{c} \cdot \boldsymbol{C} = \boldsymbol{d} \cdot \boldsymbol{D} = 1$ while all the other inner products between the two bases vanish. Geometrically, the reciprocal vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ dually represent the lines spanned by the edges $[\![\mathbf{b}, \mathbf{c}]\!]$, $[\![\mathbf{a}, \mathbf{c}]\!]$, $[\![\mathbf{a}, \mathbf{b}]\!]$ resp., in that any point $\boldsymbol{x}$ is on those lines whenever $\boldsymbol{x} \cdot \boldsymbol{A} = 0$, $\boldsymbol{x} \cdot \boldsymbol{B} = 0$, $\boldsymbol{x} \cdot \boldsymbol{C} = 0$, and their Lorentzian norms $\|\boldsymbol{A}\| = 1/h_{\mathsf{a}}$, $\|\boldsymbol{B}\| = 1/h_{\mathsf{b}}$, $\|\boldsymbol{C}\| = 1/h_{\mathsf{c}}$ are the inverse heights of the vertices over their opposite edges. The reciprocal vector $\boldsymbol{D}$ represents the circum-circle of the triangle in the same way, and its norm $\|\boldsymbol{D}\| = R$ is the circum-radius of the triangle. Second, one scales the reciprocal vectors to obtain a new basis $\hat{\boldsymbol{A}} \coloneqq h_{\mathsf{a}} \boldsymbol{A}$, $\hat{\boldsymbol{B}} \coloneqq h_{\mathsf{b}} \boldsymbol{B}$, $\hat{\boldsymbol{C}} \coloneqq h_{\mathsf{c}} \boldsymbol{C}$ & $\hat{\boldsymbol{D}} = \boldsymbol{D}/R$ the norms of which are all unity. The inner products of the null vector basis with their respective normalized reciprocals are then $h_{\mathsf{a}}, h_{\mathsf{b}}, h_{\mathsf{c}}$, whereas $\boldsymbol{d} \cdot \hat{\boldsymbol{D}} = 1/R$. Finally, one applies the inverse scale factors to the corresponding null vectors to obtain a new basis satisfying $\hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{A}} = \hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{B}} = \hat{\boldsymbol{c}} \cdot \hat{\boldsymbol{C}} = \hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{D}} = 1$, so the two renormalized bases again constitute a reciprocal pair.

The *conformal center* of the quadrangle $\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \hat{\boldsymbol{c}}, \hat{\boldsymbol{d}}$ is then defined as half the sum of these renormalized null vectors:

$$\boldsymbol{z} \;=\; \tfrac{1}{2}\left(\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}} + \hat{\boldsymbol{c}} + \hat{\boldsymbol{d}}\right) \;=\; \tfrac{1}{2}\left(\boldsymbol{a}/h_{\mathsf{a}} + \boldsymbol{b}/h_{\mathsf{b}} + \boldsymbol{c}/h_{\mathsf{c}} + \boldsymbol{n}_\infty R\right) \tag{6}$$

Using the well-known relation $1/r = 1/h_{\mathsf{a}} + 1/h_{\mathsf{b}} + 1/h_{\mathsf{c}}$ where r is the in-radius of the triangle as above, we find that

$$\begin{aligned} 4\,\|\boldsymbol{z}\|^2 \;&=\; 2\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{h_{\mathsf{a}}h_{\mathsf{b}}} + \frac{\boldsymbol{a}\cdot\boldsymbol{c}}{h_{\mathsf{a}}h_{\mathsf{c}}} + \frac{\boldsymbol{b}\cdot\boldsymbol{c}}{h_{\mathsf{b}}h_{\mathsf{c}}}\right) - 2\left(\frac{1}{h_{\mathsf{a}}} + \frac{1}{h_{\mathsf{b}}} + \frac{1}{h_{\mathsf{c}}}\right) R \\ &=\; -\,\frac{c^2}{h_{\mathsf{a}}h_{\mathsf{b}}} - \frac{b^2}{h_{\mathsf{a}}h_{\mathsf{c}}} - \frac{a^2}{h_{\mathsf{b}}h_{\mathsf{c}}} - 2\,\frac{R}{r}\;. \end{aligned} \tag{7}$$

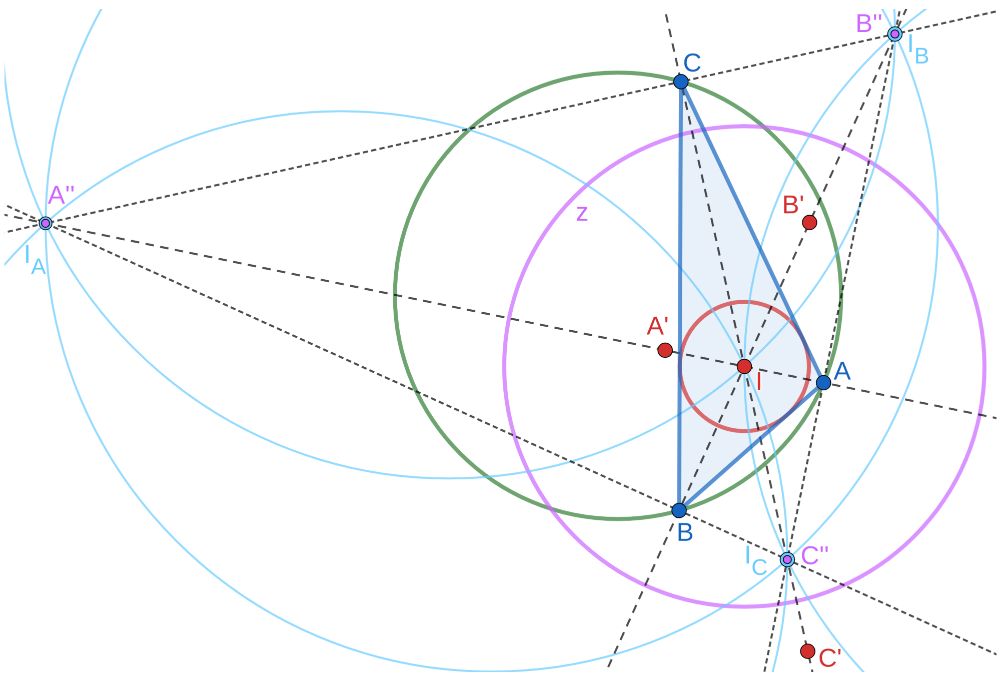

Figure 2: The conformal dual of a triangle is obtained by reflecting its vertices $\mathsf{A}, \mathsf{B}, \mathsf{C}$ in the in-center I to get $\mathsf{A}', \mathsf{B}', \mathsf{C}'$, and then inverting those points in the circle z (magenta) centered on I with radius $2\sqrt{rR}$, where r & R denote the in-radius and circum-radius, resp. The results A'', B'', C'' coincide with the ex-centers I_A, I_B, I_C, resp., while the point-at-infinity is mapped to the in-center. The reciprocal basis of the triangle consists of the lines spanned by the edges opposite each vertex and its circum-circle (green), which is "opposite" the point-at-infinity; the cyan circles are the reciprocal basis of its conformal dual.

The first term on the right can also be simplified as

$$-\frac{c^2 h_\mathsf{c} + b^2 h_\mathsf{b} + a^2 h_\mathsf{a}}{h_\mathsf{a} h_\mathsf{b} h_\mathsf{c}} \;=\; -\frac{4\,s\,t}{8\,t^3/(abc)} \;=\; -\frac{s\,abc}{2\,t^2} = -\frac{2\,s\,R}{t} \;=\; -\frac{2\,R}{r}\,, \tag{8}$$

(since $abc = 4\,tR$) and hence $\|\boldsymbol{z}\|^2 = -R/r$, a scale-independent ratio.

It follows that the reflection of e.g. $\hat{\boldsymbol{a}}$ w.r.t. $\hat{\boldsymbol{z}} \coloneqq \boldsymbol{z}/\|\boldsymbol{z}\|$ is

$$\begin{aligned}\hat{\boldsymbol{z}}\,\hat{\boldsymbol{a}}\,\hat{\boldsymbol{z}} \;=\; -\frac{r}{4\,R}\big(&\hat{\boldsymbol{b}}\,\hat{\boldsymbol{a}}\,\hat{\boldsymbol{b}} + \hat{\boldsymbol{c}}\,\hat{\boldsymbol{a}}\,\hat{\boldsymbol{c}} + \hat{\boldsymbol{d}}\,\hat{\boldsymbol{a}}\,\hat{\boldsymbol{d}} + \\ &(\hat{\boldsymbol{b}}\,\hat{\boldsymbol{a}}\,\hat{\boldsymbol{c}} + \hat{\boldsymbol{c}}\,\hat{\boldsymbol{a}}\,\hat{\boldsymbol{b}}) + (\hat{\boldsymbol{b}}\,\hat{\boldsymbol{a}}\,\hat{\boldsymbol{d}} + \hat{\boldsymbol{d}}\,\hat{\boldsymbol{a}}\,\hat{\boldsymbol{b}}) + (\hat{\boldsymbol{c}}\,\hat{\boldsymbol{a}}\,\hat{\boldsymbol{d}} + \hat{\boldsymbol{d}}\,\hat{\boldsymbol{a}}\,\hat{\boldsymbol{c}})\big)\,.\end{aligned} \tag{9}$$

The reverse-symmetric terms in this expansion can themselves be further expanded as $\hat{\boldsymbol{b}}\,\hat{\boldsymbol{a}}\,\hat{\boldsymbol{b}} = \hat{\boldsymbol{b}}\,(\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}} + \hat{\boldsymbol{a}}\wedge\hat{\boldsymbol{b}}) = 2\,(\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}})\,\hat{\boldsymbol{b}}$ etc., and $\hat{\boldsymbol{b}}\,\hat{\boldsymbol{a}}\,\hat{\boldsymbol{c}} + \hat{\boldsymbol{c}}\,\hat{\boldsymbol{a}}\,\hat{\boldsymbol{b}} = (\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}} - \hat{\boldsymbol{a}}\wedge\hat{\boldsymbol{b}})\,\hat{\boldsymbol{c}} + \hat{\boldsymbol{c}}\,(\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}} + \hat{\boldsymbol{a}}\wedge\hat{\boldsymbol{b}}) = 2\,(\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}})\,\hat{\boldsymbol{c}} + 2\,(\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{c}})\,\hat{\boldsymbol{b}} - 2\,(\hat{\boldsymbol{b}}\cdot\hat{\boldsymbol{c}})\,\hat{\boldsymbol{a}}$, etc. The inner products of the renormalized null vectors in these expressions are given by e.g.

$$\hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{b}} \;=\; \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{h_\mathsf{a}\,h_\mathsf{b}} \;=\; -\frac{abc^2}{8\,t^2} \;=\; -\frac{R\,c}{2\,t} \tag{10}$$

as well as e.g.

$$\hat{\boldsymbol{c}} \cdot \hat{\boldsymbol{d}} \;=\; \frac{\boldsymbol{c} \cdot \boldsymbol{n}_\infty}{h_{\mathsf{c}}/R} \;=\; -\frac{R\,c}{2\,t} \;=\; \hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}}\;, \tag{11}$$

Together with $r = t/s$, Eqs. (10) & (11), and their analogues for the other pairs of null vectors, this works out to:

$$\begin{aligned} \hat{\boldsymbol{z}}\,\hat{\boldsymbol{a}}\,\hat{\boldsymbol{z}} \;&=\; (4\,s)^{-1}\Big(c\,\hat{\boldsymbol{b}} + b\,\hat{\boldsymbol{c}} + a\,\hat{\boldsymbol{d}} + (c\,\hat{\boldsymbol{c}} + b\,\hat{\boldsymbol{b}} - a\,\hat{\boldsymbol{a}}) \\ &\qquad +\; (c\,\hat{\boldsymbol{d}} + a\,\hat{\boldsymbol{b}} - b\,\hat{\boldsymbol{a}}) + (b\,\hat{\boldsymbol{d}} + a\,\hat{\boldsymbol{c}} - c\,\hat{\boldsymbol{a}})\Big) \\ &=\; (-\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}} + \hat{\boldsymbol{c}} + \hat{\boldsymbol{d}})\,/\,2 \end{aligned} \tag{12}$$

In a similar fashion we obtain $\hat{\boldsymbol{z}}\,\hat{\boldsymbol{b}}\,\hat{\boldsymbol{z}} = (\hat{\boldsymbol{a}} - \hat{\boldsymbol{b}} + \hat{\boldsymbol{c}} + \hat{\boldsymbol{d}})/2$, $\hat{\boldsymbol{z}}\,\hat{\boldsymbol{c}}\,\hat{\boldsymbol{z}} = (\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}} - \hat{\boldsymbol{c}} + \hat{\boldsymbol{d}})/2$, and $\hat{\boldsymbol{z}}\,\hat{\boldsymbol{d}}\,\hat{\boldsymbol{z}} = (\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}} + \hat{\boldsymbol{c}} - \hat{\boldsymbol{d}})/2$.

These reflections w.r.t. $\hat{\boldsymbol{z}}$ are the *conformal duals* $\hat{\boldsymbol{a}}^*, \hat{\boldsymbol{b}}^*, \hat{\boldsymbol{c}}^*, \hat{\boldsymbol{d}}^*$ of the renormalized vertices and point-at-infinity. It can be shown that $\hat{\boldsymbol{a}}^* = \boldsymbol{i}_{\mathsf{a}}/(2\,r_{\mathsf{a}})$, $\hat{\boldsymbol{b}}^* = \boldsymbol{i}_{\mathsf{b}}/(2\,r_{\mathsf{b}})$, $\hat{\boldsymbol{c}}^* = \boldsymbol{i}_{\mathsf{c}}/(2\,r_{\mathsf{c}})$ and $\hat{\boldsymbol{d}}^* = \boldsymbol{i}/(2\,r)$ where $\boldsymbol{i}_{\mathsf{a}}, \boldsymbol{i}_{\mathsf{b}}, \boldsymbol{i}_{\mathsf{c}} \in \mathcal{G}_{3,1}$ are the centers of the ex-circles opposite the vertices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ resp., $\boldsymbol{i} \in \mathcal{G}_{3,1}$ is the center of the in-circle, and r_{a}, r_{b}, r_{c}, r are the corresponding radii. Since $\|\boldsymbol{z}\|^2 < 0$, this duality corresponds to inversion in a circle centered on $\boldsymbol{i}$ of radius $2\,r/\sqrt{r/R} = 2\sqrt{r\,R}$ preceded or followed by reflection in $\boldsymbol{i}$ (cf. Fig. 2).

Finally, note that the Gramian of the renormalized vectors is

$$\begin{aligned} (\hat{\boldsymbol{d}} \wedge \hat{\boldsymbol{c}} \wedge \hat{\boldsymbol{b}} \wedge \hat{\boldsymbol{a}}) \cdot (\hat{\boldsymbol{a}} \wedge \hat{\boldsymbol{b}} \wedge \hat{\boldsymbol{c}} \wedge \hat{\boldsymbol{d}}) \;&=\; \frac{R^4}{16\,t^4}\det\begin{bmatrix} 0 & c & b & a \\ c & 0 & a & b \\ b & a & 0 & c \\ a & b & c & 0 \end{bmatrix} \\ &=\; \frac{R^4}{16\,t^4}\,(a+b+c)(a+b-c)(a-b+c)(a-b-c)\;, \end{aligned} \tag{13}$$

which by Heron's formula is just $-R^4/t^2$, again a scale-independent quantity. This may be viewed as a conformally invariant version of Heron's formula, which brings us in a full circle.

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