Geometric Algebra and symmetry in crystallography and physics

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Summary of the Abstract

From viewpoints of crystallography and of elementary particles, we explore symmetries of multivectors in the geometric (Minkowski) algebra $Cl(3,1)$ *that can be used to describe space-time.*

Introduction

Recently, [9] and [8] classified multivectors based on their symmetries[1] under space inversion (main grade involution in geometric algebra) $\hat{1}$, time reversal $1'$ and reversion (called wedge reversion by them) $\tilde{1}$. [9] notes in the conclusions that *One could perhaps explore charge reversal* ($\hat{C}$)*, parity reversal* ($\hat{P}$) *and time reversal* ($\hat{T}$) *in the relativistic context [11].* In the standard model of elementary particle physics many experiments have confirmed violation of parity symmetry by the weak interaction, and of $\hat{C}\hat{P}$ symmetry. However, strong interactions by themselves do preserve $\hat{C}\hat{P}$ symmetry [5].
Therefore one aim of this research is to work in this direction by looking at the effect of these three symmetries on multivectors of $Cl(3,1)$, a (geometric) algebra that can be used to express space-time physics, with $\hat{C}$, $\hat{P}$ and $\hat{T}$ symmetries, e.g., defined by [6], but here we only focus on the effect of these transformations on the 16 basis blades that constitute the multivector basis of $Cl(3,1)$, and ignore any functional dependence of coefficients in linear combinations that might express spinors or other physical quantities. Furthermore, [11] and [6] have a clear preference for the use of $Cl(1,3)$, while in this work we prefer[2] to use $Cl(3,1)$ because its volume-time subalgebra $\{1, e_0, e_{123}, e_{0123}\}$ isomorphic to quaternions, where e_0 expresses the time direction, at the foundation of the theory of space-time Fourier transforms [17].
[9] and [8] work signature independent for all Clifford algebras of quadratic spaces. We work in an algebra of specific signature and want to take advantage of the principal reverse[3] (see e.g. [17] (2.1.12)), which applied to $Cl(3,1)$ acts like the conventional reverse, but in addition changes the sign of the time vector $e_0 \to -e_0$. So we focus on the group of eight symmetries generated by grade involution $\hat{1}$, reversion $\tilde{1}$ and principal reverse[4] $1'$.

[1]Note that [9] and [8] use for space inversion $\bar{1}$, and for (wedge) reversion $1^{\dagger}$.
[2]Another notable work using $Cl(3,1)$ in elementary particle physics is, e.g. [26].
[3]The principal reverse is in geometric algebra the equivalent of matrix transposition, see [1].
[4]The reader should be aware that therefore in this work we do not use a priori the notion of time

Table 1: Action of involutions of group (1) on all 16 basis elements (2) of $Cl(3,1)$. Tp. = type with scalar S, time vector V_0 multiple of e_0, space vector V, bivector B_0 with e_0 factor, space bivector B, trivector T_0 with e_0 factor, space trivector T and pseudoscalar quadvector Q. Bas. = basis element, e = even, o =odd.

Tp.	Bas.	$\hat{1}$	$\tilde{1}$	$\bar{1}$	$1'$	$\hat{1}'$	$\tilde{1}'$	$\bar{1}'$
S	1	e	e	e	e	e	e	e
V_0	e_0	o	e	o	o	e	o	e
	e_1	o	e	o	e	o	e	o
V	e_2	o	e	o	e	o	e	o
	e_3	o	e	o	e	o	e	o
	e_{01}	e	o	o	e	e	o	o
B_0	e_{02}	e	o	o	e	e	o	o
	e_{03}	e	o	o	e	e	o	o
	e_{23}	e	o	o	o	o	e	e
B	e_{31}	e	o	o	o	o	e	e
	e_{12}	e	o	o	o	o	e	e
	e_{023}	o	o	e	e	o	o	e
T_0	e_{031}	o	o	e	e	o	o	e
	e_{012}	o	o	e	e	o	o	e
T	e_{123}	o	o	e	o	e	e	o
Q	e_{0123}	e	e	e	o	o	o	o

Our main reference is [22].

Classification of multivectors in $Cl(3,1)$

This talk is structured as follows. First we study the symmetries of $Cl(3,1)$ multivectors under space inversion $\hat{1}M$, reversion $\tilde{1}M$, Clifford conjugation $\bar{1}M$ and principal reverse $1'M$. These involutions generate by composition the following Abelian group of involutions

$$G = \{1, \hat{1}, \tilde{1}, \bar{1}, 1', \hat{1}', \tilde{1}', \bar{1}'\}. \tag{1}$$

Their action on the blade basis of the geometric algebra $Cl(3,1)$ is listed in Table 1. In the table horizontal lines separate the eight principle types of multivectors. Combinations of them yield 43 further types, i.e. a total of 51 types of multivector symmetry.
The unit blade basis of the geometric algebra $Cl(3,1)$ is given by one scalar, four vectors, six bivectors, four trivectors and one pseudoscalar quadvector I,

$$\{1, e_0, e_1, e_2, e_3, e_{01}, e_{02}, e_{03}, e_{23}, e_{31}, e_{12}, e_{023}, e_{031}, e_{012}, e_{123}, I = e_{0123}\}. \tag{2}$$

reversal of [9] and [8], which there also has the symbol $1'$. Although we do obtain it for multivectors that have e_0 as a factor, by the product of reversion $\tilde{1}$ and principal reverse $1'$ (our notation).

Table 2: Table of all compositions of symmetry operators $\hat{C}, \hat{P}$ and $\hat{T}$, where operations in the top row are applied first to M followed by an operation from the first column. For example: combining $\hat{T}\hat{C}$ from the top row with $\hat{C}\hat{P}$ from the first column (6th row) shows that $\hat{C}\hat{P}\hat{T}\hat{C}M = \hat{P}\hat{T}M$.

1st: 2nd:	1	$\hat{C}$	$\hat{P}$	$\hat{T}\hat{C}$	$\hat{C}\hat{P}$	$\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$
1	1	$\hat{C}$	$\hat{P}$	$\hat{T}\hat{C}$	$\hat{C}\hat{P}$	$\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$
$\hat{C}$	$\hat{C}$	1	$\hat{C}\hat{P}$	$-\hat{T}$	$\hat{P}$	$-\hat{T}\hat{C}$	$\hat{P}\hat{T}$	$\hat{C}\hat{P}\hat{T}$
$\hat{P}$	$\hat{P}$	$-\hat{C}\hat{P}$	1	$\hat{C}\hat{P}\hat{T}$	$-\hat{C}$	$\hat{P}\hat{T}$	$\hat{T}\hat{C}$	$\hat{T}$
$\hat{T}\hat{C}$	$\hat{T}\hat{C}$	$\hat{T}$	$-\hat{C}\hat{P}\hat{T}$	1	$\hat{P}\hat{T}$	$\hat{C}$	$-\hat{P}$	$\hat{C}\hat{P}$
$\hat{C}\hat{P}$	$\hat{C}\hat{P}$	$-\hat{P}$	$\hat{C}$	$\hat{P}\hat{T}$	-1	$\hat{C}\hat{P}\hat{T}$	$-\hat{T}$	$-\hat{T}\hat{C}$
$\hat{T}$	$\hat{T}$	$\hat{T}\hat{C}$	$\hat{P}\hat{T}$	$-\hat{C}$	$-\hat{C}\hat{P}\hat{T}$	-1	$\hat{C}\hat{P}$	$-\hat{P}$
$\hat{C}\hat{P}\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$	$-\hat{T}\hat{C}$	$\hat{P}$	$\hat{T}$	$-\hat{C}\hat{P}$	-1	$-\hat{C}$
$\hat{P}\hat{T}$	$\hat{P}\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{T}$	$\hat{C}\hat{P}$	$-\hat{T}\hat{C}$	$-\hat{P}$	$-\hat{C}$	-1

Multivectors in $Cl(3,1)$ under $\hat{C}$, $\hat{P}$ and $\hat{T}$ symmetries

Next, we consider aspects of charge conjugation, parity reversal and time reversal, when $Cl(3,1)$

$$\hat{C}M = Me_1e_0, \qquad \hat{P}M = e_0Me_0, \qquad \hat{T}M = Ie_0Me_1, \tag{3}$$

is applied in the description of elementary particle physics. The composition of the symmetries $\hat{C}$, $\hat{P}$ and $\hat{T}$ is shown in Table 2. The table does not change when the algebra $Cl(1,3)$ is employed instead. Furthermore, the table is also the same, when the full $\hat{C}$, $\hat{P}$ and $\hat{T}$ symmetries are applied (including the operations on the space-time vector argument of multivector functions as found in [6], equations (8.90)). An interesting feature is that Table 2 is isomorphic to the mutliplication table of the basis elements of $Cl(3,0) \cong Cl_+(3,1) \cong Cl_+(1,3)$.
The application of the symmetries $\hat{C}$, $\hat{P}$ and $\hat{T}$ to the multivector basis of $Cl(3,1)$ in Table 3. In contrast to Table 1, not only signs are changed, but also permutations occur limited to four subsets (including one subgroup) of four elements each.

Summary

In this presentation we have pursued the application of elementary symmetries of the geometric algebra $Cl(3,1)$ that can describe space-time. Inspired by [9] and [8], we chose three involutions of space inversion, reverse and *principal reverse* and studied the Abelian group thus generated and its action on the multivectors of $Cl(3,1)$. We found that similar to [8], a classification in eight principal and further 43 types of multivectors is thus possible, leading to a total of 51 types. Then we looked at algebraic aspects of applying charge conjugation, parity reversal and time reversal to the multivector basis of $Cl(3,1)$. We found that the composition of the symmetry operations $\hat{C}, \hat{P}$ and $\hat{T}$ forms an algebra isomorphic to $Cl(3,0)$ and $Cl_+(3,1)$, and we commented on the structures

Table 3: Application of charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$ (top row) defined in (3), to all elements of the basis (first column) of $Cl(3,1)$ given in (2).

Basis	1	$\hat{C}$	$\hat{P}$	$\hat{T}\hat{C}$	$\hat{C}\hat{P}$	$\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$
1	1	$-e_{01}$	-1	e_{0123}	e_{01}	e_{23}	e_{0123}	$-e_{23}$
e_0	e_0	e_1	$-e_0$	e_{123}	$-e_1$	$-e_{023}$	e_{123}	e_{023}
e_1	e_1	e_0	e_1	$-e_{023}$	e_0	e_{123}	e_{023}	e_{123}
e_2	e_2	$-e_{012}$	e_2	$-e_{031}$	$-e_{012}$	e_3	e_{031}	e_3
e_3	e_3	e_{031}	e_3	$-e_{012}$	e_{031}	$-e_2$	e_{012}	$-e_2$
e_{01}	e_{01}	-1	e_{01}	$-e_{23}$	-1	$-e_{0123}$	e_{23}	$-e_{0123}$
e_{02}	e_{02}	e_{12}	e_{02}	$-e_{31}$	e_{12}	$-e_{03}$	e_{31}	e_{03}
e_{03}	e_{03}	$-e_{31}$	e_{03}	$-e_{12}$	$-e_{31}$	e_{02}	e_{12}	$-e_{02}$
e_{23}	e_{23}	$-e_{0123}$	$-e_{23}$	$-e_{01}$	e_{0123}	-1	$-e_{01}$	1
e_{31}	e_{31}	$-e_{03}$	$-e_{31}$	$-e_{02}$	e_{03}	e_{12}	$-e_{02}$	$-e_{12}$
e_{12}	e_{12}	e_{02}	$-e_{12}$	$-e_{03}$	$-e_{02}$	$-e_{31}$	$-e_{03}$	e_{31}
e_{023}	e_{023}	e_{123}	$-e_{023}$	$-e_1$	$-e_{123}$	e_0	$-e_1$	$-e_0$
e_{031}	e_{031}	e_3	$-e_{031}$	$-e_2$	$-e_3$	$-e_{012}$	$-e_2$	e_{012}
e_{012}	e_{012}	$-e_2$	$-e_{012}$	$-e_3$	e_2	e_{031}	$-e_3$	$-e_{031}$
e_{123}	e_{123}	e_{023}	e_{123}	e_0	e_{023}	$-e_1$	$-e_0$	$-e_1$
e_{0123}	e_{0123}	$-e_{23}$	e_{0123}	1	$-e_{23}$	e_{01}	-1	e_{01}

found when $\hat{C}, \hat{P}$ and $\hat{T}$ are applied to the complete set of basis blades of $Cl(3,1)$. Analogous results hold when the algebra $Cl(1,3)$ is employed and when the full $\hat{C}, \hat{P}$ and $\hat{T}$ symmetries of [6], equations (8.90), are used. It may be interesting to apply both approaches in Clifford space gravity [3], and the study of elementary particles using a new embedding of octonions in geometric algebra [23, 21].

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