Notes on geometric algebras in geometric control theory (sub-Riemannian geometry)

J. Hrdina[a]

[a]Institute of Mathematics
Faculty of Mechanical Engineering, Brno University of Technology
Technická 2896/2, 616 69 Brno, Czech Republic
hrdina@fme.vutbr.cz

Summary of the Abstract

"For who can make that straight, which he hath made crooked?" Ecclesiastes 7:13

In a sub-Riemannian space we can neither move nor send information in all the directions, nor can we receive information from everywhere. There are constraints (imposed by God, by a moral imperative, by a government, or just by the laws of Nature). We study the role of geometric algebras in sub-Riemannian space. We mainly discuss control problems on free Carnot groups of step 2*, specially Heisenberg group.*

Actual Abstract Sections

1 Introduction

Geometric control theory utilizes various geometric approaches to manage different dynamical systems [10, 4], with a particular focus on sub-Riemannian geometry within the Hamiltonian framework [2, 1]. Our goal is to expand these methodologies by integrating geometric algebras (GA) that are in line with the essence of these challenges. In this pursuit, we reframe certain control problems using GA concepts [14, 15, 16], leveraging the natural $SO(n)$-invariant operations within geometric algebras to effectively handle sets of optimal solutions [6].

2 Carnot groups

Carnot groups are a subclass of Lie groups characterized by having a nilpotent Lie algebra. A Lie group is essentially a smooth manifold endowed with a group structure. Important examples of Lie groups include matrix groups such as $SO(n)$ and $O(n)$, as well as their coverings, i.e. Clifford groups $Spin(n)$ and $Pin(n)$, which we'll delve into later. The Lie algebra corresponding to a Lie group G consists of the tangent space T_eG at the identity element, along with the Lie bracket operation $[,]$. An algebra $\mathfrak{g}$ is considered nilpotent if it satisfies the condition $[\mathfrak{g},[\mathfrak{g},[\mathfrak{g}\dots]]]=0$, where the brackets are nested. If $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]=0$, the nilpotency step is two. A fundamental example of a Carnot group with step two is the three-dimensional Heisenberg group, denoted as $\mathbb{H}_3$. The terms "Lie group," "Lie algebra," and "Lie bracket" pay homage to the Norwegian mathematician

Sophus Lie (1842 - 1899), whose contributions laid the groundwork for these concepts. Additionally, Sophus Lie's legacy extends to the inception of the Abel Prize, often regarded as the Nobel Prize equivalent in the realm of Mathematical Sciences. Today, Lie groups hold significant importance in various fields, including mechanics and quantum mechanics.

2.1 Heisenberg group $\mathbb{H}_3$

The Heisenberg group is named after Werner Heisenberg (1901 - 1976) German theoretical physicist, one of the main pioneers of the theory of quantum mechanics. It can be seen as the subset of matrices

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\} \tag{1}$$

which forms a group with the usual matrix multiplication:

$$\begin{pmatrix} 1 & a_1 & c_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 & c_2 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_1 + a_2 & c_1 + c_2 + a_1 b_2 \\ 0 & 1 & b_1 + b_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly, the bijection $\psi(x_1, x_2, t) = \begin{pmatrix} 1 & x_1 & t \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$ induces a group structure on $\mathbb{R}^3$ and the left-invariant fields on this group (Proposition 1.2 in [5])

$$X = \partial_{x_1}, \quad Y = \partial_{x_2} + x_1 \partial_t, \quad T = \partial_t$$

form a nilpotent Lie algebra $\langle X, Y, T = [X, Y] \rangle$. The form of the vector fields was determined by the choice of coordinates. With a straightforward change of coordinates (as demonstrated in Proposition 1.3 in [5]), we can transform them into a vector field represented in a symmetric form

$$X = \partial_{y_1} - 2y_2 \partial_\tau, \quad Y = \partial_{y_2} - 2y_1 \partial_\tau, \quad T = 4\partial_\tau,$$

where the group operation on $\mathbb{R}^3$ will then be of the form

$$(y_1, y_2, \tau) \circ (\bar{y}_1, \bar{y}_2, \bar{\tau}) = (y_1 + \bar{y}_1, y_2 + \bar{y}_2, \tau + \bar{\tau} - 2(y_1 \bar{y}_2 - y_2 \bar{y}_1)). \tag{2}$$

The Lie group $(\mathbb{R}^3, \circ)$ is called *symmetric three dimensional Heisenberg group* $\mathbb{H}_3$, the element $(0, 0, 0)$ is origin and the inverse is $(y_1, y_2, \tau) = (-y_1, -y_2, -\tau)$.

The group $\mathbb{H}_3$ can be realized in the Grassmannian algebra $Gr(2)$ by the identification

$$(y_1, y_2, t) \mapsto 1 + y_1 e_1 + y_2 e_2 + t e_1 \wedge e_2,$$

where the multiplication coincides with wedge operation

$$\begin{aligned} &(1 + y_1 e_1 + y_2 e_2 + t e_1 \wedge e_2) \wedge (1 + \bar{y}_1 e_1 + \bar{y}_2 e_2 + \bar{t} e_1 \wedge e_2) \\ &= 1 + (y_1 + \bar{y}_1) e_1 + (y_2 + \bar{y}_2) e_2 + (t + \bar{t} + (y_1 \bar{y}_2 - y_2 \bar{y}_1)) e_1 \wedge e_2). \end{aligned} \tag{3}$$

Note that, the difference in the last term in (2) and (3) is caused by slightly different definition of Lie bracket and wedge product on vectors $[a, b] = ab - ba = -2a \wedge b$.

2.2 Free Carnot groups of step two.

The multiplication (2) can be interpreted that the Heisenberg group is isomorphic to the semidirect product $\mathbb{R}^2 \rtimes \mathfrak{so}(2)$ where the mentioned multiplication can be seen as

$$((y_1, y_2), \tau) \circ ((\bar{y}_1, \bar{y}_2), \bar{\tau}) = ((y_1, y_2) + (\bar{y}_1, \bar{y}_2), (\tau + \bar{\tau}) - 2(y_1, y_2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\bar{y}_1, \bar{y}_2)^T),$$

where $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is generator of $\mathfrak{so}(2)$.

In general we denote free Carnot groups of step two as Lie groups $\mathbb{C}_n$ which are isomorphic to $\mathbb{R}^n \rtimes \mathfrak{so}(n)$. In dimension $n = 3$ we speak of so called Cartan group $\mathbb{C}_3 = \mathbb{R}^3 \rtimes \mathfrak{so}(3)$ together with an operation defined by

$$(u, A) \circ (\tilde{u}, \tilde{A}) = (u + \tilde{u}, A + \tilde{A} - 2(uE_1\tilde{u}^T + uE_2\tilde{u}^T + uE_3\tilde{u}^T)), \tag{4}$$

where $E_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ are generators of $\mathfrak{so}(3)$.

Free Carnot group $\mathbb{C}_3 = \mathbb{R}^n \rtimes \mathfrak{so}(n)$ of step two can be realized in the Grassmannian algebra $Gr(n)$ by the identification $\mathbb{R}^m \oplus \mathbb{R}^{\frac{n(n-1)}{2}} \cong \mathbb{R}^m \oplus \wedge^2\mathbb{R}^m$, i.e.

$$(y_1, \cdots y_p, t_{12}, \dots t_{n-1n}) \mapsto 1 + y_1e_1 + \cdots + y_ne_n + t_{12}e_1 \wedge e_2 + \cdots + t_{n-1n}e_{n-1} \wedge e_n. \tag{5}$$

3 Nilpotent control problem

By nilpotent control problems we mean the invariant control problems on Carnot groups and we consider the free Carnot groups $\mathbb{C}_n$ of step 2, [1, 13, 11]. If we denote the local coordinates by $(x, z) \in \mathbb{R}^m \oplus \wedge^2\mathbb{R}^m$, we can model the corresponding Lie algebra $\mathfrak{g}$ as

$$\begin{aligned} X_i &= e_i + e_i \wedge (e_1 + \cdots + e_m), \\ X_{ij} &= e_i \wedge e_j. \end{aligned} \tag{6}$$

We discuss the related optimal control problem

$$\dot{q}(t) = u_1X_1 + \cdots + u_mX_m \tag{7}$$

for $t > 0$ and q in G and the control $u = (u_1(t), \dots, u_m(t)) \in \mathbb{R}^m$ with the boundary condition $q(0) = q_1$, $q(T) = q_2$ for fixed points $q_1, q_2 \in G$, where we minimize the cost functional $\frac{1}{2}\int_0^T(u_1^2 + \cdots + u_m^2)\mathrm{d}t$. The solutions $q(t)$ then correspond to the sub–Riemannian geodesics, i.e. admissible curves parametrized by a constant speed whose sufficiently small arcs are the length minimizers.

The cost function mentioned above corresponds to the energy and in the square to the length of the curve. In fact, it is a quadratic form on $\mathbb{R}^n$ and identification (5) can be seen as

$$\mathbb{R}^m \oplus \mathbb{R}^{\frac{n(n-1)}{2}} \hookrightarrow \mathbb{G}_n,$$

where $\mathbb{G}_n$ is a geometric algebra with positive definite signature.

3.1 Controls - vertical (fiber) system

We use the Hamiltonian approach to this control problem [1]. There are no strict abnormal extremals for the step 2 Carnot groups, so we focus on the normal geodesics and address them just as geodesics. The left–invariant vector fields X_i, $i = 1, \dots, m$ form a basis of TG and determine the left–invariant coordinates on G. We define the corresponding left–invariant coordinates h_i, $i = 1, \dots, m$ and w_i, $i = 1, \dots, \binom{m}{2}$ on the fibres of T^*G by $h_i(\lambda) = \lambda(X_i)$ and $w_i(\lambda) = \lambda(X_{m+i})$, for arbitrary 1–forms λ on G. Thus we use (x_i, w_i) as the global coordinates on T^*G. The geodesics are exactly the projections of normal Pontryagin extremals. Using $u_j(t) = h_j(\lambda(t))$ and the equation $\dot{\lambda}(t) = \vec{H}(\lambda(t))$ for the normal extremals, we write the fiber system as

$$
\begin{aligned}
\dot{h}_i &= -\sum_{l=1}^{m-n} \sum_{j=1}^{m} c_{ij}^l h_j w_l, \quad && i = 1, \dots, m, \\
\dot{w}_j &= 0, && j = 1, \dots, \binom{m}{2},
\end{aligned}
\tag{8}
$$

where c_{ij}^l are the structure constants of the Lie algebra $\mathfrak{g}$ for the basis X_i. The solutions w_i, $i = 1, \dots, n-m$ are constants that we denote by

$$
w_1 = K_1, \ \dots, w_{n-m} = K_{\binom{m}{2}}. \tag{9}
$$

If at least one of K_i is non–zero, the first part of the fibre system (8) forms a homogeneous system of ODEs $\dot{h} = -\Omega h$ with constant coefficients for $h = (h_1, \dots, h_m)^T$ and the system matrix Ω. Its solution is given by $h(t) = e^{-t\Omega} h(0)$, where $h(0)$ is the initial value of the vector h at the origin.

In the geometric algebra framework, we have

$$
\Omega = K_i e_1 \wedge e_2 + \cdots K_{\binom{m}{2}} e_{m-1} \wedge e_m
$$

and $e^{-t\Omega} \in Spin(m)$.

For example in $\mathbb{C}_2 = \mathbb{H}_3$ *and* $\mathbb{C}_3$*, we have* $\omega = K e_1 \wedge e_2$ *and* $\omega = K_1 e_1 \wedge e_2 + K_2 e_1 \wedge e_3 + K_3 e_2 \wedge e_3$ *respectively. So*

$$
h_1 = g_1 h(0) \bar{g}_1, \quad g_1 = \cos(Kt) + \sin(Kt) e_1 \wedge e_2 \tag{10}
$$

$$
h_2 = g_1 h(0) \bar{g}_1, \quad g_2 = \frac{1}{K}(\cos(Kt) + \sin(Kt)(K_1 e_1 \wedge e_2 + K_2 e_2 \wedge e_3 + K_3 e_1 \wedge e_3)), \tag{11}
$$

where $K = K_1^2 + K_2^2 + K_3^2$

3.2 Geodesics - base (horizontal) systems

Assume that $\lambda(t) = (x_i(t), z_i(t), h_i(t), w_i(t))$ in T^*G is a normal extremal. Then the controls u_j to the system (7) satisfy $u_j(t) = h_j(\lambda(t))$ and the base system takes the form of

$$\begin{aligned} \dot{x}_i &= h_i, \quad i = 1, \dots, m \\ \dot{z}_j &= -\frac{1}{2}\sum_{i=1}^{m} c_{ik}^{j} h_i x_k, \quad j = 1, \dots, n-m \end{aligned} \tag{12}$$

for $q = (x_i, z_i)$.

In $\mathbb{C}_2$ and $\mathbb{C}_3$ we have

$$x_1(t) = \int g_1(t) h_0 g_1(t)\, dt, \qquad x_2(t) = \int g_2(t) h_0 g_2(t)\, dt, \tag{13}$$

$$z_1(t) = \int h_1(t) \wedge \dot{x}_1(t)\, dt \qquad z_2(t) = \int h_2(t) \wedge \dot{x}_1(t)\, dt, \tag{14}$$

where $K = K_1^2 + K_2^2 + K_3^2$.

4 Mechanical motivation: vertical rooling disc

As a motivation, we consider mechanisms moving in the plane, typically wheeled mechanisms like cars (with or without trailers) robotic snakes [7, 9] or trident snake mechanisms, [9, 8]. To control the mechanisms locally, we consider the nilpotent approximations of the original control systems [3]. Although the configuration spaces and their approximations have the same filtration, the approximations form Carnot groups generally endowed with more symmetries [12]. One gets the symmetries generated by the right–invariant vector fields, and there may be additional symmetries acting non–trivially on the distribution. This observation leads to the idea of local control in the geometric algebra approach.

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