Symmetries of the boundary theorem and electrodynamics

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Symmetries of Maxwell's equations of electrodynamics are studied from the point of view of Stokes' boundary theorem in geometric calculus. We show how considering the magnetic vector potential leads to a more direct correspondence and elaborate on its properties using hydrodynamical analogies.

I. INTRODUCTION

For the past one and a half century, Maxwell's equations for electrodynamics have been heavily investigated and applied in a wide range of systems and settings[1-7]. Although the their vector derivative format is more commonly studied, their integral form can be seen to contain more information, as the dimensionality of the manifolds in which the fields are summed up in are made explicit,

$$\oint_{\partial M} \mathbf{E} \cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_M \mathbf{B} \cdot \mathrm{d}\mathbf{S}, \tag{1}$$

$$\oint_{\partial M} \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \mu_0 \iint_M \mathbf{J} \cdot \mathrm{d}\mathbf{S} + \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \iint_M \mathbf{E} \cdot \mathrm{d}\mathbf{S}, \tag{2}$$

$$\oiint_{\partial N} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0} \iiint_N \rho \, \mathrm{d}V, \tag{3}$$

$$\oiint_{\partial N} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0. \tag{4}$$

Furthermore, the time derivative outside of the integral in Eq. (1) encodes the Lorentz force law, if Reynolds transport theorem is used[8]. The vector derivative form of the equations is obtained by applying the relevant vector integral theorems[9], and omitting the integrals and differential elements of both sides.

The integral vector theorems of vector calculus can be generalized and combined in a single theorem commonly known as Stokes' theorem, using differential forms along with the exterior derivative[10,11]. Hestenes has argued[12,13] that Stokes' theorem via the exterior derivative contains half of the information content of the boundary theorem of geometric calculus[14], where the other half can be obtained from dual forms. In a previous work, this difference was investigated and a systematic way of visualization formulated for the basic cases in one, two, and three dimensions[15]. In this work, we compare the symmetries of the integral form of Maxwell's equations, Eqs. (1)-(4), with those obtained from the maximally symmetric cases of the boundary theorem of geometric calculus in two and three dimensions. We show how a more direct correspondence between the two is found in the electromagnetic potentials and discuss the implications thereof.

II. THE BOUNDARY THEOREM OF GEOMETRIC CALCULUS

Compared with vector calculus and calculus of differential forms, geometric calculus differs in that the inner and outer derivative are combined in a single operator

$$\boldsymbol{\nabla} F = \boldsymbol{\nabla} \cdot F + \boldsymbol{\nabla} \wedge F, \tag{5}$$

where $\boldsymbol{\nabla}$ denotes the multivector derivative operator. For a differentiable multivector field F with an orthonormal basis $\mathbf{e}_i$ on an open set in $\mathbb{R}^n$, it is defined by

$$\boldsymbol{\nabla} F = \mathbf{e}_i \partial_i F = \mathbf{e}_1 \partial_1 F + \mathbf{e}_2 \partial_2 F + ... + \mathbf{e}_n \partial_n F, \tag{6}$$

with the shorthand notation $\partial_i = \frac{\partial}{\partial_i}$. For differentiating fields on manifolds in $\mathbb{R}^n$, one has to use the component of the vector derivative lying within the manifold, as it can have components parallel and perpendicular to the manifold in general[16], respectively,

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_{||} + \boldsymbol{\nabla}_{\perp}. \tag{7}$$

Using the pseudoscalar, I_m, of the smooth m-dimensional oriented manifold M, the vector derivative component parallel to the manifold can be obtained with

$$\boldsymbol{\nabla}_{||} F = I_m^{-1}(I_m \cdot \boldsymbol{\nabla} F), \tag{8}$$

which leads to the definition of the tangential vector derivative on manifolds

$$\boldsymbol{\partial} = \boldsymbol{\nabla}_{||} = I_m^{-1}(I_m \cdot \boldsymbol{\nabla}). \tag{9}$$

Note, to distinguish it from the boundary operator ∂, bold notation is used for the vector derivative on manifolds. If the vector derivative lies entirely within the manifold, i.e. for the special case of $m = n$ for M^n in $\mathbb{R}^n$, then $\boldsymbol{\nabla}_{\perp} = 0$ and thus from Eqs. (7) and (9), $\boldsymbol{\partial} = \boldsymbol{\nabla}$.

The boundary theorem of geometric calculus for a multivector field F in a flat manifold[14,16,17] can be written as

$$\int_M \mathrm{d}\mathbf{x}^m \, \boldsymbol{\partial} F = \oint_{\partial M} d\mathbf{x}^{m-1} \, F, \tag{10}$$

where $\mathrm{d}\mathbf{x}^m$ and $\mathrm{d}\mathbf{x}^{m-1}$ are directed differential elements of the manifold M and boundary ∂M, respectively. The boundary theorem includes separate components of inner and outer derivatives[12], in a similar manner to Eq. (5),

the derivative splits into generalized divergence and curl components,

$$\int_M d\mathbf{x}^m \left(\boldsymbol{\partial} \cdot F + \boldsymbol{\partial} \wedge F\right) = \oint_{\partial M} d\mathbf{x}^{m-1} F . \qquad (11)$$

To obtain a graphical representation of the symmetries of the theorem, scalars can be denoted with dots, vectors with oriented lines, bivectors with oriented planes, trivectors with oriented volumes and so forth. For simplicity, manifolds are shown to be of maximal symmetry (spherical) and, unless stated otherwise, M^n is considered in $\mathbb{R}^n$ so that $\boldsymbol{\partial} = \boldsymbol{\nabla}$. For the present purpose, it suffices to explore homogeneous multivector fields of grade $n-1$.

III. TWO-DIMENSIONAL CASES

For a vector field on a two-dimensional manifold in $\mathbb{R}^2$, the boundary theorem leads to Figs. 1 and 2. From a purely visual perspective, Fig. 1 can be seen to share the vector structure of Gauss's law, Eq. (3). The dimensionality of the integrals is however off by one degree, as Gauss's law applies in three dimensional space. At this point, the comparison may seem superficial, but it is worth keeping in mind that Maxwell's equations are universal for locally conserved quantities[18].
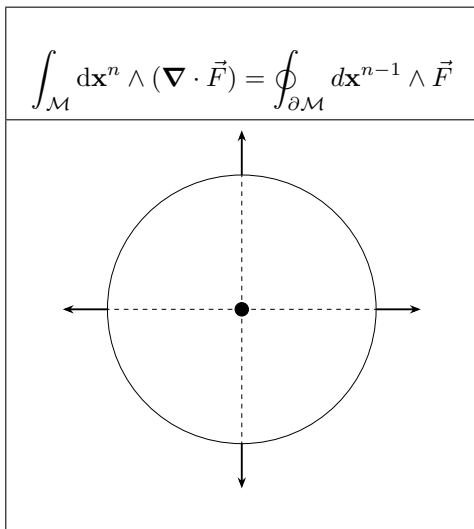

FIG. 1. 2D graphical representation of the divergence component of Stokes' theorem using Geometric calculus for the case of $\dim(F) = 1 = dim(\partial\mathcal{M})$.

Considering the dynamics implied by Eq. (2), sketched in Fig. 3, no direct correspondence to Fig. 2 is evident at first sight. However, the current density and displacement current vectors are dual to the curl bivector. Comparing Eq. (1) with Fig. 2, the boundary circulation and the bivector component within the manifold are shared, but Eq. (1) involves an additional time derivative component of the bivector. Furthermore, the relationship
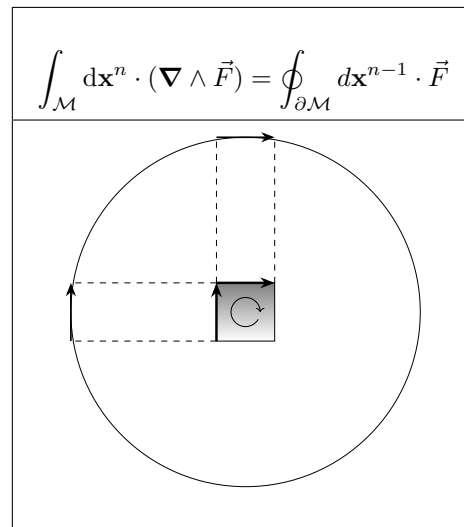

FIG. 2. 2D graphical representation of the curl component of Stokes' theorem using Geometric calculus for the case of $\dim(F) = 1 = dim(\partial\mathcal{M})$.

between the fields and their derivatives is not compatible between the two. This not surprising, as Maxwell's equations have been shown to correspond to a space-time split of the Faraday bivector[17], so more complex relations are expected. It's nonetheless worth noting how the two dimensional Stokes' theorem shares some common elements, and observe in what way they differ moving forward.
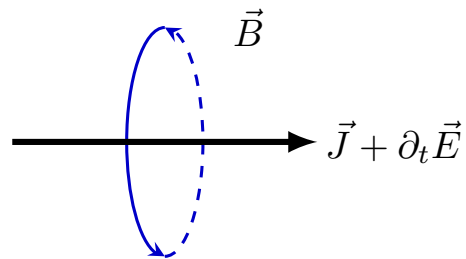

FIG. 3. Sketch of the magnetic field, $\vec{B}$, around a current carrying element $\vec{J}$ and/or displacement current $\partial_t \vec{E}$.

It has long been argued that the electromagnetic potentials are more fundamental than the force fields derived from them[19,20]. Therefore, we pose the question if a more straight forward correspondence to the boundary theorem cases can be found by considering the electromagnetic potentials?

Due to being an axial (bivector-dual) field, the magnetic field has a vector potential, $\vec{A}$, so that $\vec{B} = \boldsymbol{\nabla} \times \vec{A}$, rather than a scalar potential. The integral form of this expression combined with Green's curl theorem gives

$$\iint_S \vec{B} \cdot d\vec{S} = \iint_S \boldsymbol{\nabla} \times \vec{A} \cdot d\vec{S} = \oint_{\partial S} \vec{A} \cdot d\vec{\ell}. \qquad (12)$$

This field configuration is directly manifested by a cur-

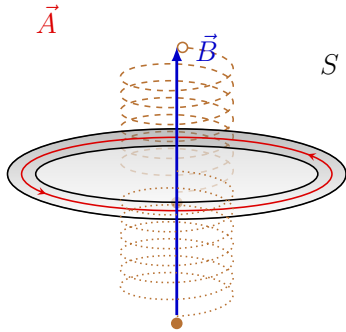rent carrying solenoid, Fig. 4. In a cornerstone proposal

FIG. 4. The magnetic vector potential, $\vec{A}$ and magnetic field $\vec{B}$, around a solenoid S.

by Aharanov and Bohm[21], and follow-up experiment by Tonomura[20], a phase-shift of the electron wavefunction was shown to occur as it propagates around the solenoid, even in the absence of a magnetic field strength in the space surrounding the solenoid. Although still somewhat debated, the effect has a straight forward hydrodynamical analogue as shown by Berry et al.[22], where the fluid velocity corresponds to the magnetic vector potential, and the vorticity to the magnetic field. This analogy holds through the whole system of electromagnetic and hydrodynamical equations[8], see Table I.

Looking back at the two dimensional curl case of the boundary theorem, Fig. 2, the resemblance is near exact. The major point gained from this comparison is that the bivector within the manifold, responsible for the non-zero circulation, is a kind of singularity, albeit a bivector-valued one. Keeping in mind that Cauchy's residue theorem of complex analysis is contained within the boundary theorem of geometric calculus[14], this is an interesting notion worthy of more rigorous study, since bivectors are pseudoscalars in two dimensional spaces.

IV. THREE-DIMENSIONAL CASES

By considering bivector fields on three-dimensional manifolds in a similar manner as before, the next set of cases of the boundary theorem are obtained[15]. To conveniently distinguish the inner and outer components, the coordinate-free definition of the tangential vector derivative can be applied,

$$\boldsymbol{\partial} F = \lim_{|d\mathbf{x}^m| \to 0} \frac{I_m^{-1}}{|d\mathbf{x}^m|} \oint_{\partial M} d\mathbf{x}^{m-1} F, \qquad (13)$$

the definition of which is enabled by the boundary theorem[13]. Since the field is now bivector valued, denoted by $\widehat{F}$, more components are obtained under the integral,

$$\boldsymbol{\partial}\widehat{F} = \lim_{|dV| \to 0} \frac{I_3^{-1}}{|dV|} \oiint_{\partial V} (d\mathbf{x}^2 \cdot \widehat{F} + d\mathbf{x}^2 \times \widehat{F} + d\mathbf{x}^2 \wedge \widehat{F}). \quad (14)$$

Assuming the manifold M^3 to be strictly in $\mathbb{R}^3$, the last term of Eq. (14) vanishes. Its existence is allowed if M is embedded in a larger space, and although interesting, this is set aside for now. With dimensional analysis the vector-valued divergence $\boldsymbol{\nabla} \cdot \widehat{F}$ and the trivector-valued curl $\boldsymbol{\nabla} \wedge \widehat{F}$ can be matched with the corresponding boundary integrals,

$$\iiint_M d\mathbf{x}^3 \cdot (\boldsymbol{\nabla} \cdot \widehat{F}) = \oiint_{\partial M} d\mathbf{x}^2 \times \widehat{F}, \qquad (15)$$

and

$$\iiint_M d\mathbf{x}^3 \cdot (\boldsymbol{\nabla} \wedge \widehat{F}) = \oiint_{\partial M} d\mathbf{x}^2 \cdot \widehat{F}. \qquad (16)$$

The graphical representation of Eq. (15) is shown in Fig. 5.

Electromagnetism	Hydrodynamics
magnetic vector potential [$\mathbf{A}$: ML/TQ]	velocity [$\mathbf{v}$: L/T]
magnetic field [$\mathbf{B}$: M/TQ]	vorticity [$\boldsymbol{\omega}$: 1/T]
electric field [$\mathbf{E}$: ML/T^{2}Q]	acceleration [$\mathbf{L}$: L/T^2]
electric scalar potential [ϕ : ML^2/T^{2}Q]	kinematic pressure [ϕ : ML^2/T^{2}Q]
phase function [χ: ML^2/TQ]	velocity potential [Φ: L^2/T]
charge [q: Q]	mass [m: M]
charge density [ρ_q: Q/L^3]	fluid density [ρ_f: M/L^3]
current density [$\mathbf{J}$: Q/TL^2]	mass flux [$\mathbf{j}_m$: M/TL^2]

TABLE I. Analogous concepts in electrodynamics and hydrodynamics. Key: Concept [**symbol**: Dimension in SI-units (in terms of the basic quantities mass (M), length (L), time (T) and charge (Q))].
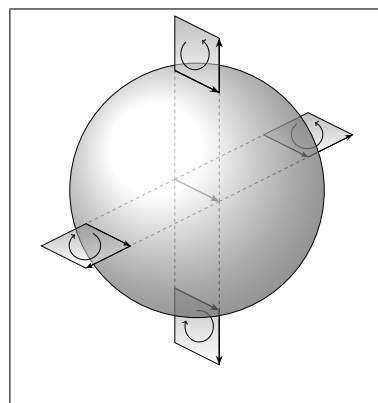

FIG. 5. Graphical representation of the divergence component of Stokes' theorem using Geometric calculus for the cases of $\dim(F) = 2 = \dim(\mathcal{M}) - 1 = dim(\partial \mathcal{M})$.

At first glance, the emerging structure is quite unfamiliar, and no correspondence with Maxwell's equations is evident. Recall, resemblance to Eqs. (2) and (4) is yet to be seen. Although the divergence of the magnetic field is zero and often considered in conjunction with Gauss's law, the magnetic field differs in being an axial vector, so we do not consider the two dimensional divergence case to apply for the magnetic divergence.

Continuing in the same spirit as before, consider the magnetic vector potential for a current carrying element, Fig. 6a, setting the displacement current to zero for simplification. The circulation (curl) of the vector potential has the exact same form as the vector divergence of the bivector case of the boundary theorem, Fig. 5. This cor-
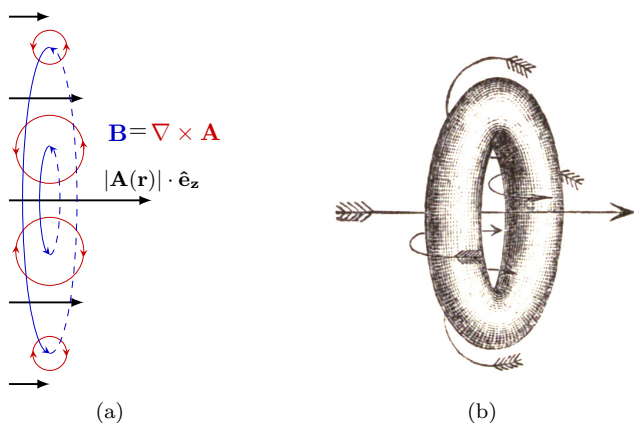

FIG. 6. (a) The magnetic vector potential and magnetic field around a current carrying element. (b) Sketch of a smoke ring, by Tait[23], ano 1876.

respondence goes to show that considering the magnetic field as a vector, rather than a bivector, hides the richer dynamical structure in this case. It is from circulation in the momentum (magnetic vector potential), that the axial essence of the magnetic field vector lies. Furthermore, a straight forward hydrodynamical analogue exists, the smoke ring, when considering the magnetic vector potential in the role of fluid velocity, Fig. 6b.

Moving on to the trivector valued curl component, Eq. (16), the graphical representation is shown in Fig. 7. As before, the form is somewhat unfamiliar. If we however consider the dual of the magnetic field, the bivector $\widehat{B}$, Fig. 7 can be seen to correspond to a magnetic monopole, but as a generalized curl of a bivector field, rather than a divergence of a vector field, as in Eq. (4). Once again, a more direct correspondence is found in a fresh context by considering the electromagnetic potential.
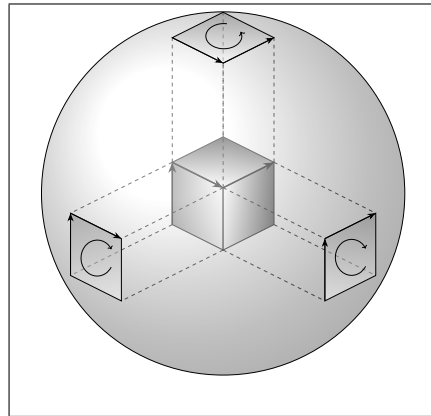

FIG. 7. Graphical representation of the curl component of Stokes' theorem using Geometric calculus for the cases of $\dim(F) = 2 = \dim(\mathcal{M}) - 1 = dim(\partial\mathcal{M})$.

V. DISCUSSION

In the current investigation, the application of the boundary theorem has been restricted to unembedded manifolds with maximal symmetry. The theorem however applies equally to both curved and irregular spaces[24]. In continuation of the current work, more complex gauge symmetries and interior degrees of freedom can be considered. Higher dimensional cases can also be mapped out, and the effect of subspaces investigated. The space-time integration elements and corresponding dynamics of the cases shown also need to be studied.

It should be noted that the correspondence between geometric calculus and differential geometry is still being worked out in full[25], application of geometric algebra and calculus is however well underway[26–28]. Arguably both Clifford analysis and tensor analysis are more general than geometric calculus but its main strength is the geometry aspect, allowing for ease of visualization. The accessibility to advanced tools of analysis is highly valuable as well, both for students and researchers.

VI. CONCLUSION

Symmetries of Maxwell's equations of electromagnetism have been compared to corresponding instances of the boundary theorem in geometric calculus. A more direct correspondence between the two has be shown to be obtainable by considering the electromagnetic potentials. The comparison highlights the importance of the bivector nature of the magnetic field, and naturally leads to consideration of the boundary theorem for more complex gauge symmetries and internal degrees of freedom, worthy of further investigation.

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