From Null Monomials to Versors in Conformal Geometry

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Abstract. Extended Abstract

In Conformal Geometric Algebra (CGA), a point (including the conformal point at infinity \(e_\infty\)) is represented by a null vector, and the representation is unique up to scale. A versor is the geometric product of finitely many invertible vectors in \(\mathbb{R}^{n+1,1}\). A conformal transformation acts on points via the (left) adjoint action of the corresponding versor \(V\) upon the null vectors representing the points:

\[ Ad_V(x) = V x V^{-1}, \]

and the versor inducing the conformal transformation is unique up to a nonzero scalar or pseudoscalar factor.

Although in conformal geometric constructions, starting from points one can construct lines and circles by 3-vectors, 2-planes and 2-spheres by 4-vectors, etc., which are outer products of null vectors representing the incident points involved in the construction, in symbolic geometric computing it is the geometric product of these null vectors that prove to be much more efficient than the graded parts of the geometric product, such as the outer product, the meet product, the inner product, etc. Hence, monomials that are the geometric product of null vectors, called null monomials, turn out to be the basic algebraic terms in symbolic computing. Their geometric interpretations and applications are an important topic of research.

In Grassmann algebra, a blade as the outer product of vectors, represents a linear subspace, and in CGA when focusing on the null vectors and positive vectors in the linear subspace, the blade can be interpreted geometrically as lines, circles, etc., or dual to such geometric entities. However, if a homogeneous multi-vector is not a blade, then its geometric interpretation is not clear in general.

In CGA, the geometric product of invertible vectors is a versor, and can be interpreted geometrically as the generator of a conformal transformation, the latter being the composition of the conformal transformations each being induced by one invertible vector factor of the versor. However, when some vector factors are not invertible, the geometric interpretation of such geometric products is still not clear.

In the first part of this talk, we address the problem of the geometric interpretation of null monomials in CGA, which is the extreme case where in the geometric product all the vector factors are not invertible. We only highlight some guiding examples below.

In CGA, we do not distinguish between a non-zero entity and its nonzero scalings. We use \(x \equiv y\) to denote that \(x, y\) are equal up to a nonzero scalar factor. Sometimes to take the sign into consideration, we use \(x \equiv^{+} y\) to denote that \(x, y\) are equal up to a positive scalar factor.

Example 1. Let null vector \(n\) represent an affine point. Then \(e_\infty n\) is a null monomial of length (or degree) 2. Let there be another null monomial \(n_1n_2\) of length 2 such that \(n_1n_2 \equiv e_\infty n\). It is easy to see that \(n_1 \equiv n\) and \(n_2 \equiv e_\infty\). So null monomial \(n_1n_2\) represents an ordered pair of points.

In contrast, \(e_\infty \land n\) represents an unordered pair of points, because \(e_\infty \land n \equiv n \land e_\infty\). When the sign is taken into account, then \(e_\infty \land n\) also represents the ordered pair of points \(e_\infty, n\).

Example 2. Let null vectors \(n_1, n_2\) represent two different affine points. Then \(e_\infty n_1n_2\) is a null monomial of length 3. If there is another null monomial \(n'_1n'_2n'_3\) of length 3 such that \(n'_1n'_2n'_3 \equiv e_\infty n_1n_2\), direct arguments show that \(n'_1 \equiv e_\infty, n'_3 \equiv n_2\), and \(n'_2\) can be any affine point on line \(n_1n_2\) other than \(n_2\). So \(e_\infty n_1n_2\) represents a pair of ordered points \(e_\infty, n_2\), together with line \(n_1n_2\).

In contrast, \(e_\infty \land n_1 \land n_2\) only represents the line passing through points \(n_1, n_2\). To represent this line with the outer product, \(e_\infty, n_1, n_2\) can be replaced by any other three points on the line.
If the sign of $e_\infty n_1 n_2$ is taken into account, then any affine point $p$ has a unique null-vector representation, denoted by the same symbol, that satisfies $e_\infty \cdot p = -1$. A point $n_3$ on line $n_1 n_2$ has the following unique null vector representation:

$$n_3 = \lambda n_1 + (1 - \lambda) n_2 + \epsilon e_\infty,$$

for some scalar $\epsilon$ that ensures $n_3^2 = 0$. In fact,

$$\epsilon = -\lambda(1 - \lambda) n_1 \cdot n_2 = \frac{d^2 n_1 n_2}{2} \lambda(1 - \lambda).$$

So $e_\infty n_3 n_2 = \lambda e_\infty n_1 n_2$, and $e_\infty n_1 n_2 \equiv + e_\infty n_2 n_1$ if and only if point $n_3$ is on line $n_1 n_2$ and on the same side with $n_1$ relative to $n_2$. So $e_\infty n_1 n_2$ represents an ordered pair of points $e_\infty$, and the ray from $n_2$ towards $n_1$.

Similarly, $n_2 n_1 e_\infty$ represents an ordered pair of points $n_2$, $e_\infty$, and line $n_1 n_2$. When the sign is taken into account, then in the geometric interpretation, the line is replaced by the ray from $n_2$ towards $n_1$.

Formally, over a base numbers field $K$, the null monomial algebra on an alphabet $A = \{n_i\}_{i \in I}$ is the quotient of the free associative algebra $K[A]$ modulo the two-sided ideal generated by elements of the form $n_i n_i$. Informally, we allow the alphabet $A$ to take all null vectors (up to scale) of $\mathbb{R}^{n+1}$.

A null monomial has two ends. If the two ends are identical, then the null monomial is said to be isotropic. The degree (or length) of a null monomial is the number of null vector factors in the sequence of the monomial. By definition, no two adjacent vector factors in the monomial are identical up to scale. An anisotropic monomial has degree $\geq 1$, while an isotropic monomial has degree $\geq 1$.

In the first part of this talk, we present geometric interpretations for all null monomials in CGA, together with their normal forms leading to the geometric interpretations. The main conclusion is the following, called the conformal split of null monomials.

Theorem 1. In CGA, any isotropic null monomial $n_1 \cdots n_k n_1$ equals up to scale $n_1 V \equiv V n_1$, where $V$ is a versor whose vector factors anticommute with $n_1$, and can be chosen to anticommute with any fixed second null vector. Any anisotropic null monomial $n_1 \cdots n_k$ equals up to scale $n_1 n_k V_r$ (or $V n_1 n_k$), where $V_r$ (or $V_l$) is a versor whose vector factors anticommute with $n_k$ (or $n_1$), and can be chosen to anticommute with $n_1$ (or $n_k$) as well.

Geometrically, if versor $V$ has its vector factors anticommute with both $n_1$ and $n_k$, then it induces a conformal transformation fixing the two points represented by $n_1, n_k$ respectively. For example, when $n_1 = e_\infty$, if $V$ is even, then it induces a rotation fixing point $n_k$, else it induces the composition of a rotation fixing point $n_k$ and a mirror reflection with respect to a hyperplane passing through $V_l$.

Let there be two anisotropic null monomials $M = n_0 n_1 \cdots n_k$ and $M' = n_0' n_1' \cdots n'_k$. Then $M \equiv V M n_0 n_k$ and $M' \equiv n_0 V M' n'_k$ for some versors $V M, V M'$ fixing point $n_0$. So

$$MM' \equiv V M n_0 n_k n_0 V M' n'_k \equiv V M V M' n_0 n'_k.$$  

The product of the two monomials is equivalent to the geometric product of the two versors, followed by the common leading vector factor $n_0$, and the ending vector factor of $M'$.

For an anisotropic null monomial ended by $n_0$, say $x_k x_{k-1} \cdots x_1 n_0$, we can also start from the right side of the monomial to insert copies of $n_0$, turn it into the normal form $x_k (n_0 x_{i_2} x_{i_3} \cdots x_{i_j}) n_0$, and provide a geometric interpretation similar to that in Theorem 1.

For two anisotropic null monomials $M = n_0 n_1 \cdots n_k$ and $M' = n'_1 \cdots n'_k$, we have $M \equiv V M n_0 n_k$ and $M \equiv n'_k n_0 V M'$ for some versors $V M, V M'$ fixing point $n_0$. So

$$MM' \equiv V M n_0 n_k n_0 V M' n_0' \equiv V M V M' n_0 n_0'.$$

The product of the two monomials is equivalent to the geometric product of the first versor, the displacement vector between the two ends other than $n_0$ in the two monomials, the second versor, and the common end $n_0$.

The second part of the talk is the core content. It addresses the problem of generating versors by anisotropic null monomials. Let

$$M = x_1 x_2 \cdots x_r$$

be an anisotropic null monomial. We shift the first vector factor $x_1$ to the end of the monomial to get $x_2 \cdots x_r x_1$, then we multiply the result by a nonzero scalar $\lambda^{-1}$ and add it up with $M$. The result is the following null binomial
that we call a shifted-scaled null binomial:

\[ V := x_1x_2 \cdots x_r + \lambda^{-1}x_2x_3 \cdots x_rx_1. \]

(7)

The pair \((M, \lambda)\) is called a shifted-scaled pair. The leading vector factor \(x_1\) of \(M\) is the shifted vector factor, called the index of the binomial. Vector factor \(x_2\) is called the left end of the binomial, and \(x_r\) is called the right end. \(\lambda\) is called the scaling factor of the binomial.

In binomial \(V\) there are all together 4 ends: vector \(x_1\) that occurs twice, and vectors \(x_2, x_r\) each occur once. When \(x_2 = x_r\), then vector \(x_2\) also occurs twice as ends. In this case, binomial \(V\) can be rewritten in a form where \(x_2\) serves as the index.

Proposition 1. If a shifted-scaled null binomial has two vector factors each occurring twice as ends of the binomial, then any of them can serve as the index of the binomial.

Proof. Let

\[ W = x_2x_1x_3 \cdots x_{r-1}x_1 + \lambda^{-1}x_1x_3 \cdots x_{r-1}x_1x_2. \]

(8)

be a shifted-scaled null binomial indexed by \(x_2\), where \(x_1\) also occurs twice as ends. Write \(W\) as

\[ W \equiv x_1x_2x_1x_3 \cdots x_{r-1}x_1x_2 + \lambda x_2x_1x_3 \cdots x_{r-1}x_1x_2x_1. \]

(9)

The right side is a binomial generated by shifting \(x_1x_2x_1x_3 \cdots x_{r-1}x_1x_2\) and then scaling by \(\lambda\). Q.E.D.

Theorem 2. Every shifted-scaled null binomial is a versor in CGA. Conversely, up to a nonzero scalar or pseudoscalar factor, every versor in CGA equals a shifted-scaled null binomial. Because of this, when a versor takes the form of a shifted-scaled null binomial, we call it a null versor.

We raise some examples below.

Example 3. Let

\[ V = e_\infty n + \lambda^{-1}ne_\infty. \]

(10)

Then

\[ V V^\dagger = \lambda^{-1}(e_\infty ne_\infty n + ne_\infty ne_\infty) = 4\lambda^{-1}(e_\infty \cdot n)^2 \neq 0, \]

(11)

where we have used the contraction identity

\[ n_1n_2n_1 = 2(n_1 \cdot n_2)n_1, \]

(12)

for any null vector \(n\). So \(V\) is a versor in the Clifford algebra over the 2-space \(e_\infty \wedge n\), and generates a dilation centered at affine point \(n\).

In details, let \(n = e_0\) be the origin, and let \(n_x = e_0 + x + e_\infty x^2/2\) be the null vector representation of point \(x \in (e_\infty \wedge n)^\perp = \mathbb{R}^n\), then with \(e_0 \cdot e_\infty = -1\), we have

\[ A_d(V(n_x)) \equiv (e_\infty e_0 + \lambda^{-1}e_0e_\infty)(e_0 + x + e_\infty x^2/2)(e_0e_\infty + \lambda^{-1}e_\infty e_0) \]

\[ = 4(\lambda^{-2}e_0 + \lambda^{-1}x + e_\infty x^2/2) \]

\[ = e_0 + \lambda x + e_\infty(\lambda x)^2/2. \]

(13)

So \(\lambda\) is the dilation ratio. In particular, if \(\lambda = 1\), then \(e_\infty n + ne_\infty \equiv 1\) induces the identity transformation.

Example 4. Let

\[ V = e_\infty n_2e_\infty n_1 + \lambda^{-1}n_2e_\infty n_1e_\infty \equiv \frac{e_\infty n_1}{e_\infty \cdot n_1} + \lambda^{-1}\frac{n_2e_\infty}{e_\infty \cdot n_2}. \]

(14)

Then

\[ V V^\dagger \equiv e_\infty n_1e_\infty n_2 + n_2e_\infty n_1e_\infty = 4(e_\infty \cdot n_1)(e_\infty \cdot n_2) \neq 0. \]

(15)

Expanding \(V\) into graded terms, we get

\[ V \equiv (1 + \lambda^{-1}) + e_\infty \wedge \left( \frac{n_1}{e_\infty \cdot n_1} - \lambda^{-1}\frac{n_2}{e_\infty \cdot n_2} \right). \]

(16)

By

\[ e_\infty \cdot \left( \frac{n_1}{e_\infty \cdot n_1} - \lambda^{-1}\frac{n_2}{e_\infty \cdot n_2} \right) = 1 - \lambda^{-1}, \]

(17)
if \( \lambda = 1 \), then \( V \) induces a translation, else it induces a dilation.

In details, if \( \lambda = 1 \), by \( e_\infty \cdot n_1 = -1 \), we have
\[
V_n V^{-1} = (e_\infty n_1 + n_2 e_\infty) n_1 (n_1 e_\infty + e_\infty n_2) = n_2 e_\infty n_1 e_\infty n_1 \equiv n_2,
\]
so \( V \) induces the translation from point \( n_1 \) to point \( n_2 \).

If \( \lambda \neq 1 \), by (10), let \( V = (e_\infty x + \mu^{-1} x e_\infty) / e_\infty \cdot x \) for some unknown null vector \( x \) and scalar \( \mu \), so that \( x \) is the dilation center and \( \lambda \) is the dilation ratio. The 0-graded part and 2-graded part of the equality respectively give
\[
\frac{\lambda n_1 / e_\infty \cdot n_1 - n_2 / e_\infty \cdot n_2}{\lambda - 1} + e_\infty,
\]
where scalar \( \epsilon \) is chosen to make \( x^2 = 0 \). Indeed,
\[
\epsilon = \frac{\lambda n_1 \cdot n_2}{(\lambda - 1)^2 (e_\infty \cdot n_1)(e_\infty \cdot n_2)}.
\]

Geometrically, \( x \) is the point on affine line \( n_1 n_2 \) with affine ratio \( \frac{n_2 x}{n_1} : \frac{n_1 x}{n_2} = \lambda \).

Example 5. Let
\[
V = e_\infty n_0 e_\infty n_1 n_2 e_\infty n_0 + \lambda^{-1} n_0 e_\infty n_1 n_2 e_\infty n_0 e_\infty \equiv e_\infty n_1 n_2 e_\infty n_0 + \lambda^{-1} n_0 e_\infty n_1 n_2 e_\infty e_\infty.
\]
It is easy to verify that \( V V^\dagger \equiv 1 \). Take \( n_0 \) as the origin of \( \mathbb{R}^n \). Then \( e_\infty n_1 n_2 e_\infty = \overline{n_1 n_2} e_\infty \), where \( \overline{n_1 n_2} \in \mathbb{R}^n \) is the displacement vector from point \( n_1 \) to point \( n_2 \), and as a vector in \( \mathbb{R}^{n+1,1} \), \( n_1 n_2 \) is orthogonal to both \( e_\infty \) and \( n_0 \). So
\[
V \equiv \overline{n_1 n_2} e_\infty n_0 + \lambda^{-1} n_0 \overline{n_1 n_2} e_\infty = \overline{n_1 n_2} (e_\infty n_0 - \lambda^{-1} n_0 e_\infty) = (e_\infty n_0 - \lambda^{-1} n_0 e_\infty) \overline{n_1 n_2}.
\]

By (10), \( e_\infty n_0 - \lambda^{-1} n_0 e_\infty \) induces the dilation of ratio \(-\lambda\) centered at \( n_0 \). So when \( \lambda = -1 \),
\[
e_\infty n_0 e_\infty n_1 n_2 e_\infty n_0 = n_0 e_\infty n_1 n_2 e_\infty n_0 e_\infty \equiv \overline{n_1 n_2}
\]
induces the reflection with respect to the affine hyperplane normal to \( \overline{n_1 n_2} \) and passing through point \( n_0 \).

Theorem 3. All shifted-scaled pairs with the same index form a group under the component-wise product, called the connecting group, and the product is called the connecting product. For two pairs \( (n_0 n_1 \cdots n_r, \lambda) \) and \( (n_0 n'_1 \cdots n'_s, \lambda') \), their connecting product \( ((n_0 n_1 \cdots n_r) (n_0 n'_1 \cdots n'_s), \lambda \lambda') \) generates a null versor that equals the geometric product of three versors up to scale: the versor generated by the first pair, the versor generated by \( (n_0 n_1 n_0 n'_1, 1) \), and the versor generated by the second pair. Geometrically, the connecting group of shifted-scaled pairs with the same two indices is surjectively homomorphic to the group of conformal transformations fixing the two points represented by the two indices respectively.

For two shifted-scaled pairs \( (n_0 n_1 \cdots n_r, \lambda) \) and \( (n_0 n'_1 \cdots n'_s, \lambda') \) indexed by \( n_0 \), their \textit{prepending product} is of the form
\[
\left(n_0 x (n_0 n_1 \cdots n_r) (n_0 n'_1 \cdots n'_s), \lambda \lambda' \right),
\]
where \( x \) is a null vector determined by the two pairs, whose explicit expression will not be presented here.

Theorem 4. All shifted-scaled pairs with the same index form a group under the prepending product, called the \textit{prepending group} of scaled null monomials. This group is homomorphic to the group of null versors each being generated by such a pair. Geometrically, the prepending group of shifted-scaled pairs with the same index is surjectively homomorphic to the group of conformal transformations fixing the point represented by the index.

Finally, consider the geometric product of a sequence of invertible vectors and null vectors. Let \( V_1, \ldots, V_{k+1} \) be versors, and let \( x_1, \ldots, x_k \) be null vectors. Then
\[
W = V_1 x_1 V_2 x_2 \cdots V_k x_k V_{k+1} = (V_1 x_1 V_1^{-1}) V_1 V_2 x_2 (V_1 V_2)^{-1} \cdots (V_1 \cdots V_k) x_k (V_1 \cdots V_k)^{-1} V_1 \cdots V_k V_{k+1} = x_1' x_2' \cdots x_k' V_1 \cdot \cdots \cdot V_k V_{k+1},
\]
where \( x'_i = (V_1 \cdots V_i) x_i (V_1 \cdots V_i)^{-1} \) is a null vector, for all \( 1 \leq i \leq k \). By this and Theorem 1, we get
\[
W = V_1 x_1 V_2 x_2 \cdots V_k x_k V_{k+1} \equiv x'_1 x'_2 x'_k V',
\]
where \( V' \) is a versor. By shifting and scaling the subsequence of null vectors, the following versor can be generated from \( W \):
\[
V_1 x_1 V_2 x_2 \cdots V_k x_k V_{k+1} + \lambda^{-1} V_1 (V_2 x_2 \cdots V_k x_k) x''_k V_{k+1},
\]
where \( x''_k = (V_2 \cdots V_k)^{-1} x_1 (V_2 \cdots V_k) \).