Construction of Special Conics in Bundles of Conics Using Geometric Algebras

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Summary of the Abstract

Using Geometric Algebra for Conics (GAC), we inspect the geometric problems related to a bundle of conics which is a linear combination of two (possibly intersecting) conics. In particular, we focus on construction of special conics from the bundle – three pairs of lines, and two conjugate parabolas, by means of GAC. Furthermore, we discuss possible usage of these conics in finding the intersection of two conics.

1 Geometric Alegbra for Conics

Geometric Algebra for Conics (GAC), originally introduced in [12], and consequently elaborated in [6], is already acknowledged to be useful for conic manipulation, e.g. for intersections [2], and for simple conic fitting [7], as well as conic fitting with additional geometric constraints such as axial alignment. [10, 11]

Let us recall that GAC comprises a Clifford algebra $\mathcal{C}l(5,3)$ with an embedding $C : \mathbb{R}^2 \to \mathbb{R}^{5,3}$ of a proper point $\mathbf{x} = xe_1 + ye_2$ from the plane $\mathbb{R}^2$ to a six-dimensional subspace of one-vectors in GAC, in the form

$$C(x,y) = \bar{n}_+ + xe_1 + ye_2 + \frac{1}{2}(x^2+y^2)n_+ + \frac{1}{2}(x^2-y^2)n_- + xyn_\times \tag{1.1}$$

where $\{\bar{n}_\times, \bar{n}_-, \bar{n}_+, e_1, e_2, n_+, n_-, n_\times\}$ is the eight-dimensional vector basis of $\mathcal{C}l(5,3)$, [7], together with an associated bilinear form of the inner product of vectors in GAC given by the matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consequently, the inner product null space (IPNS) representation of a general conic section Q in GAC is given by

$$Q_I = \bar{v}^{\times}\bar{n}_{\times} + \bar{v}^{-}\bar{n}_{-} + \bar{v}^{+}\bar{n}_{+} + v^1 e_1 + v^2 e_2 + v^+ n_+. \tag{1.2}$$

Equations of particular conics in GAC can be found in [6, 2]. It is also well known that the type and features of conic Q can be read off its matrix representation [8], which is obtained using (1.2) as a matrix

$$M = \begin{pmatrix} -\frac{1}{2}(\bar{v}^{+} + \bar{v}^{-}) & -\frac{1}{2}\bar{v}^{\times} & \frac{1}{2}v^1 \\ -\frac{1}{2}\bar{v}^{\times} & -\frac{1}{2}(\bar{v}^{+} - \bar{v}^{-}) & \frac{1}{2}v^2 \\ \frac{1}{2}v^1 & \frac{1}{2}v^2 & -v^+ \end{pmatrix}.$$

Let us also note that a proper point embedded into GAC using mapping $C(x, y)$ of form (1.1) can be represented as a vector

$$P_I = \begin{pmatrix} 0 & 0 & 1 & x & y & \frac{1}{2}(x^2+y^2) & \frac{1}{2}(x^2-y^2) & xy \end{pmatrix}^T, \tag{1.3}$$

the improper point (point at infinity) introduced in [9] as a vector

$$P_{\infty I} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(s^2+t^2) & \frac{1}{2}(s^2-t^2) & st \end{pmatrix}^T$$

and the IPNS conic section (1.2) as a vector

$$Q_I = \begin{pmatrix} \bar{v}^{\times} & \bar{v}^{-} & \bar{v}^{+} & v^1 & v^2 & v^+ & 0 & 0 \end{pmatrix}^T$$

2 Construction of Conics from 5 Points Using GAC

As shown in [6], an outer product null space (OPNS) representation of a conic Q can be constructed from five GAC points $P_1, \ldots, P_5$ of the form (1.3) using the wedge product:

$$Q_O = P_1 \wedge P_2 \wedge P_3 \wedge P_4 \wedge P_5.$$

If an OPNS conic (or generally, any object in GAC) needs to be converted into IPNS representation, or, vice versa, it can be done using two "pseudoscalars"

$$\begin{aligned} I_{OI} &= \bar{n}_{+}\bar{n}_{-}\bar{n}_{\times}e_1e_2n_{+}, \\ I_{IO} &= \bar{n}_{+}e_1e_2n_{+}n_{-}n_{\times}. \end{aligned} \tag{2.1}$$

Then,

$$\begin{aligned} A_O &= A_I \cdot I_{IO}, \\ A_I &= A_O \cdot I_{OI}. \end{aligned} \tag{2.2}$$

For a conic constructed with a wedge product to be unique and non-degenerate, the points $P_1, \ldots, P_5$ must be in *general linear position*, i.e. no three of them are collinear. If the greatest number of the collinear points in the set is three, then the conic is still uniquely determined but is degenerate. Moreover, if four or five points of the set are collinear, the spanned conic is not unique, [14, 13].
Additionally, there is no need for all the five points of the set to be *proper*, since one or two of them may be improper as it is in the case of parabola and hyperbola or even a pair of lines, respectively. Using this knowledge and the representation of improper points in GAC, we can wedge one or two improper points with proper points to get a parabola with a prescribed direction of the axis of symmetry, a hyperbola with a given direction of the asymptote or a pair of asymptotes, or a pair of lines with given directions.

2.1 Four-point and Bundle of Conics

It is generally known that two distinct conics may create up to four distinct real points of intersections and there are various algorithms to compute them—one of them described in [14], another one (partially using GAC) can be found in [2]. By means of GAC, we can represent the intersection of two conics as a *four-point*. According to [6], intersections of two conics are given by the wedge product of their IPNS representations, and, given two distinct conics Q^1, Q^2, the IPNS representation of the associated four-point is computed as

$$\left(Q^1 \cap Q^2\right)_I = Q^1_I \wedge Q^2_I.$$

Even though the way to decompose a four-point into the intersection points is yet unknown, the concept of a four-point itself can prospectively be of great help in computing them. For the sake of simplicity, let us further assume the cases when two conics have four real intersection points. Some of the algorithms for computation of the conics intersection exploit *bundle* (or *pencil*) *of conics*, i.e. a set of all conics passing through the four intersections of two conics, as seen in Fig. 1 (left). The algorithm described in [14] goes basically as follows:

1. Compute three degenerate cases in the bundle of conics, i.e. three pairs of lines going through the points of intersection (see Fig. 1 (right)),
2. Decompose one of the pairs of lines into two distinct lines,
3. Intersect each of the two lines with one of two intersecting conics.

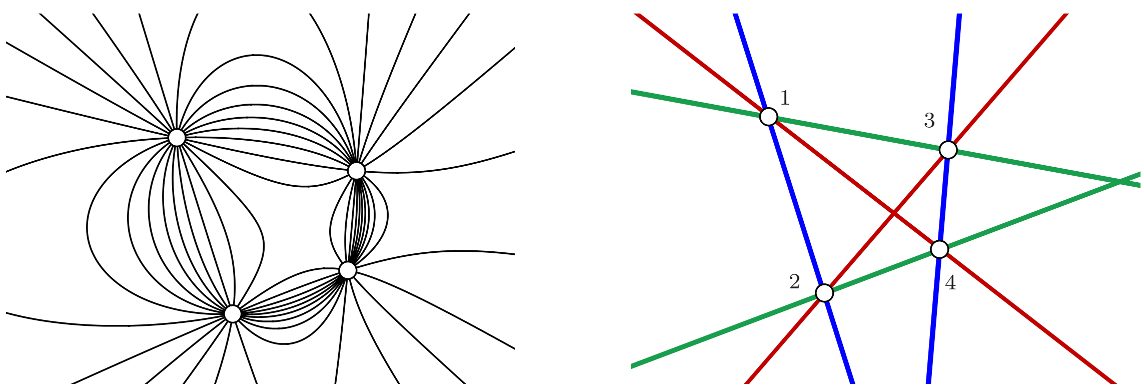

Figure 1: Bundle of conics through four points and three degenerate special cases (taken from [14])

As stated in [2], four-point $Q^1_I \wedge Q^2_I$ is a bi-vector, but when converted into OPNS, it becomes a four-vector which corresponds to a wedge of four points. Therefore, we can wedge an OPNS representation of the intersections of two conics and further wedge it with another point P_5 of the form (1.3), thus obtaining an OPNS conic Q_O as:

$$Q_O = \left(Q^1_I \wedge Q^2_I\right)^* \wedge P_5, \tag{2.3}$$

where * signifies duality between IPNS and OPNS given by transition equations (2.2) and (2.1). Also, as apparent from Fig. 1 (right), each pair of lines in a bundle of conics has its own point of intersection. Consequently, finding any pair of lines in a bundle of conics can be carried out by wedging the intersections of two conics with the same intersection of the line-pair. Fortunately, intersections of the line-pairs can be computed by means of matrix representation of the intersecting conics and a generalised eigenproblem, as described in [3]. Given two distinct conics Q^1, Q^2 with four real points of intersection and associated matrices M_1, M_2 of the form (1), the points of intersection of the line-pairs correspond to eigenvectors v_1, v_2, v_3 of eigenproblem

$$M_1 v = \lambda M_2 v. \tag{2.4}$$

Let us note that the obtained eigenvectors have three coordinates corresponding to homogeneous coordinates. Thus, they have to be embedded into GAC using embedding $C\mathbb{P}$ before wedging with the four-point.
Let us further examine an example of the construction of the line-pairs with particular conics[1]:

Example 1 *Let us consider two ellipses E^1, E^2 with IPNS representations*

$$E_I^1 = \bar{n}_+ - \frac{8}{17}\bar{n}_- - \frac{225}{34}n_+,$$
$$E_I^2 = \bar{n}_+ + \frac{3}{5}\bar{n}_- + \frac{16}{5}e_1 + \frac{2}{5}e_2 + \frac{1}{5}n_+,$$

and the associated matrices

$$M_1 = \begin{pmatrix} -\frac{9}{34} & 0 & 0 \\ 0 & -\frac{25}{34} & 0 \\ 0 & 0 & \frac{225}{34} \end{pmatrix}, \quad M_2 = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{8}{5} \\ 0 & -\frac{1}{5} & \frac{1}{5} \\ \frac{8}{5} & \frac{1}{5} & -\frac{1}{5} \end{pmatrix}.$$

Consequently, the solution to generalised eigenproblem (2.4) *is given by*

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \approx \begin{pmatrix} 1.9191 \\ 3.1839 \\ 0.4117 \end{pmatrix}, \quad \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \approx \begin{pmatrix} 2.4167 & 2.232 & 10.1865 \\ -1.0921 & -6.4631 & -0.1261 \\ 1 & 1 & 1 \end{pmatrix},$$

and the IPNS representation of the points of intersection R_1, R_2, R_3 of the sought line-pairs are computed as an embedding of the obtained eigenvectors to GAC, i.e.

$$R_i = C\mathbb{P}(v_i), \qquad i = 1, 2, 3.$$

Finally, each of the line-pairs is computed using a wedge product according to (2.3)*:*

$$LP_O^i = \left(E_I^1 \wedge E_I^2\right)^* \wedge R_i, \qquad i = 1, 2, 3.$$

[1]Problems given in Example 1 was computed in MAPLE using library CLIFFORD by Abłamowicz & Frauser. For more details see [1].

After conversion to IPNS, the representations of the acquired line-pairs read

$$LP_I^1 \approx 2.009\bar{n}_+ + 3.5455\bar{n}_- + 13.4233e_1 + 1.6779e_2 + 15.3036n_+,$$
$$LP_I^2 \approx -17.4331\bar{n}_+ - 19.0061\bar{n}_- - 81.3307e_1 - 10.1663e_2 - 57.9101n_+,$$
$$LP_I^3 \approx 29.8078\bar{n}_+ - 36.3612\bar{n}_- - 66.7566e_1 - 8.3446e_2 - 339.4835n_+.$$

Both conics, the intersection four-point and the constructed line-pairs with their points of intersection can be seen in Fig. 2.

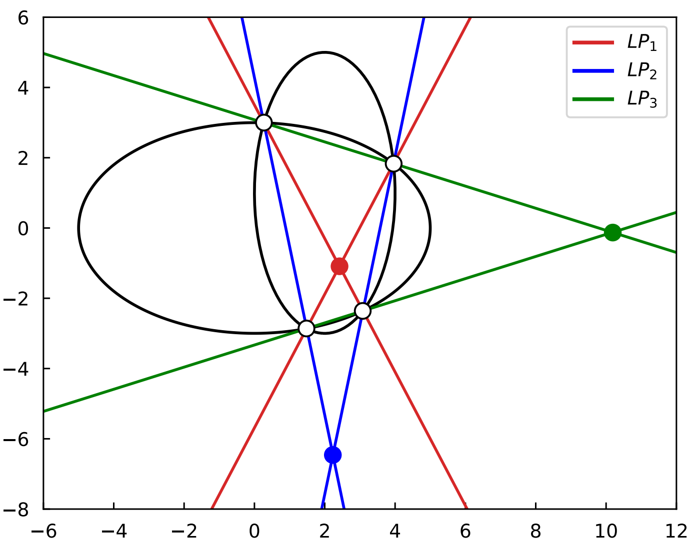

Figure 2: Four-point obtained as an intersection of two conics from Example 1. Each of the three line-pairs was constructed by wedging the four-point and the corresponding point of intersection.

Unfortunately, subsequent decomposition of a line-pair cannot be carried out yet by means of GAC. On the other hand, intersecting a line with a conic could be carried out by some kind of subalgebra of GAC, in particular, CRA, [4], or PGA, [5]. Moreover, if decomposition of a line-pair using GAC was possible, then it could also be advantageous—instead of intersecting the lines with the conic—to directly intersect the lines of one line-pair with the lines of another line-pair. Let us note that the decomposition algorithm can be applied by means of standard linear algebra [14, 2], but a fully GAC operational procedure similar to the point pair decomposition in CGA is desirable. The topic will be the subject of further research.

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