A Novel Line Alignment Algorithm using Geometric Algebra

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Abstract. This work explores the use of Geometric Algebra in aligning sets of $3D$ lines. We work in spherical space where bivectors represent the lines and rotors represent the rotations and translations. The process takes its inspiration from the ICP point matching algorithm and relies on the formation of cross-correlation and auto-correlation matrices. We investigate the performance of the algorithm in the presence of noise and compare with rotor extraction from points.

Keywords: Geometric Algebra; Line Registration; Spatial Analysis; Spatial Transformations; Photogrammetry; Computer Vision

1 Introduction

The fields of Photogrammetry and Computer Vision (CV) have traditionally based their models and algorithms on 'points'. Points provide mathematical simplicity which leads to efficient algorithms. However, an object's shape is often defined much more effectively via lines and surfaces. According to Bookstein [1], [2], the traditional way of describing a surface with triads of coordinates for many points (which are largely chosen at random) is not the most suitable way to accurately capture or convey the geometry of the surface. On the contrary, he considers the use of lines, planes, circles, spheres, etc. more convenient for this purpose, as they offer a better approach for both the rendering and understanding of geometric shapes. Similarly, Torlegârd [3] notes that in environments where the geometry of structures (like buildings and bridges) can typically be represented with combinations of straight lines, the use of algorithms that rely on lines rather than points can streamline the rendering process.

It is worth noting that in the early 80s there was a lot of criticism of the reliance on landmarks for orienting images, given that finding suitable control points can be quite challenging. Characteristically, Masry in his pioneering work on line photogrammetry [4] questioned why researchers in photogrammetry continue to ignore the wealth of information offered by linear elements (roads, railway lines, large buildings, etc.) in aerial photographs and satellite images. He also highlighted potential problems in small-scale images (such as those from satellites) where typically intersection points of linear features (e.g. roads) are sought to correlate the image with the geographic space:

- A sufficient number of linear features is necessary, but the number of intersection points may be inadequate for practical applications.
- Linear features should not intersect within the image boundaries.
- The distribution of intersection points can lead to ill-conditioned solutions.

Computer vision researchers also recognised the advantage of using features such as straight lines as carriers of $2D$ and $3D$ information. Huang and Netravali [5] have provided a valuable and general overview of such relevant research during the 90s. In cases where automation plays a primary role, the use of lines as observables instead of points shows significant advantages, as pointed out from the mid 90s by Heikkilä [6], Mikhail and Weerawong [7], Weng et al. [8], [9] and also earlier in the 70s by Hottier [10]. Key benefits identified in the literature include:

1. Linear features are detected more easily than points through image processing algorithms, such as edge and boundary detection. This means that their position and orientation can be estimated with subpixel accuracy (which is finer than the image pixel itself) by fitting a line equation to the redundant boundary pixel observations. Finally, the inclusion of depth information in $3D$ space enhances the precision of these estimations and allows for more accurate modeling and analysis of objects.

2. For homologous lines[3] across overlapping images or between an image and space, defining homologous points is unnecessary. Theoretically, it is not necessary to depict or correlate the same line segment in both images or between image and space. This principle extends to $3D$ space where the geometric properties of lines can be used to match and reconstruct features across multiple viewpoints without the need for explicit point matching and thus simplifying the process of $3D$ modeling and reconstruction.

3. In applications related to the built environment and human constructions (e.g. architectural applications, urban area map updates, industrial applications, AI applications, autonomous navigation), the linear features and especially the straight lines, outnumber points. Specifically, in $3D$ modeling lines are crucial because they help represent the edges, corners, and key parts of buildings and infrastructure.

4. Line elements have greater descriptive power than points. This basically means that objects' shapes are more effectively described through relationships between lines rather than points. Straight lines use geometric constraints (like parallelism, perpendicularity, coplanarity) to strengthen the

[3] Homologous lines refer to lines that correspond to each other in different images or between an image and a physical space. They represent the same physical feature or edge from different perspectives or viewpoints.

final solution. In $3D$ space, these constraints are critical in defining the spatial relationships and orientations of objects.

Geometric Algebra (GA) provides a robust mathematical framework for addressing complex geometric transformations and spatial relationships in a coherent and intuitive manner. We propose a GA-based algorithm to align sets of lines in $3D$ space. This algorithm exploits GA's ability to handle rotations and translations by encapsulating the duality of these movements through the use of motors and offers a framework for precise and efficient computations.

2 Methodology

We will work in $4D$ spherical space [11] – which we will also call '$1D$Up' space – this space has three spatial vectors $\{e_i\}$, $i = 1, 2, 3$, which all square to $+1$, and a 4th, e_4, which also squares to $+1$. Similar algorithms can be constructed in PGA and CGA. To construct our lines we will use pairs of points in $3D$ Euclidean space, transfer these to spherical space (with a given curvature factor λ) and form lines in the $1D$Up space (by wedging the spherical points). For a point p in $3D$ Euclidean space, its representation in the $4D$ Spherical space is given by:

$$P = \left(\frac{2\lambda}{\lambda^2 + p^2}\right) p + \left(\frac{\lambda^2 - p^2}{\lambda^2 + p^2}\right) e_4, \quad P^2 = 1 \tag{1}$$

Then, a spherical line $L = qe_{12} + re_{13} + se_{14} + te_{23} + ue_{24} + v_{34}$ consisting of 6 bivectors, where q, r, s, t, u, v are scalar coefficients can be formed by wedging two points in the $4D$ Spherical space P_1 and P_2. Moreover, L can be transformed (rotation + translation) using a motor M as $L^{'} = ML\tilde{M}$ with $L^{'} = q^{'}e_{12} + r^{'}e_13 + s^{'}e_14 + t^{'}e_23 + u^{'}e_24 + v^{'}e_34$, where $q^{'}, r^{'}, s^{'}, t^{'}, u^{'}, v^{'}$ are also scalar coefficients.

2.1 Recovering the Transformation

Suppose we have a set of 6 lines $L_i, i = 1, \dots, 6$, which are rotated and translated via a rotor R to another set of 6 lines, $L_i', i = 1, \dots, 6$. so that:

$$L_i^{'} = RL_i\tilde{R} \tag{2}$$

Now, consider reciprocal sets of line $\{L^i\}$ and $\{L^{i'}\}$, such that:

$$L_i \cdot L^j = \delta_{ij}, \quad L_i' \cdot L^{j'} = \delta_{ij} \tag{3}$$

It is then also the case that $L^{i'} = RL^i\tilde{R}$. The rotor R is made up of scalar, bivector and quadvector (pseudoscalar, $I = e_1e_2e_3e_4$) parts which we can write as $R = \alpha + B + \beta I$. It then follows that (sum over repeated indices):

$$L_iRL^i = 6\alpha + 6\beta I - 2B = -2R + 8(\alpha + \beta I) \tag{4}$$

Hence:

$$L_i R L^i \tilde{R} = L_i L^{i'} = -2 + 8(\alpha + \beta I)\tilde{R} \tag{5}$$

Note that we can form $L_i L^{i'}$ a priori. This tells us that we can form the quantity X from our observations as:

$$X = \frac{1}{8}(L_i L^{i'} + 2) \tag{6}$$

and set it equal to Y:

$$Y = (\alpha + \beta I)\tilde{R} \tag{7}$$

We can thus recover R if we can find α and β. α and β can be found by forming $X\tilde{X}$ and noting that:

$$X\tilde{X} = \alpha^2 + \beta^2 + 2\alpha\beta I \tag{8}$$

The details of forming α and β will be given in the full paper.

2.2 Forming the Reciprocal Lines

The bivectors representing both L and $L^{'}$ can be mapped onto vectors in a $6D$ Euclidean space to allow algebraic manipulation and analysis. This is achieved by simply treating the scalar coefficients of the bivectors as components of vector 6D vectors.

If we denote the $6D$ Euclidean basis as $\{f_1, f_2, f_3, f_4, f_5, f_6\}$. The mapping of the original line (bivectors) to vectors is given by:

$$qe_{12} + re_{13} + se_{14} + te_{23} + ue_{24} + v_{34} \longrightarrow qf_1 + rf_2 + sf_3 + fe_4 + uf_5 + vf_6 \tag{9}$$

and similarly for the transformed lines. Call these vector representations v_i and ${v_i}'$ – we then form the reciprocal vectors v^i and $v^{i'}$ via the usual formulae. v^i and $v^{i'}$ are then mapped back into bivectors in the $1D$Up space using the inverse mapping, to give the reciprocal lines.

2.3 Extending to n Lines

Now we move to the case where we have n lines $\{L_i\}$ and n transformed lines $\{{L_i}'\}$, $i = 1, ..., n$. Taking inspiration from [12], we form a cross-correlation matrix F and an autocorrelation matrix G, both of which will be 6×6, as follows:

$$F_{ij} = \sum_{k=1}^{N} (f_i \cdot v_k) \left(f_j \cdot v_k^{'}\right), \quad G_{ij} = \sum_{k=1}^{N} \left(f_i \cdot v_k^{'}\right) \left(f_j \cdot v_k^{'}\right) \tag{10}$$

where, as above, the v_i and ${v_i}'$ are the vector mappings of the lines. We now treat the columns of F as a set of vectors mapped from a set of 6 lines, and the columns of G as a set of vectors mapped from a set of 6 transformed lines. We then apply the process described in subsection 2.1 to recover the rotor R.

3 Preliminary Results

The analysis so far has been for the no-noise case and assumes we know the correspondences between lines. In the final paper we will look at some preliminary experiments to see how estimating the transformation with lines compares to estimating the translation with points, in the case of added noise. Care needs to be taken to ensure that we choose a method of putting *noise on lines* that is somehow equivalent to putting *noise on points* (which is much more well defined). Preliminary results indicate that that line method works well but more work is required to give detailed comparisons.

4 Conclusion

This research work aims to address the $3D$ line registration problem in GA a coherent and systematic manner. Lines are significant for both theoretical research and practical applications, as highlighted in our bibliography. Due to their clear structures, lines might be better suited for automation and real-time tasks compared to points and could offer reliable matching across images and accurate registration amidst noise or viewpoint shifts. We aim to thoroughly investigate this *Spherical Lines* method, expecting it that it will provide new insights in the field of line registration.

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