ON MULTIDIMENSIONAL DIRAC–HESTENES EQUATION

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Summary of the Abstract

The four-dimensional Dirac–Hestenes equation is equivalent to the four-dimensional Dirac equation. One of the advantages to investigate the Dirac–Hestenes equation is that solutions of this equation are real. We present the multidimensional Dirac–Hestenes equation in the geometric algebra formalism. Since the matrix representation of the complexified geometric algebra $\mathbb{C}\otimes \mathrm{Cl}_{1,n}$ depends on a parity of n, we explore even and odd cases separately. We present a lemma about the unique decomposition of an element of the left ideal into the product of the idempotent and an element of the auxiliary real subalgebra of the geometric algebra. We use this subalgebra and properties of the idempotent to present the multidimensional Dirac–Hestenes equation. We present that we might obtain a solution of the multidimensional Dirac–Hestenes equation using a solution of the multidimensional Dirac equation and conversely in the case $n = 2d-1$. We also get that the multidimensional Dirac–Hestenes equation has gauge invariance.

1 Geometric algebra formalism

One of the ways to explore problems in modern mathematical physics is to use the geometric algebra [1, 2, 3]. We consider the real geometric algebra $\mathrm{Cl}_{1,n}$. The generators $e^0, e^1, \dots, e^n$ satisfy the following anticommutation relations:

$$e^\mu e^\nu + e^\nu e^\mu = 2\eta^{\mu\nu} e, \quad \mu,\nu \in \{0,1,\dots,n\},$$

where e is the identity element and η is the diagonal matrix in which the first element is 1 and the remaining elements lying on the main diagonal are -1.
The basis of the considered geometric algebra $\mathrm{Cl}_{1,n}$ consists of all possible ordered products of the generators:

$$e^{\mu_1}e^{\mu_2}\cdots e^{\mu_k} = e^{\mu_1\mu_2\dots\mu_k}, \quad 0 \le \mu_1 < \mu_2 < \dots < \mu_k \le n.$$

Hence, the basis decomposition of the element $U \in \mathrm{Cl}_{1,n}$ is:

$$U = \sum_A u_A e^A, \quad u_A \in \mathbb{R}, \tag{1}$$

where A is an ordered multi-index of length from 0 to $n+1$, $A = \mu_1\mu_2 \dots \mu_k$. We denote the length of a multi-index A by $|A| = k$. If a multi-index has even length, it would be called an even multi-index otherwise an odd multi-index. Note, the dimension of $\mathrm{Cl}_{1,n}$ is 2^{n+1}.

Let $\mathrm{Cl}^{(0)}_{1,n}$ be an even subspace of $\mathrm{Cl}_{1,n}$ that is a linear span of the basis elements with even multi-indices:

$$\mathrm{Cl}^{(0)}_{1,n} = \{U \in \mathrm{Cl}_{1,n} \,|U = \sum_{|A|=2k} u_A e^A\}, \quad \dim \mathrm{Cl}^{(0)}_{1,n} = 2^n.$$

We also consider the complexified geometric algebra $\mathbb{C} \otimes \mathrm{Cl}_{1,n}$. The basis decomposition of $U \in \mathbb{C} \otimes \mathrm{Cl}_{1,n}$ is similar to decomposition (1) but the constants u_A are complex scalars in this case.

Let us introduce the operation of Hermitian conjugation acting on $U \in \mathbb{C} \otimes \mathrm{Cl}_{1,n}$ [4]:

$$U^\dagger = e^0 U^* e^0, \quad U \in \mathbb{C} \otimes \mathrm{Cl}_{1,n},$$

where the star denotes the superposition of the reversion and the complex conjugation:

$$U^* = \sum_A (-1)^{\frac{|A|(|A|-1)}{2}} \bar{u}_A e^A, \quad u_A \in \mathbb{C}.$$

We consider the Hermitian idempotent t

$$t^2 = t, \quad t^\dagger = t,$$

and the corresponding left ideal $L(t)$ generated by t

$$L(t) = \{U \in \mathbb{C} \otimes \mathrm{Cl}_{1,n} \,|Ut = U\}.$$

If a left ideal $L(t)$ does not contain another left ideal except itself and $L(0)$, it is called a minimal left ideal. The corresponding idempotent t is called the primitive idempotent.

We consider the Dirac equation in the geometric algebra formalism. It is convenient to investigate n-dimension Dirac spinor as an element of the left ideal $L(t)$ [5, 6]:

$$\varphi(x) : \mathbb{R}^{1,n} \to L(t).$$

Actually, there is a difference between cases $n = 2d - 1$ and $n = 2m$. If we consider the first case, then a spinor should belong to the minimal left ideal. In another case, a spinor could belong not only to the minimal left ideal. We discuss this fact below.

We denote the mass of a particle by m. For convenience, we assume that Planck constant, the charge of a particle, and the speed of light are equal to 1. The electromagnetic vector-potential $\mathbf{a}(x)$ depends on a point x of the pseudo–Euclidean space $\mathbb{R}^{1,n}$. That is, $\mathbf{a}(x) = (a_0(x), \dots, a_n(x)) : \mathbb{R}^{1,n} \to \mathbb{R}^{n+1}$. The multidimensional Dirac equation in the geometric algebra formalism is:

$$\sum_{\mu=0}^{n} i e^\mu (\partial_\mu + i a_\mu(x)) \varphi(x) = m\varphi(x), \tag{2}$$

where $\partial_\mu = \partial/\partial x^\mu$ and i is the imaginary unit.
It is known that the classical four-dimension Dirac equation is equivalent to the Dirac–Hestenes equation [7]. It means that we can obtain a solution of the Dirac–Hestenes equation using a solution of the Dirac equation and conversely. The Dirac–Hestenes equation gives a deeper understanding of a geometry in different tasks since the considered wave function turns out to be completely real. Further, we present the multidimensional Dirac–Hestenes equation in the case $n = 2d - 1$. Also, it is a fact that the multidimensional Dirac equation has gauge invariance. We get the same fact for the multidimensional Dirac–Hestenes equation.

2 Decomposition of an element of the left ideal

First, we remind several facts for the special case $n = 3$. The following decomposition of an element of the minimal left ideal $L(t)$ is well-known [6]:

Lemma 1 *Let $L(t)$ be the minimal left ideal, generated by the idempotent t:*

$$t = \frac{1}{4}(e + e^0)(e + ie^{12}) \in \mathbb{C} \otimes \mathrm{Cl}_{1,3}\,.$$

Then there is a unique decomposition:

$$\forall \varphi(x) \in L(t)\ \exists! \Psi(x) \in \mathrm{Cl}^{(0)}_{1,3} :\ \varphi(x) = \Psi(x)t.$$

Using Lemma 1, it is possible to obtain the solution $\Psi(x)$ of the Dirac–Hestenes equation via the solution $\psi(x)$ of the Dirac equation and conversely [2, 7]. The Dirac–Hestenes equation has the form:

$$\sum_{\mu=0}^{3} e^\mu(\partial_\mu \Psi(x) + \Psi(x) a_\mu(x) e^{12}) e^0 = m\Psi(x)e^{12}, \quad \Psi(x) \in \mathrm{Cl}^{(0)}_{1,3}\,.$$

In this section, we introduce Lemma 3 that is a generalization of Lemma 1 for the multidimensional case. Since the matrix representation of the complexified geometric algebra differs for the cases $n = 2d - 1$ and $n = 2d$, we investigate them separately.
The real dimensions of the minimal left ideal $L(t)$ and the even real subalgebra $\mathrm{Cl}^{(0)}_{1,3}$ are the same and equal to 8. However, the equality of the dimensions does not hold for $n > 4$ since:

$$\dim L(t) = 2^{[\frac{n+2}{2}]+1}, \quad \dim \mathrm{Cl}^{(0)}_{1,n} = 2^n,$$

$$\left[\frac{n+2}{2}\right] + 1 < n, \quad \forall n > 4.$$

It follows for $n > 4$ that there cannot be an element of the even real subalgebra in the similar statement to Lemma 1. Therefore, it is necessary to introduce another real subalgebra to which $\Psi(x)$ belongs. The subalgebra must have smaller dimension than the even real subalgebra $\mathrm{Cl}^{(0)}_{1,n}$.

2.1 Case $n = 2d - 1$

One of the ways to fix the primitive Hermitian idempotent $t \in \mathbb{C} \otimes \mathrm{Cl}_{1,2d-1}$ is [4]:

$$t = \frac{1}{2}(e + e^0) \prod_{\mu=1}^{d-1} \frac{1}{2}(e + ie^{2\mu-1}e^{2\mu}). \tag{3}$$

In this subsection, we consider the minimal left ideal $L(t)$ generated by (3).
One of the main advantages of using geometric algebra is that it is easier to interpret geometrically results when considering the Dirac equation. We use the variable I for reducing the Dirac equation to the form where all values are real:

$$I = -e^{12}, \quad it = It = tI. \tag{4}$$

To introduce the Dirac–Hestenes equation where the wave function $\Psi(x)$ should consist only basis elements with even indices, we also use another variable E:

$$E = e^0, \quad t = Et = tE. \tag{5}$$

For an explicit description of the minimal left ideal $L(t)$, it is convenient to consider an additional algebra Q which is generated by the generators with odd indices:

$$Q = \mathrm{Cl}(e^1, e^3, e^5, e^7, \ldots, e^{2d-1}) \subset \mathbb{C} \otimes \mathrm{Cl}_{1,2d-1}\,.$$

Explicit form of this basis will be presented in a talk. Since the Dirac–Hestenes equation contains variables E and I, we have to consider the following real subalgebra Q':

$$Q' = \mathrm{Cl}(e^0, e^1, e^2, e^3, e^5, e^7, \ldots, e^{2d-1}) \subset \mathrm{Cl}_{1,2d-1}\,. \tag{6}$$

It means that its generators are e^0, e^2, and the generators with odd indices. Note that for $n = 3$ we get the algebra $Q' = \mathrm{Cl}_{1,3}$.
In Lemma 3 for the multidimensional case, which is generalization of Lemma 1, the even subalgebra $Q'^{(0)}$ is used instead of the even subalgebra $\mathrm{Cl}^{(0)}_{1,3}$. Lemma 2 is used to prove the uniqueness of the decomposition in Lemma 3 and to construct multidimensional Dirac–Hestenes equation.

Lemma 2 *Let Q' be $\mathrm{Cl}(e^0, e^1, e^2, e^3, e^5, e^7, \ldots, e^{2d-1})$ and t have form (3). If $Y \in Q'^{(0)}$ and $Yt = 0$, then $Y = 0$.*

We could not change $Q'^{(0)}$ into $\mathrm{Cl}^{(0)}_{1,2d-1}$ in Lemma 2. Let us present an example for the case $d = 3$. If $Y = e^{12} - e^{34} \in \mathrm{Cl}^{(0)}_{1,5}$, then we get $Yt = -it + it = 0$.

Lemma 3 *Let Q' be $\mathrm{Cl}(e^0, e^1, e^2, e^3, e^5, e^7, \ldots, e^{2d-1})$ and $L(t)$ be the minimal left ideal generated by idempotent t (3). Then:*

$$\forall \varphi \in L(t) \ \ \exists! \Psi \in Q'^{(0)} : \varphi = \Psi t.$$

It follows from the uniqueness of the decomposition in Lemma 3 that the Dirac equation and the Dirac–Hestenes equation are equivalent.

2.2 Case $n = 2d$

There are two types of a Dirac spinor for the considering case ($\mathbb{C}\otimes \mathrm{Cl}_{1,2d}$): a semi-spinor and a double spinor [5]. The semi-spinor belongs to the minimal left ideal. The double spinor belongs to the direct sum of two minimal left ideals.
There is an isomorphism between complexified geometric algebras and complex matrix algebras [4, 6]:

$$\mathbb{C}\otimes \mathrm{Cl}_{1,2d} \simeq \mathrm{Mat}(2^d, \mathbb{C}) \oplus \mathrm{Mat}(2^d, \mathbb{C}).$$

We construct a primitive Hermitian idempotent t and a real algebra Q' via the matrix representation of $\mathbb{C}\otimes \mathrm{Cl}_{1,2d-1}$ and $\mathbb{C}\otimes \mathrm{Cl}_{1,2d+1}$. If φ is a semi-spinor, then we can fix t as following:

$$t = \frac{1}{2}(e+e^0)\prod_{\mu=1}^{d}\frac{1}{2}(e+ie^{2\mu-1}e^{2\mu}) \in \mathbb{C}\otimes \mathrm{Cl}_{1,2d}\,.$$

It is convenient to use algebra Q' (6) in this case. If φ is a double spinor, then we can fix Hermitian idempotent t (3), which is not primitive in this case, and the algebra Q' as follows:

$$Q' = \mathrm{Cl}(e^0, e^1, e^2, e^3, e^5, \ldots, e^{2d-1}, e^{2d}).$$

Note that properties (4) and (5) are valid for all considered cases. Lemma 3 might be transferred to the case $n = 2d$ if we use the Hermitian idempotents t and the real algebras Q' constructed in this subsection.

3 Multidimensional Dirac–Hestenes equation

We have described the way to construct Hermitian idempotents and real algebras, which solutions of the multidimensional Dirac–Hestenes equation belong, in the previous section. Let us present several theorems about the multidimensional Dirac–Hestenes equation in the case $n = 2d-1$.

Theorem 4 *Let fix the primitive Hermitian idempotent t as in formula (3) and the real algebra Q' be:*

$$Q' = \mathrm{Cl}(e^0, e^1, e^2, e^3, e^5, e^7, \ldots, e^{2d-1}) \subset \mathrm{Cl}_{1,2d-1}\,.$$

If $\varphi(x) \in L(t)$ is a solution of multidimensional Dirac equation (2), then $\Psi(x) \in Q'^{(0)}$: $\varphi(x) = \Psi(x)t$ is a solution of the multidimensional Dirac–Hestenes equation:

$$\sum_{\mu=0,1,2,3,5,7,\ldots,2d-1} e^{\mu}(\partial_\mu\Psi(x) + \Psi(x)a_\mu(x)I)E + \\ + \sum_{\mu=3,5,\ldots,2d-3}(\partial_{\mu+1}\Psi(x)I - \Psi(x)a_{\mu+1}(x))e^{\mu}E + m\Psi(x)I = 0, \qquad (7)$$

where $I = -e^{12}$ and $E = e^0$.
If $\Psi(x) \in Q'^{(0)}$ is a solution of equation (7), then $\varphi(x) \in L(t)$: $\varphi(x) = \Psi(x)t$ is a solution of equation (2).

It is known that the four-dimensional Dirac–Hestenes equation has gauge invariance [7, 8]. Electric and magnetic fields are invariant under transformation of an electromagnetic potential $a'_\mu(x) = a_\mu(x) - \partial_\mu\lambda(x)$ where $\lambda(x) : \mathbb{R}^{1,2d-1} \to \mathbb{R}$. In this case, the spinor $\Psi'(x) = \exp(-e^{12}\lambda(x))\Psi(x)$ is a solution of the Dirac–Hestenes equation with the shifted potential a'_μ. We get that the multidimensional Dirac–Hestenes equation also has gauge invariance.

Theorem 5 *Let $\Psi(x)$ be a solution of multidimensional Dirac–Hestenes equation (7). Then $\Psi(x)$ has gauge invariance:*

$$\Psi'(x) = \Psi(x)e^{I\lambda(x)}, \qquad a'_\mu(x) = a_\mu(x) - \partial_\mu\lambda(x),$$

where $\lambda(x) : \mathbb{R}^{1,2d-1} \to \mathbb{R}$.

It is convenient to consider n-dimensional Dirac spinors, which are used in supersymmetry theory, in a real geometric algebra. Actually, even and odd cases are important for applications. For instance, the three-dimensional Dirac equation is used to explore the properties of graphene. We have presented the multidimensional Dirac–Hestenes equation for $n = 2d - 1$, where solutions belong to the real subalgebra. The corresponding theorems for the case $n = 2d$ will be also presented in a talk. The cases of a semi-spinor and a double spinor will be considered in details. In addition, an example of a solution of the Dirac equation and a corresponding solution of Dirac–Hestenes equation will be presented in a future article.

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