ON RANK OF MULTIVECTORS IN GEOMETRIC ALGEBRA

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Summary of the Abstract

We introduce the notion of rank of multivector in Clifford geometric algebras of arbitrary dimension without using the corresponding matrix representations and using only geometric algebra operations. We use the concepts of characteristic polynomial in geometric algebras and the method of SVD. The results can be used in various applications of geometric algebras in computer science, engineering, and physics.

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1 Real and Complexified GA

Let us consider the real Clifford geometric algebra (GA) $\mathcal{G}_{p,q}$ [4, 5, 3, 11] with the identity element $e \equiv 1$ and the generators e_a, $a = 1, 2, \ldots, n$, where $n = p + q \geq 1$. The generators satisfy the conditions

$$e_a e_b + e_b e_a = 2\eta_{ab} e, \qquad \eta = (\eta_{ab}) = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}).$$

Consider the subspaces $\mathcal{G}^k_{p,q}$ of grades $k = 0, 1, \ldots, n$, which elements are linear combinations of the basis elements $e_A = e_{a_1 a_2 \ldots a_k} = e_{a_1} e_{a_2} \cdots e_{a_k}$, $1 \leq a_1 < a_2 < \cdots < a_k \leq n$, with ordered multi-indices of length k. An arbitrary element (multivector) $M \in \mathcal{G}_{p,q}$ has the form

$$M = \sum_A m_A e_A \in \mathcal{G}_{p,q}, \qquad m_A \in \mathbb{R},$$

where we have a sum over arbitrary multi-index A of length from 0 to n. The projection of M onto the subspace $\mathcal{G}^k_{p,q}$ is denoted by $\langle M \rangle_k$.

The grade involution and reversion of a multivector $M \in \mathcal{G}_{p,q}$ are denoted by

$$\widehat{M} = \sum_{k=0}^{n} (-1)^k \langle M \rangle_k, \qquad \widetilde{M} = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \langle M \rangle_k. \tag{1}$$

We have

$$\widehat{M_1 M_2} = \widehat{M_1}\widehat{M_2}, \qquad \widetilde{M_1 M_2} = \widetilde{M_2}\widetilde{M_1}, \qquad \forall M_1, M_2 \in \mathcal{G}_{p,q}. \tag{2}$$

Let us consider the complexified Clifford geometric algebra $\mathcal{G}^{\mathbb{C}}_{p,q} := \mathbb{C} \otimes \mathcal{G}_{p,q}$ [11]. An arbitrary element of $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$ has the form

$$M = \sum_A m_A e_A \in \mathcal{G}^{\mathbb{C}}_{p,q}, \qquad m_A \in \mathbb{C}.$$

Note that $\mathcal{G}^{\mathbb{C}}_{p,q}$ has the following basis of 2^{n+1} elements:

$$e, ie, e_1, ie_1, e_2, ie_2, \ldots, e_{1\ldots n}, ie_{1\ldots n}. \tag{3}$$

In addition to the grade involution and reversion, we use the operation of complex conjugation, which takes complex conjugation only from the coordinates m_A and does not change the basis elements e_A:

$$\overline{M} = \sum_A \overline{m}_A e_A \in \mathcal{G}^{\mathbb{C}}_{p,q}, \qquad m_A \in \mathbb{C}, \qquad M \in \mathcal{G}^{\mathbb{C}}_{p,q}.$$

We have

$$\overline{M_1 M_2} = \overline{M_1}\ \overline{M_2}, \qquad \forall M_1, M_2 \in \mathcal{G}^{\mathbb{C}}_{p,q}.$$

2 Unitary groups in GA

Let us consider an operation of Hermitian conjugation $\dagger$ in $\mathcal{G}^{\mathbb{C}}_{p,q}$ (see [6, 11]):

$$M^\dagger := M|_{e_A \to (e_A)^{-1},\ m_A \to \overline{m}_A} = \sum_A \overline{m}_A (e_A)^{-1}. \tag{4}$$

We have the following two equivalent definitions of this operation:

$$M^\dagger = \begin{cases} e_{1\ldots p}\widetilde{\overline{M}}e^{-1}_{1\ldots p}, & \text{if } p \text{ is odd},\\ e_{1\ldots p}\widetilde{\widehat{\overline{M}}}e^{-1}_{1\ldots p}, & \text{if } p \text{ is even}, \end{cases} \tag{5}$$

$$M^\dagger = \begin{cases} e_{p+1\ldots n}\widetilde{\overline{M}}e^{-1}_{p+1\ldots n}, & \text{if } q \text{ is even},\\ e_{p+1\ldots n}\widetilde{\widehat{\overline{M}}}e^{-1}_{p+1\ldots n}, & \text{if } q \text{ is odd}. \end{cases} \tag{6}$$

The operation[1]

$$(M_1, M_2) := \langle M_1^\dagger M_2 \rangle_0$$

is a (positive definite) scalar product with the properties

$$(M_1, M_2) = \overline{(M_2, M_1)}, \tag{7}$$
$$(M_1 + M_2, M_3) = (M_1, M_3) + (M_2, M_3), \quad (M_1, \lambda M_2) = \lambda (M_1, M_2), \tag{8}$$
$$(M, M) \geq 0, \quad (M, M) = 0 \Leftrightarrow M = 0. \tag{9}$$

Using this scalar product we introduce inner product space over the field of complex numbers (unitary space) in $\mathcal{G}^{\mathbb{C}}_{p,q}$.

We have a norm

$$||M|| := \sqrt{(M, M)} = \sqrt{\langle M^\dagger M \rangle_0}. \tag{10}$$

Let us consider the following faithful representation (isomorphism) of the complexified geometric algebra

$$\beta : \mathcal{G}^{\mathbb{C}}_{p,q} \quad \to \quad \begin{cases} \mathrm{Mat}(2^{\frac{n}{2}}, \mathbb{C}), & \text{if } n \text{ is even},\\ \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C}) \oplus \mathrm{Mat}(2^{\frac{n-1}{2}}, \mathbb{C}), & \text{if } n \text{ is odd}. \end{cases} \tag{11}$$

Let us denote the size of the corresponding matrices by

$$N := 2^{[\frac{n+1}{2}]},$$

where square brackets mean taking the integer part.

Let us present an explicit form of one of these representations of $\mathcal{G}^{\mathbb{C}}_{p,q}$ (we use it also for $\mathcal{G}_{p,q}$ in [7] and for $\mathcal{G}^{\mathbb{C}}_{p,q}$ in [12]). We denote this fixed representation by β'. Let us consider the case $p = n$, $q = 0$. To obtain the matrix representation for another signature with $q \neq 0$, we should multiply matrices $\beta'(e_a)$, $a = p+1, \ldots, n$ by imaginary unit i. For the identity element, we always use the identity matrix $\beta'(e) = I_N$ of the corresponding dimension N. We always take $\beta'(e_{a_1 a_2 \ldots a_k}) = \beta'(e_{a_1})\beta'(e_{a_2}) \cdots \beta'(e_{a_k})$. In the case $n = 1$,

[1] Compare with the well-known operation $M_1 * M_2 := \langle \widetilde{M_1} M_2 \rangle_0$ in the real geometric algebra $\mathcal{G}_{p,q}$, which is positive definite only in the case of signature $(p, q) = (n, 0)$.

we take $\beta'(e_1) = \mathrm{diag}(1, -1)$. Suppose we know $\beta'_a := \beta'(e_a)$, $a = 1, \ldots, n$ for some fixed odd $n = 2k + 1$. Then for $n = 2k + 2$, we take the same $\beta'(e_a)$, $a = 1, \ldots, 2k + 1$, and

$$\beta'(e_{2k+2}) = \begin{pmatrix} 0 & I_{\frac{N}{2}} \\ I_{\frac{N}{2}} & 0 \end{pmatrix}.$$

For $n = 2k + 3$, we take

$$\beta'(e_a) = \begin{pmatrix} \beta'_a & 0 \\ 0 & -\beta'_a \end{pmatrix}, \qquad a = 1, \ldots, 2k + 2,$$

and

$$\beta'(e_{2k+3}) = \begin{pmatrix} i^{k+1}\beta'_1 \cdots \beta'_{2k+2} & 0 \\ 0 & -i^{k+1}\beta'_1 \cdots \beta'_{2k+2} \end{pmatrix}.$$

This recursive method gives us an explicit form of the matrix representation β' for all n.

Note that for this matrix representation we have

$$(\beta'(e_a))^\dagger = \eta_{aa}\beta'(e_a), \qquad a = 1, \ldots, n,$$

where $\dagger$ is the Hermitian transpose of a matrix. Using the linearity, we get that Hermitian conjugation of matrix is consistent with Hermitian conjugation of corresponding multivector:

$$\beta'(M^\dagger) = (\beta'(M))^\dagger, \qquad M \in \mathcal{G}^{\mathbb{C}}_{p,q}. \tag{12}$$

Note that the same is not true for an arbitrary matrix representations β of the form (11). It is true the matrix representations $\gamma = T^{-1}\beta' T$ obtained from β' using the matrix T such that $T^\dagger T = I$.

Let us consider the group

$$\mathrm{U}\mathcal{G}^{\mathbb{C}}_{p,q} = \{M \in \mathcal{G}^{\mathbb{C}}_{p,q} : M^\dagger M = e\}, \tag{13}$$

which we call a unitary group in $\mathcal{G}^{\mathbb{C}}_{p,q}$. Note that all the basis elements e_A of $\mathcal{G}_{p,q}$ belong to this group by the definition.

Using (11) and (12), we get the following isomorphisms to the classical matrix unitary groups:

$$\mathrm{U}\mathcal{G}^{\mathbb{C}}_{p,q} \simeq \begin{cases} \mathrm{U}(2^{\frac{n}{2}}), & \text{if } n \text{ is even}, \\ \mathrm{U}(2^{\frac{n-1}{2}}) \times \mathrm{U}(2^{\frac{n-1}{2}}), & \text{if } n \text{ is odd}, \end{cases} \tag{14}$$

where

$$\mathrm{U}(k) = \{A \in \mathrm{Mat}(k, \mathbb{C}), \quad A^\dagger A = I\}. \tag{15}$$

3 SVD in GA

We have the following well-known theorem on singular value decomposition of an arbitrary complex matrix. For an arbitrary $A \in \mathbb{C}^{n\times m}$, there exist matrices $U \in \mathrm{U}(n)$ and $V \in \mathrm{U}(m)$ such that

$$A = U\Sigma V^\dagger, \tag{16}$$

where

$$\Sigma = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k), \qquad k = \min(n, m), \qquad \mathbb{R} \ni \lambda_1, \lambda_2, \ldots, \lambda_k \geq 0.$$

Note that choosing matrices $U \in \mathrm{U}(n)$ and $V \in \mathrm{U}(m)$, we can always arrange diagonal elements of the matrix Σ in decreasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$.

Diagonal elements of the matrix Σ are called singular values, they are square roots of eigenvalues of the matrices $AA^\dagger$ or $A^\dagger A$. Columns of the matrices U and V are eigenvectors of the matrices $AA^\dagger$ and $A^\dagger A$ respectively.

Theorem 1 (SVD in GA) *[10] For an arbitrary multivector $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$, there exist multivectors $U, V \in \mathrm{U}\mathcal{G}^{\mathbb{C}}_{p,q}$, where*

$$\mathrm{U}\mathcal{G}^{\mathbb{C}}_{p,q} = \{U \in \mathcal{G}^{\mathbb{C}}_{p,q} : U^{\dagger}U = e\}, \qquad U^{\dagger} := \sum_{A} \overline{u}_A (e_A)^{-1},$$

such that

$$M = U\Sigma V^{\dagger}, \tag{17}$$

where multivector Σ belongs to the subspace $K \in \mathcal{G}^{\mathbb{C}}_{p,q}$, which is a real span of a set of $N = 2^{[\frac{n+1}{2}]}$ fixed basis elements (3) of $\mathcal{G}^{\mathbb{C}}_{p,q}$ including the identity element e.

4 Determinant and other characteristic polynomial coefficients in GA

Let us consider the concept of determinant and characteristic polynomial [7] in geometric algebra. Explicit formulas for characteristic polynomial coefficients are discussed in [1], applications to Sylvester equation are discussed in [8], the relation with noncommutative Vieta theorem is discussed in [9], applications to calculation of elementary functions in geometric algebras are discussed in [2].

We can introduce the notion of determinant

$$\mathrm{Det}(M) := \det(\beta(M)) \in \mathbb{R}, \qquad M \in \mathcal{G}^{\mathbb{C}}_{p,q},$$

where β is (11), and the notion of characteristic polynomial

$$\begin{aligned}&\varphi_M(\lambda) := \mathrm{Det}(\lambda e - M) = \lambda^N - C_{(1)}\lambda^{N-1} - \cdots - C_{(N-1)}\lambda - C_{(N)} \in \mathcal{G}^0_{p,q} \equiv \mathbb{R},\\ &M \in \mathcal{G}^{\mathbb{C}}_{p,q}, \quad N = 2^{[\frac{n+1}{2}]}, \quad C_{(k)} = C_{(k)}(M) \in \mathcal{G}^0_{p,q} \equiv \mathbb{R}, \quad k = 1, \ldots, N.\end{aligned} \tag{18}$$

The following method based on the Faddeev–LeVerrier algorithm allows us to recursively obtain basis-free formulas for all the characteristic coefficients $C_{(k)}$, $k = 1, \ldots, N$ (18):

$$M_{(1)} := M, \qquad M_{(k+1)} = M(M_{(k)} - C_{(k)}), \tag{19}$$

$$C_{(k)} := \frac{N}{k}\langle M_{(k)}\rangle_0, \qquad k = 1, \ldots, N. \tag{20}$$

In this method, we obtain high coefficients from the lowest ones. The determinant is minus the last coefficient

$$\mathrm{Det}(M) = -C_{(N)} = -M_{(N)} = U(C_{(N-1)} - M_{(N-1)}) \tag{21}$$

and has the property

$$\mathrm{Det}(M_1 M_2) = \mathrm{Det}(M_1)\mathrm{Det}(M_2), \qquad M_1, M_2 \in \mathcal{G}^{\mathbb{C}}_{p,q}. \tag{22}$$

The inverse of a multivector $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$ can be computed as

$$M^{-1} = \frac{\mathrm{Adj}(M)}{\mathrm{Det}(M)} = \frac{C_{(N-1)} - M_{(N-1)}}{\mathrm{Det}(M)}, \qquad \mathrm{Det}(M) \neq 0. \tag{23}$$

5 Rank in GA

Let us introduce the notion of rank of a multivector $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$:

$$\mathrm{rank}(M) := \mathrm{rank}(\beta(M)) \in \{0, 1, \ldots, N\}, \tag{24}$$

where β is (11). Below we present another equivalent definition, which does not depend on the matrix representation. We use the fact that rank is the number of nonzero singular values in the SVD and Vieta formulas.

Lemma 1 *Suppose that a square matrix $A \in \mathbb{C}^{N \times N}$ is diagonalizable. Then*

$$
\begin{aligned}
&\operatorname{rank}(A) = N \quad \Leftrightarrow \quad C_{(N)} \neq 0; && (25)\\
&\operatorname{rank}(A) = k \in \{1, \ldots, N-1\} \Leftrightarrow C_{(k)} \neq 0,\ C_{(j)} = 0,\ j = k+1, \ldots, N; && (26)\\
&\operatorname{rank}(A) = 0 \quad \Leftrightarrow \quad A = 0. && (27)
\end{aligned}
$$

Lemma 2 *For an arbitrary multivector $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$, we have*

$$
\begin{aligned}
C_{(N)}(M^\dagger M) = 0 \quad &\Longleftrightarrow \quad C_{(N)}(M) = 0, && (28)\\
C_{(1)}(M^\dagger M) = 0 \quad &\Longleftrightarrow \quad M = 0. && (29)
\end{aligned}
$$

Theorem 2 (Rank in GA) *For an arbitrary $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$, we have*

$$
\operatorname{rank}(M) = \begin{cases}
N, & \text{if } C_{(N)}(M) \neq 0,\\
N-1, & \text{if } C_{(N)}(M) = 0 \text{ and } C_{(N-1)}(T) \neq 0,\\
N-2 & \text{if } C_{(N)}(M) = C_{(N-1)}(T) = 0 \text{ and } C_{(N-2)}(M^\dagger M) \neq 0,\\
\cdots & \\
2, & \text{if } C_{(N)}(M) = C_{(N-1)}(T) = \cdots = C_{(3)}(T) = 0 \text{ and } C_{(2)}(T) \neq 0,\\
1, & \text{if } C_{(N)}(M) = C_{(N-1)}(T) = \cdots = C_{(2)}(T) = 0 \text{ and } M \neq 0,\\
0, & \text{if } M = 0,
\end{cases} \tag{30}
$$

where $T := M^\dagger M$.

Example 1 *For an arbitrary $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$, $p+q=1$, we have*

$$
\operatorname{rank}(M) = \begin{cases}
2, & \text{if } M\widehat{M} \neq 0,\\
1, & \text{if } M\widehat{M} = 0 \text{ and } M \neq 0,\\
0, & \text{if } M = 0.
\end{cases} \tag{31}
$$

Example 2 *For an arbitrary $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$, $p+q=2$, we have*

$$
\operatorname{rank}(M) = \begin{cases}
2, & \text{if } M\widetilde{\widehat{M}} \neq 0,\\
1, & \text{if } M\widetilde{\widehat{M}} = 0 \text{ and } M \neq 0,\\
0, & \text{if } M = 0.
\end{cases} \tag{32}
$$

Example 3 *For an arbitrary $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$, $p+q=3$, we have*

$$
\operatorname{rank}(M) = \begin{cases}
4, & \text{if } M\widetilde{\widehat{M}}\widehat{M}\widetilde{M} \neq 0,\\
3, & \text{if } M\widetilde{\widehat{M}}\widehat{M}\widetilde{M} = 0 \text{ and } T\widetilde{\widehat{T}}\widehat{T} + T\widetilde{\widehat{T}}\widetilde{T} + T\widehat{T}\widetilde{T} + \widetilde{\widehat{T}}\widehat{T}\widetilde{T} \neq 0,\\
2, & \text{if } M\widetilde{\widehat{M}}\widehat{M}\widetilde{M} = T\widetilde{\widehat{T}}\widehat{T} + T\widetilde{\widehat{T}}\widetilde{T} + T\widehat{T}\widetilde{T} + \widetilde{\widehat{T}}\widehat{T}\widetilde{T} = 0 \text{ and}\\
 & T\widetilde{\widehat{T}} + T\widehat{T} + T\widetilde{T} + \widetilde{\widehat{T}}\widehat{T} + \widetilde{\widehat{T}}\widetilde{T} + \widehat{T}\widetilde{T} \neq 0,\\
1, & \text{if } M\widetilde{\widehat{M}}\widehat{M}\widetilde{M} = T\widetilde{\widehat{T}}\widehat{T} + T\widetilde{\widehat{T}}\widetilde{T} + T\widehat{T}\widetilde{T} + \widetilde{\widehat{T}}\widehat{T}\widetilde{T} =\\
 & = T\widetilde{\widehat{T}} + T\widehat{T} + T\widetilde{T} + \widetilde{\widehat{T}}\widehat{T} + \widetilde{\widehat{T}}\widetilde{T} + \widehat{T}\widetilde{T} = 0 \text{ and } M \neq 0,\\
0, & \text{if } M = 0,
\end{cases} \tag{33}
$$

where $T := M^\dagger M$.

Example 4 *Let us consider the $\triangle$-operation [7]*

$$
M^{\triangle} := \sum_{k=0}^{n} (-1)^{\frac{k(k-1)(k-2)(k-3)}{24}} \langle M \rangle_k = \sum_{k=0,1,2,3 \bmod 8} \langle M \rangle_k - \sum_{k=4,5,6,7 \bmod 8} \langle M \rangle_k. \tag{34}
$$

For an arbitrary $M \in \mathcal{G}^{\mathbb{C}}_{p,q}$, $p+q=4$, *we have*

$$\operatorname{rank}(M) = \begin{cases} 4, & \textit{if } M\widetilde{\widehat{M}}(\widehat{M}\widetilde{M})^{\triangle} \neq 0, \\ 3, & \textit{if } M\widetilde{\widehat{M}}(\widehat{M}\widetilde{M})^{\triangle} = 0 \textit{ and } T\widetilde{\widehat{T}}\widehat{T} + T\widetilde{\widehat{T}}\widetilde{T} + T(\widehat{T}\widetilde{T})^{\triangle} + \widetilde{\widehat{T}}(\widehat{T}\widetilde{T})^{\triangle} \neq 0, \\ 2, & \textit{if } M\widetilde{\widehat{M}}(\widehat{M}\widetilde{M})^{\triangle} = T\widetilde{\widehat{T}}\widehat{T} + T\widetilde{\widehat{T}}\widetilde{T} + T(\widehat{T}\widetilde{T})^{\triangle} + \widetilde{\widehat{T}}(\widehat{T}\widetilde{T})^{\triangle} = 0 \textit{ and} \\ & T\widetilde{\widehat{T}} + T\widehat{T} + T\widetilde{T} + \widetilde{\widehat{T}}\widehat{T} + \widetilde{\widehat{T}}\widetilde{T} + (\widehat{T}\widetilde{T})^{\triangle} \neq 0, \\ 1, & \textit{if } M\widetilde{\widehat{M}}(\widehat{M}\widetilde{M})^{\triangle} = T\widetilde{\widehat{T}}\widehat{T} + T\widetilde{\widehat{T}}\widetilde{T} + T(\widehat{T}\widetilde{T})^{\triangle} + \widetilde{\widehat{T}}(\widehat{T}\widetilde{T})^{\triangle} = \\ & = T\widetilde{\widehat{T}} + T\widehat{T} + T\widetilde{T} + \widetilde{\widehat{T}}\widehat{T} + \widetilde{\widehat{T}}\widetilde{T} + (\widehat{T}\widetilde{T})^{\triangle} = 0 \textit{ and } M \neq 0, \\ 0, & \textit{if } M = 0, \end{cases} \tag{35}$$

where $T := M^{\dagger}M$.

Corollary 1 *We have the following properties of the rank of arbitrary multivectors* $M_1, M_2, M_3 \in \mathcal{G}^{\mathbb{C}}_{p,q}$:

$$\operatorname{rank}(M_1 U) = \operatorname{rank}(U M_1) = \operatorname{rank}(M_1), \qquad \forall \textit{ invertible } U \in \mathcal{G}^{\mathbb{C}}_{p,q}, \tag{36}$$

$$\operatorname{rank}(M_1 M_2) \leq \min(\operatorname{rank}(M_1), \operatorname{rank}(M_2)), \tag{37}$$

$$\operatorname{rank}(M_1 M_2) + \operatorname{rank}(M_2 M_3) \leq \operatorname{rank}(M_1 M_2 M_3) + \operatorname{rank}(M_2), \tag{38}$$

$$\operatorname{rank}(M_1) + \operatorname{rank}(M_3) \leq \operatorname{rank}(M_1 M_3) + N. \tag{39}$$

Note that the results of this work are valid not only for complexified Clifford geometric algebras, but also for real Clifford geometric algebras, since we can use the same matrix representations in the real case (but these matrix representations will have non-minimal dimension in this case).

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