Factorizations of the Conformal Villarceau Motion

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Summary of the Abstract

Rational conformal motions can be described by polynomials over the even sub-algebra of a geometric algebra having a real norm polynomial. These polynomials are called spinor polynomials and factorizing them corresponds to splitting the rational motion they describe into sub-motions of lower degree. Generic spinor polynomials have a finite amount of factorizations. Examples of polynomials with an infinte amount of factorizations are very rare. In this abstract we investigate one such special case of a spinor polynomial namely the conformal Villarceau motion. We aim to give an intuitive introduction to its construction, calculate its factorizations and interpret them geometrically.

1 Introduction

In 2019 a peculiar conformal motion was proposed by L. Dorst [2] with interesting properties:

- All trajectories of points are circles
- The trajectory of two points intersects if and only if their trajectories are identical
- When correctly grouped the trajectories form a family of nested tori and correspond to their Villarceau circles
- The set of all unique trajectories forms the well known Hopf-fibration of $\mathbb{R}^3$

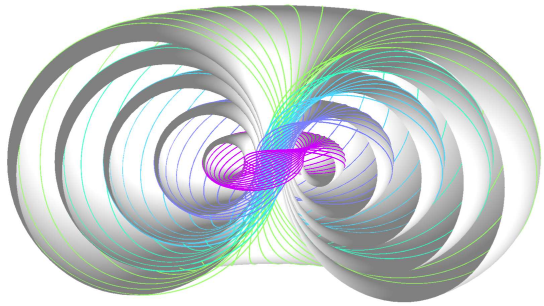

Figure 1: Hopf fibration via Villarceau circles on a family of torus surfaces.

Due to the property of it generating families of Villarceau circles we will refer to this motion as a conformal Villarceau motion. Its trajectories for a selected set of points can be seen in Figure 1.
This motion has been independently discovered by three of the authors during preparations for [8]. It stood out as motion worthy of more scrutiny since it presented interesting factorization properties. Although its degree is only two, we can find a two-parametric set of factorizations. Until now only one other motion with this many factorizations is known. The rigid body transformation translating along a circular path.
In this text we will investigate the factorizations of the Villarceau motion and try to develop an intuition for them. In Section 2 we introduce spinor polynomials, their factorization properties and characterize linear spinor polynomials. In Section 3 we investigate the Villarceau motion and its factorization.

2 Preliminaries

We will be using the conformal geometric algebra in three dimensions choosing the basis $\{e_1, e_2, e_3, e_+e_-\} \in \mathbb{R}^{4,1}$. We refer to this algebra as CGA and its even sub-algebra as CGA_+.

2.1 Factorization of Spinor Polynomials

Let us consider polynomials $C = \sum_{i=0}^{n} c_i t^i$ in the indeterminate t and coefficients $c_i \in \mathrm{CGA}_+$. We define the right evaluation of a polynomial C at a value h as

$$C(h) = \sum_{i=0}^{n} c_i h^i.$$

Let $\tilde{C}$ denote the polynomial constructed by conjugation of all coefficients of C.
We define the action of C on an element $x \in \mathrm{CGA}$ as

$$x \mapsto C x \tilde{C}.$$

If x is a point, then its image is a rational curve in Cartesian coordinates. Hence one refers to it as a *rational conformal motion*.
For an element $y \in \mathrm{CGA}_+$ to describe a conformal transformation it has to fulfill the condition $y\tilde{y} = \tilde{y}y \in \mathbb{R}$. This condition defines the *Study variety* $\mathcal{S}$ of conformal kinematics. The vanishing condition of the real part of $y\tilde{y} = \tilde{y}y$ defines the so called *null*

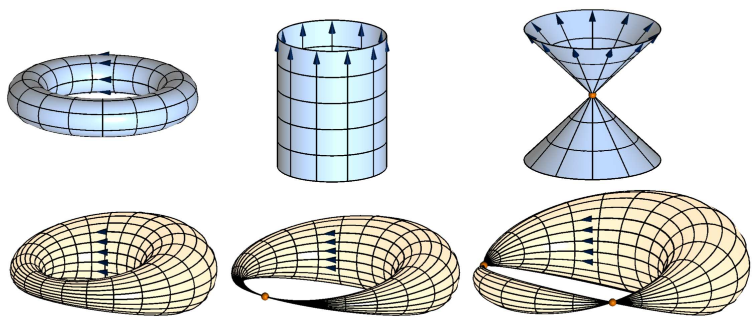

Figure 2: Elementary conformal motions: conformal rotation, translation (transversion), and scaling

quadric $\mathcal{N}$ [6]. Points in the intersection of $\mathcal{S}$ and $\mathcal{N}$ describe displacements that can be thought of as being singular, such as a scaling with scaling factor zero.
A necessary condition for a polynomial to describe a conformal motion is that $C\tilde{C}, \tilde{C}C \in \mathbb{R}[t]$ and $C\tilde{C}, \tilde{C}C$ are not the zero polynomial. In other words, C is contained in $\mathcal{S}$ but not fully in $\mathcal{N}$. If these conditions are fulfilled and its degree is positive we call C a *spinor polynomial*[8].
Factorizing spinor polynomials corresponds to splitting a conformal motion into smaller sub-motions. For applications such as kinematics and discrete differential geometry factorizations into linear factors are most important [3, 4, 5]. A lot is already known about such factorizations [7, 8]:

- For generic polynomials C the number of factorizations depends on the number of real zeros of $C\tilde{C}$. The exact number ranges between $n!$ for no real roots and $\frac{(2n)!}{2^n}$ for the maximum $2n$ roots of $C\tilde{C}$.
- There exist polynomials with no factorizations into linear factors.
- There exist polynomials with an infinite amount of factorizations into linear factors.
- The linear polynomial $t - h$ is a right factor if and only if h is a right zero.

Linear factors $at - b$ describe the most basic of transformations. They correspond to conformal rotations, translations and scalings [1]. By choosing a parametrization of $at - b$ where, a is invertible we can define $h := a^{-1}b$. We then can classify these simple transformations in the following way:

$h\tilde{h} > 0$ A conformal image of a euclidean rotation around a fixed axis and variable angle.

$h\tilde{h} = 0$ A conformal image of a euclidean translation along a fixed axis and variable distance.

$h\tilde{h} < 0$ A conformal image of a uniform scaling from a fixed center with variable scaling factor.

Figure 2 illustrates these simple motions and their conformal counterparts. The surfaces are generated by the trajectories of points on a circle in the direction of the arrows. In the top row euclidean rotation, translation and scaling are shown. The lower row shows their conformal counterparts. The orange dots highlight the singularities of the transformations. These correspond to the intersections of the motion polynomial with the null quadric.

3 The Conformal Villarceau Motion

Let us now construct and study the conformal Villarceau motion. Since we know that we are aiming for a motion tracing the Villarceau circles on a torus we want to combine two rotors. One that rotates around one axis and another simultaneously rotating around a circle perpendicular to and centered at the first axis of rotation. We choose e_3 to be our rotation axis. Simple motions in conformal kinematics can be constructed as the wedge product of any combination of two spheres, planes or points [1]. The trajectory of a point x under the simple transform given by $a \wedge b$ is then given by $x \wedge a \wedge b$. It is the object of minimal dimension hitting the constituents perpendicularly. Through this intuition we construct the rotation around the e_3 axis by wedging two perpendicular planes.

$$B_- \coloneqq e_1 \wedge e_2 = e_{12}$$

Now we construct a rotation around a circle to complete the torus on which our Villarceau circles should lie. We choose the unit circle in the e_1, e_2 plane as our axis of rotation. This rotor can be constructed by wedging the e_1, e_2 plane e_3 with the unit sphere at the origin e_+.

$$B_+ \coloneqq e_3 \wedge e_+ = e_{3+}$$

The Villarceau motion is now given by the application of both rotors.

$$C = e^{-B_- \frac{\varphi}{2}} e^{-B_+ \frac{\varphi}{2}}$$

Because B_- and B_+ have norm -1 we can write C as

$$C = (\cos(\tfrac{\varphi}{2}) - B_- \sin(\tfrac{\varphi}{2}))(\cos(\tfrac{\varphi}{2}) - B_+ \sin(\tfrac{\varphi}{2})).$$

Using the substitution $t = \cot(\frac{\varphi}{2})$ and ignoring any real factors, since they do not change the location of the image, we finally obtain the motion polynomial

$$C = (t - B_-)(t - B_+) = t^2 - t(e_{12} + e_{3+}) + e_{123+}. \tag{1}$$

By construction we already have a factorization into linear factors. We now want to find other factorizations of this polynomial. For this we will be using a general factorization algorithm [8]. First we use polynomial division to write $C = QM + R$, where $\deg R < \deg M = 2$. A linear right factor $H_2 = t - h_2$ of C necessarily has to be a factor of M and thus also of the linear remainder $R = r_1 t + r_0$. Since we want a factor, that is also a spinor polynomial, it has to fulfill the spinor polynomial condition $H_2\tilde{H}_2, \tilde{H}_2 H_2 \in \mathbb{R}[t]$. For our linear polynomial this simplifies to $h_2\tilde{h}_2, \tilde{h}_2 h_2, h_2 + \tilde{h}_2 \in \mathbb{R}$. In general h_2 can then be defined by $r_1^{-1} r_0$. In our case we have $C\tilde{C} = \tilde{C}C = M^2$, where $M = (t^2 + 1)$. From this we get

$$r_1 = -e_{12} - e_{3+} \qquad r_0 = e_{123+} - 1.$$

We see that r_1 is not invertible, so the factorizability depends on the existence of h_2 fulfilling the conditions

$$
\begin{aligned}
r_1 h_2 + r_0 = 0 &\quad (t - h_2 \text{ is right zero of } R)\\
h_2^2 + 1 = 0 &\quad (h_2 \text{ is a zero of } M)\\
h_2\tilde{h}_2, \tilde{h}_2 h_2, h_2 + \tilde{h}_2 \in \mathbb{R} &\quad (t - h_2 \text{ is a spinor polynomial})
\end{aligned}
\tag{2}
$$

Using the 16 coefficients of h_2 as variables we get a system of 43 algebraic equations, of which 17 are of degree one and 26 are of degree two. Solving the 17 linear equations results in

$$h_2 = e_{12} + s_x x + s_y y + s_z z, \tag{3}$$

where

$$s_x = 2(e_{1+} - e_{23}), \quad s_y = 2(e_{2+} + e_{13}), \quad s_z = 2(e_{3+} - e_{12}), \tag{4}$$

$$\text{and} \quad x^2 + y^2 + (z - \tfrac{1}{4})^2 - \tfrac{1}{16} = 0. \tag{5}$$

Since s_x, s_y, s_z are pairwise orthogonal, we can say that h_2 lies on a sphere given by equations (3)–(5). Let $S(u, v)$ be a parametrization of this sphere. We can then write

$$h_2(u, v) = m + \tfrac{1}{4} S(u, v), \tag{6}$$

where $m = \frac{1}{2}(e_{12} + e_{3+})$. Through polynomial division we can find $h_1(u, v)$ so that $C = (t - h_1(u, v))(t - h_2(u, v))$. It turns out that $h_1(u, v) = m - \frac{1}{4}S(u, v)$, meaning that $h_1(u, v)$ and $h_2(u, v)$ lie on the same sphere and are antipodal to each other. This implies that they commute, which can be verified by straightforward computation. By direct computation we find that the norm of h_1 and h_2 is always 1. Therefore these factors always describe conformal images of euclidean rotations.

To be able to visualize the individual factors H_1 and H_2 we assume them to have different parameters s and t giving $H_1 = (s - h_1(u, v)), H_2 = (t - h_2(u, v))$. Their product now describes a two-parametric rational motion. For any point x we can now look at its trajectory surface $D_x = H_1 H_2 x \tilde{H}_2 \tilde{H}_1$. This surface fulfills the conditions:

1. All parameter lines are circles.

2. The second fundamental form of D_x is diagonal.

This shows that the trajectory surface D_x is in fact a cyclide of Dupin. The trajectory of x under the one parametric motion C is contained as the diagonal surface curve generated by $t = s$. This curve is mapped to a Villarceau circle, when confromally transforming D_x to a torus. Therefore we follow [2] in calling this a Villarceau circle on a Dupin cyclide. For fixed x this circle stays invariant under different choices of u and v, the generated Dupin cyclide however does not. In Figure 3 we can see the Dupin cyclides D_x for different values of u and v.

Lastly let us highlight some peculiarities of the conformal Villarceau motion in the kinematic image space. The motion polynomial C parametrizes a rational curve of degree 2 in the projective space $\mathbb{P}(\mathrm{CGA}_+) = \mathbb{P}^{15}(\mathbb{R})$ and intersects the null quadric $\mathcal{N}$ in exactly two points n_1, n_2.

$$n_1 = C(\mathrm{i}) = e_{123+} - 1 - \mathrm{i}(e_{12} + e_{3+}), \quad n_2 = C(-\mathrm{i}) = e_{123+} - 1 + \mathrm{i}(e_{12} + e_{3+}).$$

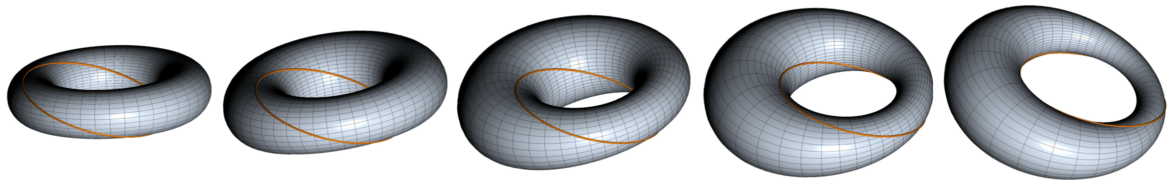

Figure 3: Different Dupin cyclides with the same Villarceau circle

Factorizability is know to be related to the connecting line between intersection points of C and $\mathcal{N}$ [9]. In our case the lines of interest are the connecting line of n_1 and n_2 and their conic tangents. Direct computation shows that none of the points on these lines are invertible, explaining why the standard factorization attempts with a finite number of factorizations failed and in the process showing the importance of non-invertible elements in CGA_+.

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