A Comprehensive Visualization of Versors of Conformal Geometric Algebra

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A visual interpretation of Conformal Geometric Algebra(CGA) is given, using multiple tools and approaches, some of them novel. First the idea of visualizing the versors A as the set of objects B such that $B = (-1)^{ab}ABA^{-1}$ for B of grade b and A of grade a. This handles many CGA elements but has the problem that not all versors have any such invariant. A solution to this is to consider what we call the n+1 dimensional *intermediate* space for n-dimensional CGA. This sets up a link between CGA and Hyperbolic Projective Geometric Algebra, which bears at least superficial similarity to the approach of [2]. Implications of this approach are outlined for Projective Geometric Algebra(PGA), oriented projective geometry, and Conformal Spacetime Algebra.

Additional Key Words and Phrases: Conformal Geometry, Space-time, dipoles

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1 INTRODUCTION

Conformal Geometric Algebra is a widely-applied approach to geometry that contains representations of geometric objects including arbitrary lines, spheres, circles, planes, points, and point-pairs, although for reasons that will become clear, it is the belief of the author that it is important to think of the representation of transformations first, with geometric objects emerging as special cases of transformations.

However, there exists a disagreement about the standardization of CGA. There are two approaches: sphere-based (IPNS)[8] and point-based (OPNS) CGA. In this paper we propose *Hybrid* CGA, in which these two approaches are combined. Loosely, the claim is that objects should be visualized with IPNS or OPNS views depending on the sign of their square; more geometrically, it is dependent on what sort of transformation they are involved in.

2 PRINCIPLES OF THE FLEMISH SCHOOL OF GEOMETRIC ALGEBRA

A specific approach has emerged within GA in recent years which we here call the Flemish school [16][4]. Algebraically, followers of the Flemish school, more than multivectors, blades or versors, emphasize *k-reflections*: geometric products of normalized, invertible 1-vectors. But Flemish principles are more rooted in geometry, with a specific way to link particular Clifford algebras Cl(p,q,r) with particular geometries, partly based on k-reflections. This approach has not been codified anywhere, but certain principles are relatively clear:

(1) The k-reflection A should be visualized as *some* set of k hyperplanes that A can be decomposed into; the set of hyperplanes can be "gauged" without implying a change to the object
(2) The geometric product is to be interpreted as transform composition
(3) Geometric objects should generally be considered especially simple *special cases* of k-reflections. For example, a planar reflection is a plane, and a 180 rotation is a line; that is, a geometric object is an invariant of the transformation it represents
(4) The relationship of a rotor to the bivector that is it's logarithm is to generalize the relationship of a rotation to its axis. For example, the logarithm of a translation is a line at infinity

A more vague but nevertheless fundamental principle is that understanding the *transformations* in an algebra is as or more important than understanding the *geometric objects* in an algebra. One motive for this is that transformations outnumber geometric objects; another is that, by principle (2) transformations turn out to be a superset of geometric objects and so geometric objects may be understood "for free" if one understands their transformations. An example illustrating both of these is that if we choose as our geometric object a line in 3D space, there is an infinite (one-parameter) family of rotations that preserve it. The line and all the rotations are included in the set of k-reflections.

The principles were developed with 2D and 3D Projective Geometric Algebra, PGA, as a motivating example. Euclidean PGA, including the idea of having 1-vectors be planes, was discovered in [18]. It was later given its name by Charles Gunn in [5], where he also explored its generalizations to Elliptic PGA Cl(4,0,0), 3D Hyperbolic PGA Cl(3,1,0), and (in a later chapter) Dual Euclidean PGA which is Cl(3,0,1) again.

3D Hyperbolic PGA Cl(3,1,0) is important to this paper. A reader used to Euclidean PGA Cl(3,0,1) will find its coordinates familiar: e1, e2 and e3 are orthogonal planes through the origin; e12, e23, e12+e31, and 0.8+0.6e12 are ordinary lines through/rotations about the origin; e4 (which squares to -1) is a plane at perceptual infinity; and e123+0.1e124 is a point displaced from the origin. However, when multiplication and conjugation are used, everything away from the origin behaves very differently to Euclidean PGA. In particular, there is a distinguished unit ball centered on the origin, known as the Klein ball (disk in 2D). This will be discussed more later and is depicted (along with other objects) in figure 3.

The Flemish school, like Gunn, aim to be able to work with exotic metrics; their recent work[15] applies the principles to Cl(3,1,1). However, Gunn's approach to the visualization of PGA objects emphasizes ideas found in traditional projective geometry including

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pencils and *bundles*; this is different from the Flemish school which, influenced by particle physics, emphasizes *gauges*. For example, the object e_{23} would be visualized by Gunn as the infinite family of planes through the line; the Flemish approach visualizes it as a single pair of orthogonal planes taken from this family - but understood to be rotateable ("gaugeable") about the line.

3 APPLYING FLEMISH SCHOOL PRINCIPLES TO 3D CGA REQUIRES THAT BIVECTORS BE BOTH CIRCLES AND POINT PAIRS

Conformal Geometric Algebra, as it is described and visualized in the work of David Hestenes[7], now sometimes known as "OPNS" CGA, does *not* follow the Flemish principles. For example, if we were following principle 1, the 1-vector e_1 would be visualized as a hyperplane, since it is a hyperplane that acts as the characterizing invariant of the transformation $e_1 A e_1^{-1}$. But Hestenes encourages practitioners to conceptualize e_1 as an arrow (similar to the "Gibbs vector" picture) sticking out of the origin "in the e_1 direction" - though note that it would not be the origin in the 3D modelling space, but instead in an "ambient" space of 5 dimensions.

Something like CGA in a Flemish style predates the discovery of gauges in PGA; this goes under the name "IPNS" or "plane-based" CGA[8], contrasted with "OPNS" or "point-based" CGA. Whether point- or plane-based CGA is better suited to a particular task is not currently widely agreed upon; a common subject of debates over it is that depending on the decision, the (algebraically defined) wedge product will be identified with the (geometrically defined) "meet" or "join".

Note that the meet (IPNS wedge) recalls the fact that a rotation made from composing two reflections will have, as its axis, the line where the two reflection planes meet. It might naively be expected, then, that the Flemish principles are already followed by IPNS CGA. In IPNS CGA, planes and spheres are always 1-vectors, lines and circles are always 2-vectors, and point pairs are always 3-vectors. This is extremely similar to the Euclidean PGA picture, with the very minor caveat that the point pairs in Euclidean PGA always have one point being the point at infinity.

In the following, the basis 1-vectors of 3D CGA will be labelled e_1, e_2, e_3, e_+, e_-(2D CGA is the same but without e_3).

The "IPNS=Flemish school" hypothesis is correct for a significant number of elements, for example e_1, e_+, $e_1 + 0.1e_+ + 0.1e_-$, e_{12}, e_{3+}, e_{123} - they should be visualized as (hyper-)spheres and circles. However, to it should not be assumed that this is a complete story, because of the following counterexample.

Consider positive-square bivectors such as e_{1-} in 2D CGA (where e_1 squares to 1 and e_- squares to -1, and so e_{1-} squares to 1). The bottom right panel of the figure 1 shows a rotor R whose bivector part is proportional to such a bivector. This rotor has the effect of pushing away from, and pulling toward, the two red points, as suggested by the arrows.

On its own, this 2D picture may seem compatible with the IPNS=Flemish hypothesis. The problem is when we consider its equivalent in 3D CGA. The equivalent there would be a pair of spheres, with the red points still as points, but in 3D space. But IPNS CGA requires that point pairs should be 3-vectors; a contradiction.

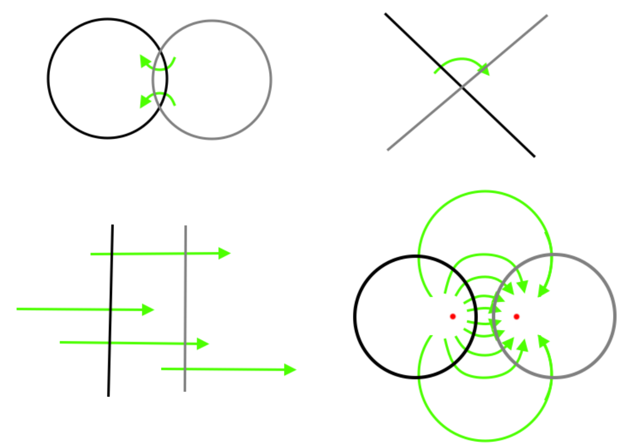

Fig. 1. Compositions of pairs of circular reflections in 2D CGA, including one that counterexample to the Flemish approach. Top left: a reflection(inversion) in the black circle followed by a reflection in the grey circle creates a "vortex pair"-esque rotor around the point pair where they meet; top right: a reflection in the black line followed by a reflection in the grey line creates a rotation; bottom left: a reflection in the black line followed by a reflection in the grey line creates a translation; bottom right: a reflection in the black circle followed by a reflection in the grey circle creates a dipole-like transformation (the counterexample)

The claim is that this pair of points *is* the bivector e_{1-} - just as it is in the OPNS representation. Like an axis line or circle, it fully defines a family of spheres, a accompanying handedness-preserving transformation (pushing away from the one point and toward the other). In particular it is the logarithm of that rotor.

Rotors characterized by such bivectors like $1.25 + 0.75e_{1-}$ should not be considered objects of mere curiosity: in the Poincare disk (or sphere) model of hyperbolic geometry, which CGA acts as an instantiation of[3], it is an important object known as a hyperbolic translation; for visualizations of hyperbolic translations in the Poincare and Klein disks, the reader is referred to [17]. Rotors like this are also related to boosts (from special relativity) and uniform scalings - a uniform scaling is a special case of the kind of transformation just described (where one of the points is at infinity).

Note that this kind of bivector (positive square) remains a point pair in any number of dimensions, whereas bivectors associated to rotations go from being point pairs in 2D CGA, to circles and lines in 3D CGA, to spheres in 4D CGA.

Therefore, a form of CGA that follows Flemish principles requires a hybrid of OPNS and IPNS CGA: bivectors can be circles or point pairs (by similar logic, trivectors can be circles or point pairs). *Null* bivectors $e_{1+} + e_{1-}$ and $e_{1+} - e_{1-}$ turn out to be *superimposed* lines and point pairs, with zero (or infinite) radius - when their radius is zero they resemble 3dz2 orbitals in chemistry.

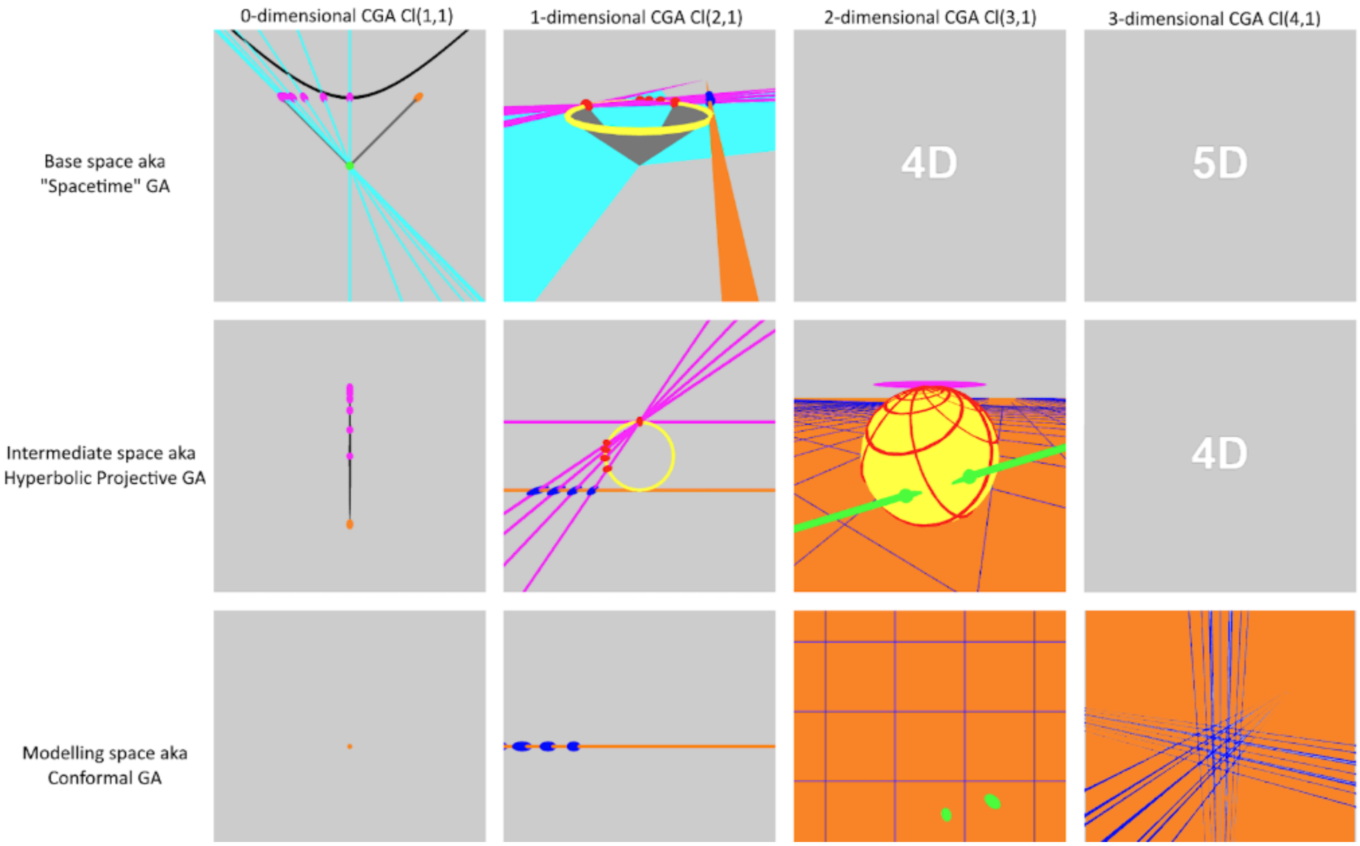

Fig. 2. Different pictures of different dimensionalities of CGA. The bottom shows the "modelling space" that we end up with; some of the other pictures will be familiar to CGA practitioners, for example the one at the top with the grey cone, which corresponds to the "horosphere" picture seen in Hestenes and Doran and Lasenby. The intermediate-space view is obtained from the horosphere view by placing one's eye at the base of the cone and looking upward. The final modelling space view is obtained by placing one's eye at the north pole of the sphere in the intermediate space. The intermediate-space view for 2D CGA also resembles "stereographic projection" or the Riemann sphere.

4 INTERMEDIATE-SPACE CGA, OR, CGA AS N+1 HYPERBOLIC PGA

In CGA literature, when two spheres do not intersect, they are sometimes said to intersect at a "negative radius" or "imaginary" circle - this is intended to call to mind a circle that can no longer be seen. In this terminology, the conclusion of the previous section could be phrased as: "imaginary circles of sphere-based 3D CGA are best thought of as point pairs, since that is the shape which characterizes the transformation gotten by composing reflections in the two spheres". But, since it *can* be seen (and it has a positive square), it is hard on any level to justify the word "imaginary" for these objects, so this term will be used no further in this paper.

With that said, there ought to be some further geometric justification for why a circle that is no longer a circle might become a point pair, or why circles and point pairs *in 3D* are of a sufficiently similar kind to be considered the same "grade".

Insight into this and other issues may be gleaned from a geometric construction relating *Conformal* geometry to n+1 *Hyperbolic* geometry, essentially due to Springborn and Bobenko[2] but which has not, to the author's knowledge, previously been applied to CGA. It is depicted in the middle rows of figure 2.

Algebraic justification for it can be seen in the fact that the signature of n-dimensional CGA precisely matches the signature of Hyperbolic Projective Geometric Algebra of n+1 dimensions - Cl(3,1,0) in the case of both 2D CGA and 3D hyperbolic PGA. Hyperbolic PGA has been discussed and visualized by Gunn[5] and De Keninck[15]. This isomorphism (which works in any dimension) turns out to match up an integral part of CGA, the "horosphere" of null objects, to the *Klein disk/ball* of hyperbolic geometry. This is a circle that divides up *axes* into those within it, which perform ordinary *rotations*, and those outside it, which perform *boosts* or hyperbolic translations. Those axes that sit precisely *on* the boundary of the klein disk (and ball, for 3D, in the middle-left column) have yet other properties.

In the case of some CGA transformations, the intermediate space turns out to be the *only* way to find an invariant for them. Examples of this include e_{-} and e_{123+} - these are, respectively, a plane and a point in the intermediate hyperbolic space ("hyperideals"[14]). If the

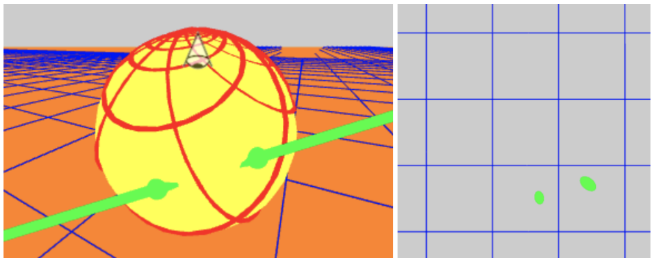

Fig. 3. To obtain the modelling-space picture of 2D CGA(right), we place our eye at the north pole of the klein ball in the intermediate hyperbolic space (left). The eye sees only those things that are on the surface of the ball - for example lines in the intermediate space like the green one appear to the eye only as a green point pair. The set of planes and lines that pass exactly through the north pole where the eye comprise the set of objects that appear in PGA (sometimes called “flats”). The geometric realization of this subalgebra marks it as a “Clifford fibre”. The precise point in the intermediate space where the eye is located is the PGA pseudoscalar. Hybrid CGA is therefore completely compatible with PGA: the only "change" is to replace PGA’s e0 with the hyperplane at infinity, eg a plane tangent at the north pole which will appear to the eye as an infinite-radius circle. This plane appears as a pink disk in figure 2

Flemish school is to insist on finding hyperplanes for k-reflections to be decomposed into, these ones will be a necessity.

If we drop the requirement that the invariant be in any sense "localizable", we find that there is a way for e_- and other negative-square 1-vectors to be visualized by creatures who are confined to the modelling space (the ones on the bottom row), as humans are with 3D CGA. They turn out to be discrete reflections (like other 1-vectors), and like any reflection they pair up all points in space. The difference is that *all* points are reflected, by them, to a *distant* partner - *none* of them are left in place. The *point-pairs* are at least preserved though - so a 3D entity can see the e_- of 3D CGA if they imagine all of space to be filled with a gas whose color varies smoothly; and in which each precise color appears at exactly two distant points. The case of this for 1D CGA is seen in figure 4.

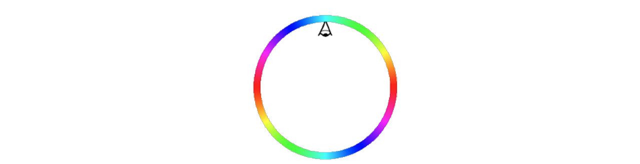

Fig. 4. The effect of a hyperbolic reflection on the horosphere of 1D CGA - all points are sent to their same-color partner. The eye at the top would see the circle stereographically projected, below. In 2D CGA, to visualize the effect of $a \rightarrow e_- a e_-$ would similarly be done with some set of colorings for points that are to be exchanged. In order for Flemish-school principles to be applied CGA in a way that is confined to the number of dimensions of the modelled space (eg 3 dimensions for 3D CGA) pictures such as the lower line may become necessary

A different approach to visualizing elements like e_- is to visualize their "carriers"[9], but this is mostly uninformative as to the nature of the e_- transformation. It also relies on the choice of point-at-infinity; this somewhat betrays the idea of CGA, because by the conformal nature of the space, the point at infinity is not *meant* to be special (different points can be continuously moved to take its place).

5 HYBRID CGA ANSWERS A PUZZLE IN ORIENTED GEOMETRY

Dorst and De Keninck have drawn attention to a puzzle in oriented geometry[4], see figure 5. The hybrid view of CGA answers this puzzle: the red lines and blue lines with rotation-implying markup (“intrinsically oriented line”; "meet line"; “pencil”[6]) is a *bivector*, while the yellow and green lines with translation-implying markup (“intrinsically oriented line”; "join line"; “spear”) are actually *trivectors*.

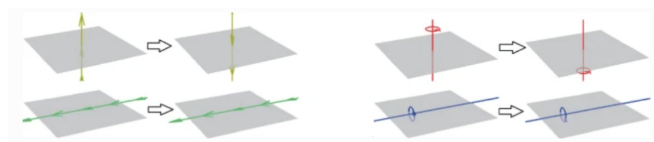

Fig. 5. Differently *oriented* lines should be expected to have different behaviour under reflections. Previous solutions[4] have focussed on PGA duality, using "doubled" PGA. The proposal of hybrid CGA is to say that the lines in these pictures, in spite of the fact that they are all lines, actually have different grades

Looking at figure 5, suppose the starting red line to be e_{13}, the blue to be e_{23}, the yellow to be e_{2+-} (dual to e_{13}), and the green to be e_{1+-} (dual to e_{23}. Supposing the reflection plane to be $e3$, the reader will find that the orientation flips (minus signs) of the figure are correctly produced, so long as they use the full sandwich product: $(-1)^{ab} ABA^{-1}$ where grade(A) = a and grade(B) = b.

Following this view, what Dorst[4] calls *intrinsic lines* and *intrinsic planes* can always be formed with wedge products (that is, grade-increasingly) of zero-radius spheres.

Table 1 shows how objects are to be understood in 3D Hybrid CGA. Note that just because a versor is involved in hyperbolic, euclidean/parabolic, or elliptic transformation does not mean it cannot be involved in the others - for example, e_1 can be involved in all of them. At the same time, quadvectors specifically are necessarily either elliptic, hyperbolic, or parabolic. A given quadvector, *together* with the objects in its outer product null space, act as a conformal model for euclidean/parabolic, elliptic, or hyperbolic geometry - in the case of hyperbolic geometry, this means the Poincaré ball or upper half plane models, with a specific quadvector defining a specific sphere or plane to act as a boundary. e_{12+-} and $-e_{12+-}$ would be two examples - following Dorst, one would be pictured as a plane (the e_3 plane) covered in small clockwise-winding arrows, the other counter-clockwise. One reason it is sensible to call the object "intrinsic" is because 2D CGA can be performed inside the plane with e_{12+-} or $-e_{12+-}$ as its governing pseudoscalar.

Quadvectors can also be visualized in the intermediate space as points. They act in some sense as "pivots" for the behaviour of planes, lines, etc passing through them. This appears strongly connected to the "pointors" of [16].

Table 1. How objects are to be understood in Hybrid CGA. "Likely" here means, *assuming* the object was found in a versor made from a product of *random* 1-vectors, what transformation the versor would implement. Note that the examples given are blades, however, *not* random versors. It is interesting to note that the geometric interpretation of objects recalls the structure of Cl(4) over the complex numbers; although this involves pairing up odd and even versors, which is less meaningful from a transformations point of view.

Grade	Squares to	Examples	Max gauges	"Likely" associated transformation	Invariant object (/un-oriented appearance)	Orientation	Notes and alternative names
0	1		1	Identity	Entirety of space		Oriented volume
1	-1	$e_-, e_- + 0.5e_1$	0	Hyperbolic reflection4	None in 3D; hyperplane in intermediate space		3D "point pair field"? Handedness-reversing in 4D
1	0	$e_0 := e_+ + e_-$	0	Annihilation	∞-radius sphere	Extrinsic	Has sidedness
1	0	$e_+ - e_-, e_1 + e_-$	0	Annihilation	0-radius sphere	Extrinsic	Has sidedness
1	1	$e_1, e_2, e_+, e_3 + e_0$	0	Planar reflection; Sphere inversion	Sphere/plane	Extrinsic	Has sidedness
2	-1	$e_{12}, e_{23}, e_{31}, e_{12} + e_{10}$	1	Rotation around line/circle	Circle/Line	Extrinsic	Set of spheres/planes through a circle/line (elliptic pencil)
2	0	$e_{01}, e_{02}, e_{01} + e_{02}$	1	Translation	∞-radius circle and ∞-radius point pair	Extrinsic; intrinsic	Set of parallel planes (plane pencil with axis at infinity)
2	0	$e_{1+} - e_{1-}$	1	Parabolic motion	0-radius circle and 0-radius point pair	Extrinsic; intrinsic	Parabolic pencil (set of spheres tangent to a plane at a point)
2	1	$e_{+-}, e_{1-}, e_{2-}, e_{3-}$	1	Hyperbolic translation or scaling	Point pair / point	Intrinsic	Hyperbolic pencil (set of spheres/planes "centered" on two points); Dipole
3	-1	$e_{123}, e_{12+}, e_{23+}$	1 or 3	Rotoreflection, point reflection	Point pair / point	Extrinsic	Elliptic bundle (set of spheres and planes passing through two points)
3	0	$e_{012}, e_{023}, e_{031}$	1	Transflection	∞-radius circle and ∞-radius point pair	Intrinsic; extrinsic	Plane bundle (set of planes through two opposing points at infinity)
3	0	$e_{12+} - e_{12-}$	1	Parabolic transflection	0-radius circle and 0-radius point pair	Intrinsic; extrinsic	Parabolic bundle (set of spheres tangent to a line at a point)
3	1	$e_{12-}, e_{23-} + e_{13-}$	2?	Hyperbolic transflection or "scaleflection"	Circle/Line	Intrinsic	Hyperbolic bundle (set of spheres and planes orthogonal to a circle); spear
4	-1	e_{123+}, e_{12+-}	2?	Hyperbolic screw or scale-and-rotation	Sphere/plane	Intrinsic	Poincare ball/upper half plane boundary; circle bundle
4	0	e_{1230}	2	Euclidean screw motion	∞-radius sphere	Intrinsic	Set of ∞-radius circles and polar dual point pairs
4	0	$e_{123+} + e_{123-}$	2	Parabolic screw motion	0-radius sphere	Intrinsic	Set of zero-radius circles around a point and polar dual point pairs
4	1	$e_{123+}, e_{123+} + 0.1e_{12+-}$	4 or 2	Elliptic screw ("isoclinic")	None in 3D; point in intermediate space		Possibly a 3D "Circle pair field"? Handedness-preserving in 4D
5	-1	e_{123+-}		Rotoreflection-and-hyperbolic-translation	None in 3D or 4D; point (origin) in 5D		"Screwflection"; can also rotoreflect-and-scale

6 APPLICATION OF THE 1-DOWN-INTERMEDIATE VISUALIZATION TO CONFORMAL SPACETIME ALGEBRA

The intermediate-space tool is useful in the context of another system, Conformal Space Time Algebra, CSTA, which is connected with Twistor theory[1] and De Sitter space[11]. The simplest CSTA algebra is for 1 dimension of space and 1 dimension of time and is represented by Cl(2,2), whose basis 1-vectors shall be labelled e1, et, e+, e-. We can therefore give 1+1 CSTA a similar treatment to the one just given to 2D CGA. Taking a Projective Geometric Algebra view of Cl(2,2), we find that, like 3D Hyperbolic PGA, it has a 2D null-manifold in it. But this time, instead of a sphere, it is in the shape of a hyperboloid-of-one-sheet.

We again mark the null 1-vector $e_+ + e_- = e0$ as being special and place our eye at the point where it is tangent to the null manifold. This eye will see 1+1 Minkowski space, just as the eye in figure 3 saw 2D conformal space. From the point of view of this eye, planes that pierce the manifold but do not contain the eye itself curves (again like CGA) - specifically, they will be the hyperbolas seen in the visualizations of [11]. Reflections in the intermediate projective-2,2-space will look like hyperbola-inversions (like circle-inversions) from the point of view of the eye. The phrase "pseudoinversive geometry" has been suggested for this.

Other objects of special relativity such as Unruh horizons, inertial world lines, wavefronts, twistors, and Penrose diagrams ("causal diamonds") appear to have realizations either as conformal objects (like point pairs) in Minkowski space, or in intermediate space.

Another, different algebra, Cl(1,1,1) "LTAP", along with higher dimensional versions like Cl(3,1,1), "STAP", is currently being researched with applications to electromagnetism and special relativity[15]. Just as PGA is a subalgebra of CGA that all pass through a point in the intermediate space, STAP is a subalgebra of CSTA whose blades and versors all pass through the point where $e0$ is tangent to the null-hyperboloid - eg the eye's position.

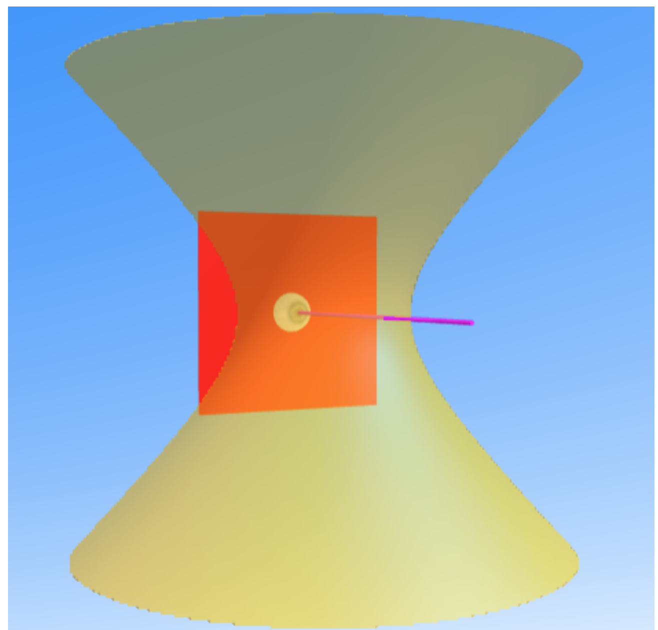

Fig. 6. The intermediate Cl(2,2,0) PGA with its null manifold in yellow - all points on it, and lines or planes tangent to it, will be null. Null planes will intersect it at crosses, which the eye will see as null "cones" in 1+1 Minkowski space - these are, precisely, lightcones, and are the 0-radius "spheres" in Minkowski space. The plane in red, tangent to the null manifold at the eye's position, is the e_0 plane of 1+1 STAP. The purple line (which will be seen as the origin point by the eye) is the e_{1t}

7 CONCLUSION

CGA was originally developed to solve problems in euclidean geometry, such as those faced by graphics programmers. But PGA has matured to fill this role, and CGA can no longer be considered a serious competitor to it. Even for the extra tasks it can do (uniform scaling and representation of spheres), its complexity, and the sparseness of its Euclidean objects, is a serious drawback.

Nevertheless, CGA may have a bright future because of the insight it offers into the group of conformal transformations. Note that the conformal group is distinguished by the fact that the sphere, and point, at infinity, can move. This is both the powerful thing about CGA and the source of the complexity that makes it less suited than PGA to *most* applied geometry. Yes, using ni and no, one can try to rid CGA of versors that move the point at infinity, to focus on "flats" which keep it in place. But this is simply to work with PGA (so long as we make the identification that PGA points are point pairs with a point at infinity), and to give up on the advantages CGA offers.

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