Applications of Geometric Algebra in Surveying and Geodesy

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Abstract

This paper introduces a novel method for solving the resection problem in two dimensions based on CGA. An efficient implementation of the recently presented VGA methods is also presented. Advantage is taken because of the characteristics of CGA, which enables the representation of points, lines, planes, and volumes in a unified mathematical framework and offers a more intuitive and geometric understanding of the problem, in contrast to existing purely algebraic methods. Several numerical examples are presented to demonstrate the efficacy of the proposed methods and to compare its validity with established techniques in the field. Our findings suggest that the proposed VGA and CGA-based methods can provide a more efficient and comprehensible solution to the two-dimensional resection problem, paving the way for further applications and advances in geodesy, surveying or navigation research. Furthermore, the method's emphasis on graphical and geometric representation makes it particularly suitable for educational purposes, allowing the reader to grasp the concepts and principles of resection more effectively.

1 Introduction

The resection problem, also known in surveying as the Snellius-Pothenot (SP) or the inverse intersection problem, involves calculating the position of an unknown point P (also called a station) using the positions of three known points A, B and C, and relative angular measurements from P. It is a relevant problem not only in geodesy and surveying, but also in other disciplines such as robot path planning, positioning, navigation or computer graphics [3, 4], and can be solved both geometrically and algebraically. Traditionally, solutions to this problem have relied on heavily algebraically loaded methods, which can be complex and challenging to comprehend. Furthermore, these methods do not always provide an intuitive understanding of the geometric relationships involved in the problem. Therefore, there is a need for a more geometric or graphical approach that simplifies the study of the resection problem. Although there are existing graphical methods for solving the problem in 2D that have been known for some time, their algebraic implementation can be quite cumbersome, hindering their widespread adoption. When the problem is approached from a geometric perspective, a better understanding of the underlying structures and relationships can be achieved, making the problem more accessible to a wider range of researchers and practitioners.

In light of the exposed ideas, the main motivation behind this paper is to develop a novel geometric method based on conformal geometric algebra (CGA) to address the resection problem in two dimensions. We also provide efficient implementations for both CGA and VGA methods.

2 Resection using Geometric Algebra

Nowadays, methods based on Geometric Algebra have been developed, providing a new solution to the resection problem while maintaining the focus on its geometrical roots.

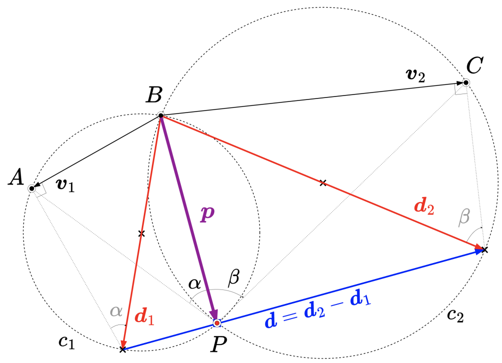

Figure 1: Vector GA-based method to solve the 2D resection problem.

2.1 Vector GA method

The 2D VGA-based method was recently proposed by J. Smith and published as disseminative material [5]. The process is mainly geometric and results in obtaining a vector $\boldsymbol{p}$ that describes the position of the point P when choosing the middle point B as the origin (see Fig. 1).

Using known data (A, B, C, α, and β) and with the help of the central angle theorem, the vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are obtained

$$\begin{aligned} \boldsymbol{d}_1 &= \boldsymbol{v}_1 + \frac{\boldsymbol{v}_1}{\tan\alpha}\boldsymbol{\sigma}_{12} = \frac{\boldsymbol{v}_1}{\sin\alpha}e^{(90-\alpha)\boldsymbol{\sigma}_{12}} \\ \boldsymbol{d}_2 &= \boldsymbol{v}_2 - \frac{\boldsymbol{v}_2}{\tan\beta}\boldsymbol{\sigma}_{12} = \frac{\boldsymbol{v}_2}{\sin\beta}e^{(\beta-90)\boldsymbol{\sigma}_{12}} \end{aligned} \tag{1}$$

The next step involves determining the vector $\boldsymbol{d}$ as $\boldsymbol{d}_2 - \boldsymbol{d}_1$. Finally, the desired vector $\boldsymbol{p}$ is the rejection of $\boldsymbol{d}_1$ or $\boldsymbol{d}_2$ on $\boldsymbol{d}$. In VGA, the above steps are summarised in the following equation

$$\boldsymbol{p} = (\boldsymbol{d}_1 \wedge \boldsymbol{d})\boldsymbol{d}^{-1} = -(\boldsymbol{d}_2 \wedge \boldsymbol{d})\boldsymbol{d}^{-1} = (\boldsymbol{d}_1 \wedge \boldsymbol{d}_2)\boldsymbol{d}^{-1} \tag{2}$$

2.2 Conformal GA Method

Given that the resection problem primarily deals with circles, utilising the conformal extension of GA, specifically the Compass Ruler Algebra (CRA), appears to be more suitable. Two traditional and well-known graphical methods are proposed to solve the resection problem using CRA: Cassini and Collins. By leveraging CRA, both methods can receive clear algebraic interpretations, as explained in the following sections.

2.2.1 Cassini Construction

The Cassini method provides a solution to the resection problem by leveraging the inscribed angle theorem. The solution is obtained by determining the intersection of two circles: one passing through points A, B, and P, and the other through points B, C, and P as shown in Fig. 1. To determine the centres of the circles, two lines must be intersected. Fig. 2 shows a concise and condensed summary of the key steps involved and discussed above. It provides valuable visual depictions that enhance the geometric intuition underlying the method, offering an algebraic interpretation of the graphical approach.

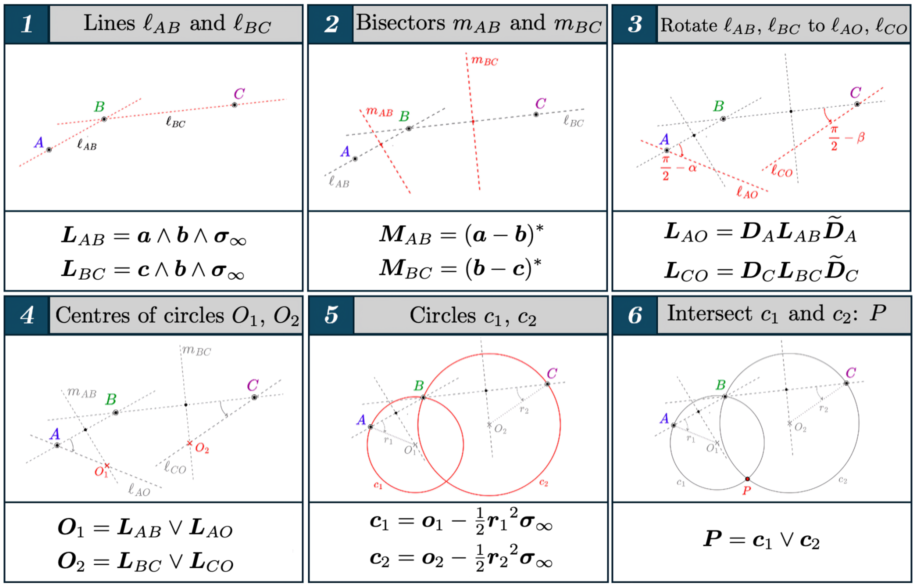

Figure 2: CRA version of Cassini's method step by step.

Method	Mean (ms)	Error (ms)	StdDev (ms)	Rank #
VGA	88.90	1.228	1.148	1
ToTal	163.00	–	–	2
Ligas	171.00	–	–	3
CollinsCGA	221.45	4.104	4.031	4
CassiniCGA	298.48	2.895	2.566	5
Font-Llagunes	228.00	–	–	6

Table 1: Performance Comparison of Resection Algorithms using Geometric Algebra and state-of-the-art algorithms (see [4]).

2.2.2 Collins Construction

The graphical method of Collins provides a solution to the resection problem using the intersection of the line passing through the point B and the so-called Collins auxiliary point E with the circle containing the points A, C and E. The step-by-step procedure for applying Collins' method graphically is depicted in Fig. 3.

3 Benchmarks

To evaluate the computational efficiency of our GA-based algorithms, extensive benchmarking tests were performed and the results were compared with state-of-the-art methods. Our implementation, which exploits the power of code generation, has achieved superior performance, outperforming the best-known algorithms to date. Moreover, sophisticated CGA-based approaches have shown excellent results and are among the most efficient in terms of execution time. Each algorithm was executed 10^6 times at random locations within the same square-shaped area used for error analysis

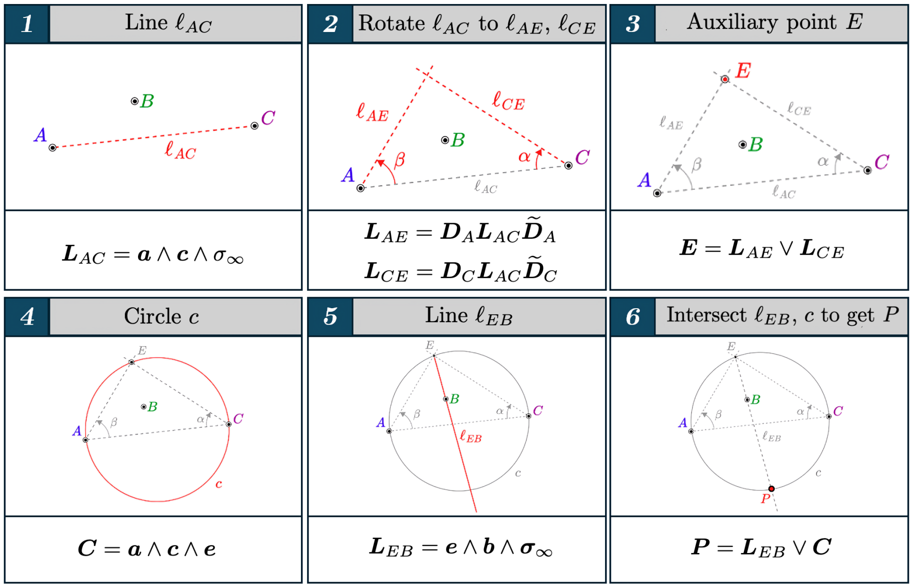

Figure 3: CRA version of Collins' Method step by step.

(see next section). The tests were performed on an Intel Core i7-9700K CPU 3.60GHz (Coffee Lake) with 8 logical and 8 physical cores (8 GB RAM, Windows 11, C#, .NET SDK 8.0.102).

Our findings reveal that our VGA-based implementation outperforms all others (see Table 1), executing approximately 83% faster than the previously best known algorithm by Pierlot (ToTal) and Ligas, and 150% faster than the one proposed by Font-Llagunes (we use the same CPU architecture as in [4]). Furthermore, the CGA versions of Collins and Cassini also rank in the top #5 of the most efficient algorithms.

4 Uncertainty analysis

This section investigates the impact of measurement uncertainties on the efficacy of the proposed GA methods. Given the intrinsic presence of noise in practical measurements, it is crucial to assess the sensitivity and resilience of the method to such perturbations [1, 4]. In order to improve the ability to evaluate the accuracy of the algorithms and determine the station position error, a new metric has been developed. Consequently, we propose three formulations that are essentially different variations of the same underlying approach, each defining the metric D based on the square of a distance to denote the proximity of the station P to the forbidden region. To validate the sensitivity of the proposed algorithms, a series of simulations have been proposed. The simulation framework is designed within a square area measuring 4 by 4 square metres, incorporating two unique configurations for three known points. Gaussian noise is introduced into the angles, characterised by a zero mean and two distinct standard deviations. The algorithms use these modified angles as input to determine the estimated position of the unknown point. The discrepancy in position (Δd) is quantified by the Euclidean distance between the exact and estimated location of the point P.

The study performs 1000 iterations for each position to determine the standard deviation of the position error. Figure 4 shows the standard deviation of the position error and the mean error measure $1/D$ in the first and second rows, in that order.

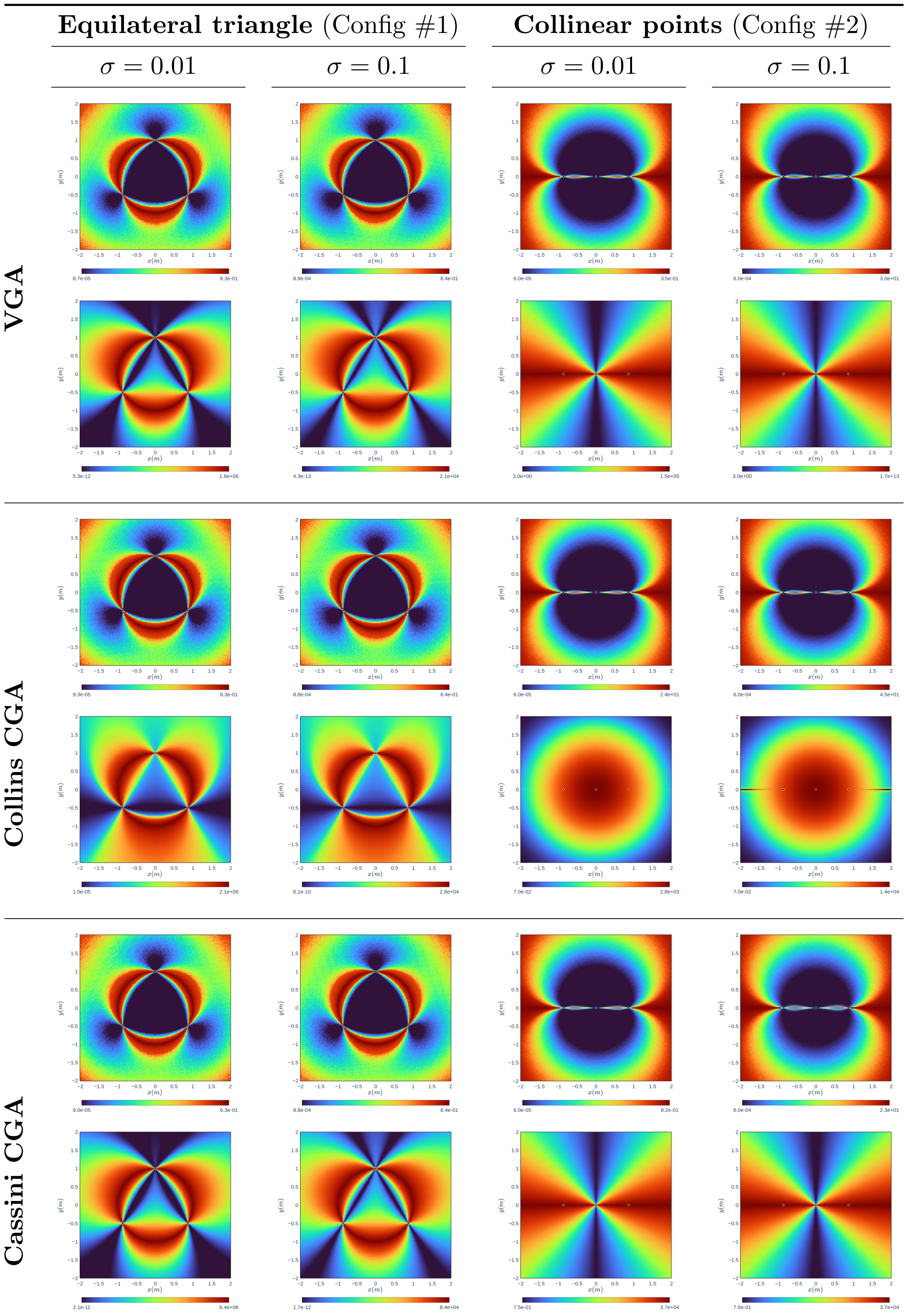

Figure 4: Error analysis for two point configurations. Config #1 is an equilateral triangle, while in config #2 the points are collinear. Both configurations have been tested with standard deviation $\sigma = 0.01$ and $\sigma = 0.1$. The three GA methods (VGA, CollinsCGA, and CassiniCGA) are compared based on their position error (first row) and metric $1/D$ (second row), respectively.

4.1 Discussion of the results

The simulations performed are in agreement with those reported in the literature and support the case studies of each of the three methods presented. Fig. 4 shows the different point configurations, where the forbidden circle is clearly identifiable due to the increase in standard deviation of the position error as it is approached. Minimal errors are observed inside the circle, while errors increase with distance outside the circle.

All methods produce identical position error plots, indicating consistency and conformity with the results obtained by most algorithms to solve the resection problem. This confirms that the sensitivity to calculate the position of the point P, even with noisy measured angles, is independent of the method used and unique, as discussed in references [4] and [2]. The metric $1/D$ can be used as an indicator of proximity to the forbidden circle. When dealing with aligned beacons, the value of D should be used directly. In other cases, the similarity between this metric and the position error suggests that the former can approximate the latter if a function of the other problem parameters is applied.

5 Conclusions

This article presents a novel approach to solving the resection problem in two dimensions using conformal geometric algebra (CGA). The CGA framework allowed for a more intuitive understanding and efficient solution of the resection problem compared to existing algebraic techniques.

The proposed method leveraged the ability of CGA to transition between different reference frames without requiring coordinate transformations. This eliminated the need for multiple calculation steps and complex algebraic manipulations that are characteristic of traditional algebraic solutions. Through extensive numerical simulations, we have demonstrated the validity and efficacy of our GA-based approach, achieving accuracy comparable to that of established algebraic techniques, while significantly improving computational efficiency and providing valuable geometric insights.

Our findings suggest that the geometric algebra framework has strong potential to solve resection-type problems not only in surveying and geodesy but also in computer graphics, robotics, computer vision, and navigation. By exploiting geometric relationships between entities, CGA paves the way for more intuitive solutions that unify computations involving different geometric primitives.

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