Closed-form inverse kinematics solutions for a class of serial robots without spherical wrist using conformal geometric algebra

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Summary of the Abstract

One of the most well-known applications of geometric algebra in engineering is providing a compact formulation of the kinematics of serial robotic manipulators. However, the use of geometric algebra in the field of robotics is still in its early stages, and there are still several open problems that can be addressed with this elegant and compact formulation. In this context, this work introduces a strategy based on conformal geometric algebra to solve the inverse kinematics problem for a class of six degrees-of-freedom (DOF) robotic manipulators without a spherical wrist, for which it is known that the inverse kinematics problem generally does not have an analytical solution. Inverse kinematics involves computing the set of all values for the joint variables (i.e., the configurations) that make the end-effector of the robot have a given position and orientation in three-dimensional space. To achieve this, a purely geometric strategy extending already existing contributions for the case where the robot has a spherical wrist is proposed. In particular, a point is assigned to each joint of the robot so that the problem reduces to computing the set of all possible joint positions for a given desired position and orientation of the end-effector of the robot. These points are found by defining and manipulating several geometric entities such as lines, planes, and spheres. Finally, validation with a real robot of the considered class is demonstrated both in simulation and experimentation.

Introduction, motivation and state of the art

A serial robot is, geometrically speaking, a sequence of rigid structural elements, called *links*, connected to each other by motor-driven kinematic pairs, called *joints*. Each joint provides relative motion between the two consecutive links it connects. An important component is the free end of the last link, called the *end-effector* of the robot. Its importance lies in the fact that all the tools the robot needs to carry out its programmed tasks, such as painting tools, screwdrivers, robotic hands, or grippers, among others, are placed on the end-effector. Therefore, it is essential to know: (1) where the end effector is located (i.e., what its position and orientation are in $\mathbb{R}^3$) for each robot *configuration*, and (2) which configuration or configurations of the robot cause its end-effector to have certain predefined position and orientation. The first problem is known as the problem of *forward kinematics*, and the second as the problem of *inverse kinematics*.

A serial robot is said to have a *spherical wrist* if the rotation or translation axes of its last three joints intersect at a single point or are parallel. While forward kinematics is a straightforward problem to tackle for any type of serial robot, inverse kinematics can be challenging when robots have more than six degrees of freedom or lack a spherical wrist.

Pieper's theorem [7] establishes that the inverse kinematics of serial robots with a spherical wrist always have an analytical solution. Furthermore, Pieper's theorem is constructive in the sense that closed-form solutions are explicitly derived for any type of robot with a spherical wrist. However, if there is misalignment between any of the axes of the last three joints (as shown in Figure 1), then the robot no longer has a spherical wrist, and therefore, Pieper's theorem cannot be applied. To solve the inverse kinematics problem in these cases, [6] developed a method based on the use of homogeneous matrices that are used to construct the forward kinematics of the robot. In fact, given the kinematic identity:

$$\boldsymbol{T}_U^1 \cdot \boldsymbol{T}_1^2 \cdots \boldsymbol{T}_{n-1}^n = \boldsymbol{T}_U^n, \tag{1}$$

where $\boldsymbol{T}_U^n$ is the homogeneous transformation matrix describing the position and orientation of the end-effector with respect to its base, and where $\boldsymbol{T}_{i-1}^i$ only depends on the joint variable q_i, Paul's method consists of analyzing each of the following matrix equations:

$$\boldsymbol{T}_{i-1}^i \cdots \boldsymbol{T}_{n-1}^n = \left(\boldsymbol{T}_{i-2}^{i-1}\right)^{-1} \cdots \left(\boldsymbol{T}_0^1\right)^{-1} \cdot \boldsymbol{T}_0^n \quad \text{for } i = 2, \ldots, n \tag{2}$$

with the aim of isolating known trigonometric equations that can be analytically solved for one or more joint variables. However, the large number of different combinations along with the complications to analytically solve arbitrary trigonometric equations make this method unsuitable for kinematic chains with non-trivial geometry. Most of the

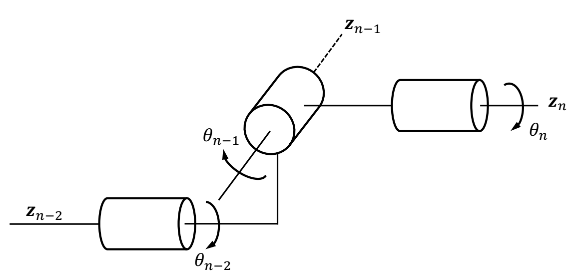

Figure 1: An example of a serial robot with an offset between the $n-2$ and $n-1$ joints.

works on inverse kinematics for robots without a spherical wrist found in the literature focus on numerical methods or particular geometric methods [4, 5, 11, 14]. Although the latter can only be applied to the specific robots for which they have been designed, they provide the complete set of solutions, unlike the former which only approximate one of the solutions. However, geometric methods are difficult to design, especially for robots without a spherical wrist. This is one of the reasons why conformal geometric algebra can be useful for addressing this problem. For instance, the works of [1, 2, 3, 9, 10] solve the inverse kinematics of different types of robots using conformal geometric algebra. The idea developed in all of them is to assign one point to each joint of the robot and then define various geometric objects whose intersections coincide with these points. This approach allows for the calculation of the joint variables and, in turn, the determination of the configuration or configurations associated with a predefined position and orientation of the end-effector. However, none of the mentioned contributions address the problem of inverse kinematics for robots without a spherical wrist using conformal geometric algebra.

Therefore, this work extends all these previous contributions. In particular, its extends a previous study by one of the authors [12]. Specifically, we introduced a first solution to the problem for a particular class of robots without a spherical wrist, which do not actually correspond to any existing industrial robot. The intention was to pave the way for more specific solution strategies by demonstrating how this problem could be approached. Indeed, the same concept is applied in this work, but the developed solution strategy is differs from the one presented there. In addition, the strategy developed in this work has been validated with a real robot, both in simulation and experimentation.

Sketch of the solution strategy

In this section, we present a summary of the main building blocks of the proposed closed-form solution for the inverse kinematics of a six degrees of freedom serial robot without a spherical wrist using conformal geometric algebra.

The starting point is the predefined position, denoted by p_6, and orientation, denoted by R_6, of the end-effector. We can assume that p_6 is already the null vector representation of the predefined position, whereas R_6 is the rotor encoding the predefined orientation. Then, point p_5 can be obtained directly by a translation of point p_6 along the z-axis of the predefined orientation, a translation of distance $-\mathrm{d}_6$, which is the length of the last link. Therefore, $p_5 = T_5^6 p_6 \widetilde{T}_5^6$, where:

$$T_5^6 = 1 - \mathrm{d}_6 \frac{e_\infty z_6}{2}$$

Additionally, p_1, which lies between links 1 and 2, is not affected by any of the joint variables. Hence, similarly to p_5, p_1 can be obtained by a translation of point p_0 along the z-axis of the reference frame a distance d_1, which is the length of the first link. Therefore, $p_1 = T_1^0 p_0 \widetilde{T}_1^0$, with:

$$T_1^0 = 1 + \mathrm{d}_1 \frac{e_\infty e_3}{2}$$

To obtain p_4 we proceed as follows. The Euclidean point $\mathbf{p_4}$ should lie on the intersection between an sphere with center $\mathbf{p_5}$ and radius d_5 (the length of the fifth link), a plane with normal vector $\mathbf{v}_{56} = \mathbf{p_6} - \mathbf{p}_5$ that contains $\mathbf{p_5}$, and a vertical plane that contains $\mathbf{p_5}$ with $\delta = \mathrm{d}_4$, the length of the fourth link. The inner representation of the first two geometric elements is straightforward:

$$\Pi_5 = \mathbf{v}_{56} + (\mathbf{p_5} \cdot \mathbf{v}_{56})e_\infty, \quad S_5 = p_5 - \frac{1}{2}{\mathrm{d}_5}^2 e_\infty \tag{3}$$

We now intersect the following three elements, whose inner representations are:

$$S_m = m - \frac{1}{2}{\mathrm{d}_m}^2 e_\infty, \quad S_0 = e_0 - \frac{1}{2}{\mathrm{d}_4}^2 e_\infty, \quad \Pi_{xy} = e_3 \tag{4}$$

where d_m is the norm of the middle point of the projection of the segment $\overline{\mathbf{Op_5}}$ to the $x - y$ plane. The intersection is the bivector $B_{\tan} = (S_m \wedge S_0 \wedge \Pi_{xy})^*$, which is the outer representation of the pair of points $\mathbf{v}_1$,$\mathbf{v}_2$ representing the normal vectors of the two possible vertical planes that contains $\mathbf{p_5}$ and have a distance d_4 to the origin. The inner representations of these two planes is:

$$\Pi_4 = \mathbf{v}_i + \mathrm{d}_4 e_\infty \quad i = 1, 2$$

Then, p_4 can be extracted from the bivector $B_4 = (\Pi_5 \wedge S_5 \wedge \Pi_4)^*$. Note that since there are two possible vertical planes Π_4 and each bivector B_4 represents a pair of points, in total we have four different possibilities for p_4.

Now, p_3 can be easily found as the translation along $\mathbf{v}_4$ an amount d_4 of any of the possible p_4 points. If we define:

$$T_3^4 = 1 - \mathrm{d}_4 \frac{e_\infty \mathbf{v}_4}{2}$$

then $p_3 = T_3^4 p_4 \widetilde{T}_3^4$. Therefore, we also have four different p_3 points. The last step involves defining the following geometric entities for each possible point p_3:

$$\begin{aligned} S_1 &= p_1 - \frac{1}{2}{\mathrm{a}_2}^2 e_\infty \\ S_3 &= p_3 - \frac{1}{2}{\mathrm{a}_3}^2 e_\infty \\ \Pi &= (p_0 \wedge p_1 \wedge p_3 \wedge e_\infty)^* \end{aligned} \tag{5}$$

where a_2 and a_3 denote the lengths of the second and third links, respectively. The intersection of these three geometric entities, $B_2 = (S_1 \wedge S_3 \wedge \Pi)^*$, is the outer representation of the two possible values for p_2. Since two p_2 points have been obtained for each of the four p_3 and p_4 points, a total of eight different solutions for the inverse kinematics will be obtained. This constitutes the set of all possible solutions, expressed in terms of the predefined position and orientation of the end-effector of the robot, i.e., a closed-form solution for this problem.

Once all points have been calculated, it is possible to compute the values of the joint variables, for which the following additional geometric entities are needed:

$$\begin{aligned} \Pi^*_{XZ} &= e_0 \wedge e_1 \wedge e_2 \wedge e_\infty \\ \Pi^* &= p_0 \wedge p_1 \wedge p_3 \wedge e_\infty \\ L^*_{01} &= p_0 \wedge p_1 \wedge e_\infty \\ L^*_{12} &= p_1 \wedge p_2 \wedge e_\infty \\ L^*_{23} &= p_2 \wedge p_3 \wedge e_\infty \end{aligned} \qquad \begin{aligned} L^*_{34} &= p_3 \wedge p_4 \wedge e_\infty \\ L^*_{45} &= p_4 \wedge p_5 \wedge e_\infty \\ L^*_{56} &= p_5 \wedge p_6 \wedge e_\infty \\ L^*_{x6} &= p_6 \wedge p'_6 \wedge e_\infty \end{aligned} \tag{6}$$

Here, point p_6' is the point p_6 plus the unit vector corresponding to the x-axis of the predefined orientation of the end-effector. Finally, the angles can be computed as follows:

$$\begin{aligned} q1 &= \angle(\Pi_{XZ}, \Pi) & q4 &= \angle(L_{23}, L_{45}) \\ q2 &= \angle(L_{01}, L_{12}) & q5 &= \angle(L_{34}, L_{56}) \\ q3 &= \angle(L_{12}, L_{23}) & q6 &= \angle(L_{45}, L_{x6}) \end{aligned} \tag{7}$$

Validation

The validation was performed with a Universal Robot 5 (UR5), illustrated in Figure 2. This robot has six degrees of freedom and an offset between the fifth and sixth joint axes, which is incompatible with a spherical wrist. The developed solution was implemented in Python using the Clifford library[1] and validated in simulation using Gazebo, a widely-used open-source robotics simulation software. For experimentation, the Mobile Anthropomorphic Dual-Arm Robot (MADAR), a mobile platform equipped with two UR5 robots [8], was utilized. Two videos showcasing the simulation results can be viewed here: Simulation trial 1, and Simulation trial 2.

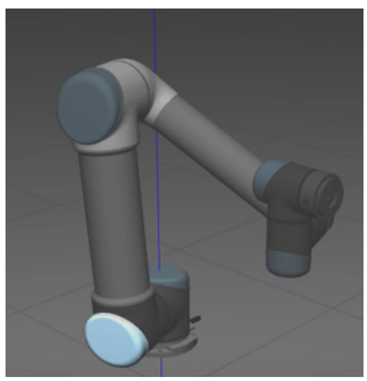

Figure 2: 3D model of the selected robot for validation, the UR5 robot.

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