Geometric Algebra: A Computational Framework for Geometrical Applications

Part 1

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In geometric algebra, you can compute directly with subspaces, which simplifies many geometrical operations. This two-part tutorial gives a brief introduction to the principles.

The traditional method of defining geometrical objects in fields like computer graphics, robotics, and computer vision routinely uses vectors to characterize constructions. Doing this effectively means extending the basic concept of a vector as an element of a linear space by an inner product and a cross product and by some additional constructions such as homogeneous coordinates. This compactly encodes the intersection of, for instance, offset planes in space. Many of these techniques work well in 3D space, but some problems exist, such as the difference between vectors and points[1] and characterizing planes by normal vectors (which may require extra computation after linear transformations because a transformed plane's normal vector is not the normal vector's transform). Application programmers typically fix these problems by introducing data structures and combination rules, possibly using object-oriented programming to implement this patch.[2]

Yet, deeper issues in programming geometry exist that many practitioners still accept. For instance, when intersecting linear subspaces, it seems unavoidable that we need to split our intersection algorithms to treat the intersection of lines and planes, planes and planes, lines and lines, and so on in separate pieces of code. After all, the outcomes themselves can be points, lines, or planes, which are essentially different in their further processing.

This need not be so. If we could see subspaces as basic computational elements and do direct algebra with them, then algorithms and their implementation would not need to split their cases on dimensionality. For instance, $A \wedge B$ could be the subspace spanned by the spaces A and B, and the expression $A \rfloor B$ could be the part of B perpendicular to A. Then, we would always have the computation rule $(A \wedge B) \rfloor C = A \rfloor (B \rfloor C)$ because we can compute the part of C perpendicular to the span of A and B in two steps: perpendicularity to B followed by perpendicularity to A. Subspaces therefore have computational rules of their own that we can use immediately, independent of how many vectors we use to span them. In this view, we can avoid the split in cases for the intersection operator because intersection of subspaces always leads to subspaces. We should consider using this structure because it would enormously simplify the specification of geometric programs.

This article (in parts one and two) intends to convince you that subspaces form an algebra with well-defined products that have direct geometric significance. Researchers can then use this algebra as a language for geometry, which we claim is a better choice than a language always reducing everything to vectors (which are just 1D subspaces). Along the way, we will see that this framework lets us divide by vectors (in fact, we can divide by any subspace), and we will see several familiar computer graphics constructs (such as quaternions, normals, and Plücker coordinates) that fold in naturally with the framework and no longer need to be considered clever but as extraneous tricks. This algebra is called *geometric algebra*. Mathematically, it is like Clifford algebra but carefully wielded to have a clear geometrical interpretation, which excludes some constructions and suggests others. Most literature uses the two terms interchangeably.

This first article primarily introduces subspaces (the basic computational element in geometric algebra) and the products of geometric algebra. We introduce these ideas but do not always give proofs of what we present. The proofs we do give are intended to illustrate using geometric algebra. (Readers can find the missing proofs in the references.) In part two of this article (to be published in a subsequent issue of *CG&A*), we will give some

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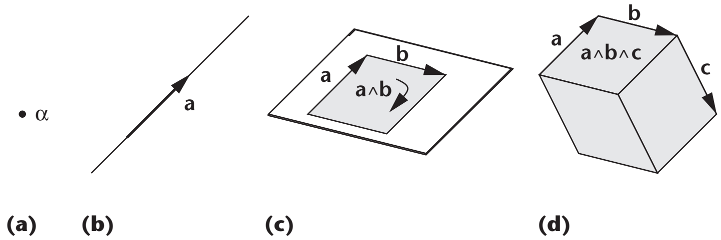
examples of how to use these products in elementary but important ways and look at more advanced topics such as differentiation, linear algebra, and homogeneous representation spaces.

Because subspaces are the main objects of geometric algebra, we introduce them first. We then introduce the geometric product and look at products derived from the geometric product. Some of the derived products, like the inner and outer products, are so basic that it is natural to treat them here also, even though the geometric product is all we need to do geometric algebra. Other products are better introduced in the context of their geometrical meaning, so we develop them in part two. This approach reduces the amount of new notation, but it might make it seem like geometric algebra must invent a new technique for every new kind of geometrical operation we want to embed. This is not the case. All you need is the geometric product and its (anti-)commutation properties.



1 **With the outer product of vectors, you can span subspaces. (a) Zero terms give a 0D subspace (a point). (b) One term gives a 1D subspace (a vector). (c) Two terms give a 2D subspace, an oriented plane element. (d) Three terms span an oriented volume element.**

Subspaces as computational elements

As in the classical approach, we start with a real vector space V^m that we use to denote 1D directed magnitudes. Typical usage would be to employ a vector to denote a translation in such a space to establish the location of a point of interest. (Points are not vectors, but their locations relative to a fixed point are.[1]) We now want to extend this capability of indicating directed magnitudes to higher dimensional directions such as facets of objects, or tangent planes. We will start with the simplest subspaces—a linear vector space's proper subspaces, which are lines, planes, and so on through the origin—and develop their algebra of spanning and perpendicularity measures. In part two, we show how to use the same algebra to treat offset subspaces and spheres.

Constructing subspaces

We start with a real m-dimensional linear space V^m, of which the elements are called vectors. Many approaches to geometry explicitly use coordinates. Although coordinates are necessary for input and output, and they are also needed to perform low-level operations on objects, most formulas and computations in geometric algebra can work directly on subspaces without resorting to coordinates. Thus, we will always view vectors geometrically: a vector denotes a 1D direction element, with a certain *attitude* or stance in space, and a *magnitude*, a measure of length in that direction. We can characterize these properties by calling a vector a directed line element, as long as we mentally associate an orientation and magnitude with it—that is, $\mathbf{v}$ is not the same as $-\mathbf{v}$ or $2\mathbf{v}$. These properties are independent of any coordinate system, and we will not refer to coordinates, except for times when we feel a coordinate example clarifies an explanation. The algebraic properties of these geometrical vectors are that they can be added and weighted with real coefficients in the usual way to produce new vectors, and they can be multiplied using an *inner product* to produce a scalar $\mathbf{a} \cdot \mathbf{b}$. (We use a metric vector space with a well-defined inner product.)

In geometric algebra, higher dimensional oriented subspaces are also basic computational elements and they are called *blades*. We use the term k-blade for a k-dimensional homogeneous subspace. Therefore, a vector is a 1-blade.

A common way to construct a blade is from vectors, using a product that constructs the span of vectors. This product is called the *outer product* (sometimes the wedge product) and denoted $\wedge$. It is codified by its algebraic properties, which we choose to ensure we get m-dimensional space elements with an appropriate magnitude (area element for $m = 2$ and volume elements for $m = 3$, see Figure 1). As in linear algebra, such magnitudes are determinants of matrices representing the basis of vectors spanning them. But such a definition would be too specifically dependent on that matrix representation. Mathematically, a determinant is an antisymmetric linear scalar-valued function of its vector arguments. That gives the clue to the outer product's rather abstract definition in geometric algebra:

> The outer product of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_k$ is antisymmetric, associative, and linear in its arguments. It is denoted $\mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_k$, and called a k-blade.

The only thing that is different from a determinant is that the outer product is not forced to be scalar-valued. This gives it the capability of representing the attitude of a k-dimensional subspace element as well as its magnitude.

2-blades in 3D space

Let us see how this works in the geometric algebra of a 3D space V^3. For convenience, let us choose a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in this space, relative to which we denote any vector. Now compute $\mathbf{a} \wedge \mathbf{b}$ for $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$. By linearity, we can write this as the sum of six terms of the form $a_1b_2\, \mathbf{e}_1\wedge\mathbf{e}_2$ or $a_1b_1\, \mathbf{e}_1\wedge\mathbf{e}_1$. By antisymmetry, the outer product of any vector with itself must be zero, so the term with $a_1b_1\, \mathbf{e}_1\wedge\mathbf{e}_1$ and other similar terms disappear. Also by antisymmetry, $\mathbf{e}_2 \wedge \mathbf{e}_1 = -\mathbf{e}_1 \wedge \mathbf{e}_2$, so we can group some terms. You can verify that the final result is

$$\begin{aligned}\mathbf{a}\wedge\mathbf{b} &= \left(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3\right)\wedge\left(b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3\right)\\ &= \left(a_1b_2 - a_2b_1\right)\,\mathbf{e}_1\wedge\mathbf{e}_2 + \left(a_2b_3 - a_3b_2\right)\,\mathbf{e}_2\wedge\mathbf{e}_3\\ &\quad + \left(a_3b_1 - a_1b_3\right)\,\mathbf{e}_3\wedge\mathbf{e}_1 \end{aligned} \tag{1}$$

We cannot simplify this further. Apparently, the outer product's axioms let us decompose any 2-blade in 3D space onto a basis of three elements. This 2-blade basis (also called *bivector basis*) $\{\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_1\}$ consists of 2-blades spanned by the basis vectors. Linearity of the outer product implies that the set of 2-blades forms a linear space on this basis. We will interpret this as the space of all plane elements (or area elements).

Let us show that $\mathbf{a} \wedge \mathbf{b}$ has the correct magnitude for an area element. That is particularly clear if we choose a specific orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, chosen such that $\mathbf{a}$ lies in the $\mathbf{e}_1$ direction and $\mathbf{b}$ lies in the $(\mathbf{e}_1, \mathbf{e}_2)$ plane—we can always do this. Then, $\mathbf{a} = a\mathbf{e}_1$, $\mathbf{b} = b \cos\phi\, \mathbf{e}_1 + b \sin\phi\, \mathbf{e}_2$ (with ϕ the angle from $\mathbf{a}$ to $\mathbf{b}$), so that

$$\mathbf{a} \wedge \mathbf{b} = (ab \sin\phi)\, \mathbf{e}_1 \wedge \mathbf{e}_2 \tag{2}$$

This single result contains both the correct magnitude of the area $ab \sin\phi$ spanned by $\mathbf{a}$ and $\mathbf{b}$ and the plane in which it resides—for we recognize $\mathbf{e}_1 \wedge \mathbf{e}_2$ as the unit directed area element of the $(\mathbf{e}_1, \mathbf{e}_2)$ plane. Because we can always adapt our coordinates to vectors in this way, this result is universally valid: $\mathbf{a} \wedge \mathbf{b}$ is an area element of the plane spanned by $\mathbf{a}$ and $\mathbf{b}$ (see Figure 1c). Denoting the unit area element in the $(\mathbf{a}, \mathbf{b})$ plane by $\mathbf{I}$, the coordinate-free formulation is

$$\mathbf{a} \wedge \mathbf{b} = (ab \sin\phi)\, \mathbf{I} \tag{3}$$

The result extends to blades of higher grades. Each is proportional to the unit hypervolume element in its subspace, by a factor that is the hypervolume.

Volumes as 3-blades

We can also form the outer product of three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Considering each of those decomposed onto their three components on some basis in our 3D space, we obtain terms of three different types, depending on how many common components occur—for example, terms like $a_1b_1c_1\, \mathbf{e}_1 \wedge \mathbf{e}_1 \wedge \mathbf{e}_1$, $a_1b_1c_2\, \mathbf{e}_1 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$, and $a_1b_2c_3\, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. Because of associativity and antisymmetry, only the last type survives in all its permutations. The final result is

$$\begin{aligned}\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \big(&a_1b_2c_3 - a_1b_3c_2 + a_2b_1c_3 \\ &- a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1\big)\, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3\end{aligned}$$

The scalar factor is the determinant of the matrix with columns $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, which is proportional to the signed volume spanned by them (as is well known from linear algebra). The term $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ is the denotation of which volume is used as a unit—that spanned by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. The order of the vectors gives its orientation, so this is a signed volume. In 3D space, there is no other choice for constructing volumes than (possibly negative) multiples of this volume (see Figure 1d). However, in higher dimensional spaces, the volume element's attitude must be indicated just as much as we needed to denote the attitude of planes in 3-space.

Pseudoscalar as hypervolume

Forming the outer product of four vectors $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d}$ in 3D space will always produce zero because they must be linearly dependent. The highest order blade that is nonzero in an m-dimensional space is an m-blade. For historical reasons, such a blade representing an m-dimensional volume element is called a *pseudoscalar* for that space. Unfortunately, this is a rather abstract term for the elementary geometric concept of an oriented hypervolume element.

Scalars as subspaces

To make scalars fully admissible elements of the algebra we have so far, we can define the outer product of two scalars, and a scalar and vector, by identifying it with the familiar scalar product in the vector space we started with: $\alpha \wedge \beta = \alpha\beta$ and $\alpha \wedge \mathbf{v} = \alpha\mathbf{v}$.

Because the scalars are constructed by the outer product of no vectors at all, we can interpret them geometrically as the representation of 0-dimensional subspace elements. These are like points with masses. So scalars are geometrical entities as well, if we are willing to stretch the meaning of subspace a little. We denote scalars mostly with Greek lowercase letters.

Linear space of subspaces

So far, we have constructed a geometrically significant algebra containing only two operations: the addition $+$ and the outer multiplication $\wedge$ (subsuming the usual scalar multiplication). Starting from scalars and a 3D vector space, we have generated a 3D space of 2-blades and a 1D space of 3-blades (because all volumes are proportional to each other). In total, therefore, we have a set of elements that naturally group by their dimensionality. Choosing some basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write what we have as spanned by the set

$$\left\{\begin{aligned}&\underbrace{1}_{\text{scalars}}, \underbrace{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3}_{\text{vector space}}, \\ &\underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_1}_{\text{bivector space}}, \underbrace{\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3}_{\text{trivector space}}\end{aligned}\right\} \tag{4}$$

Every k-blade formed by $\wedge$ can be decomposed on the k-vector basis using $+$. The dimensionality k is often called the *grade* or *step* of the k-blade or k-vector, reserving the term *dimension* for the vector space that generated them. A k-blade represents a k-dimensional oriented subspace element.

If we allow the scalar-weighted addition of arbitrary elements in this set of basis blades, we get an 8D linear space from the original 3D vector space. This space, with $+$ and $\wedge$ as operations, is called the Grassmann algebra of 3-space. In an m-dimensional space, there are $\binom{m}{k}$ basis elements of grade k, for a total basis of 2^m elements for the Grassmann algebra. We use the same basis for the space's geometric algebra, although we will construct the objects in it differently.

Products of geometric algebra

The geometric product is the most important product of geometric algebra. The fact that we can apply the geometric product to k-blades and that it has an inverse considerably extends algebraic techniques for solving geometrical problems. We can use the geometric product to derive other meaningful products. The most elementary are the inner and outer products. We treat the useful but less elementary products giving reflections, rotations, and intersection later on in this article and in more detail in part two.

Geometric product

For vectors in our metric vector space V^m, we define the geometric product in terms of the inner and outer product as

$$\mathbf{ab} \equiv \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \tag{5}$$

So the geometric product of two vectors is an element of mixed grade; it has a scalar (0-blade) part $\mathbf{a} \cdot \mathbf{b}$ and a 2-blade part $\mathbf{a} \wedge \mathbf{b}$. Therefore, it is not a blade; rather, it is an operator on blades. Changing the order of **a** and **b** gives

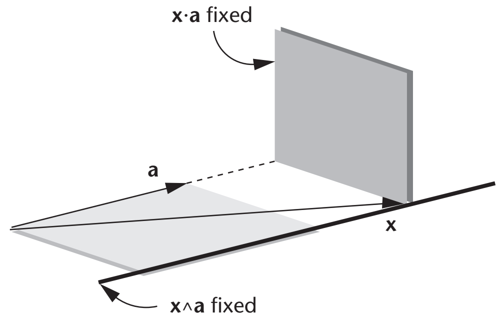
$$\mathbf{ba} \equiv \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \wedge \mathbf{a} = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \wedge \mathbf{b}$$

The geometric product of two vectors is therefore neither fully symmetric (unlike the inner product) nor fully antisymmetric (unlike the outer product). However, the geometric product is invertible.

A simple drawing might convince you that the geometric product is invertible, whereas the inner and outer product separately are not. In Figure 2, we have a given vector **a**. We indicate the set of vectors **x** with the same value of the inner product $\mathbf{x} \cdot \mathbf{a}$—this is a plane perpendicular to **a**. We also show the set of all vectors with the same value of the outer product $\mathbf{x} \wedge \mathbf{a}$—this is the line of all points that span the same directed area with **a** (because for the position vector of any point $\mathbf{p} = \mathbf{x} + \lambda\mathbf{a}$ on that line, we have $\mathbf{p} \wedge \mathbf{a} = \mathbf{x} \wedge \mathbf{a} + \lambda\mathbf{a} \wedge \mathbf{a} = \mathbf{x} \wedge \mathbf{a}$ by the antisymmetry property). Neither of these sets is a singleton (in spaces of more than one dimension), so the inner and outer products are not fully invertible. The geometric product provides both the plane and the line and lets us determine their unique intersection **x**, as Figure 2 illustrates. The geometric product then is thus invertible—from **xa** and **a**, we can retrieve **x**.

Equation 5 defines the geometric product only for vectors. For arbitrary elements of our algebra, it is defined using linearity, associativity, and distributivity over addition. We make it coincide with the usual scalar product in the vector space, as the notation already suggests. That gives the following axioms (where α and β are scalars; **x** is a vector; and A, B, and C are general elements of the algebra):

Scalars	$\alpha\,\beta$ and $\alpha\mathbf{x}$ have their usual meaning in V^m	(6)
Scalars commute	$\alpha A = A\,\alpha$	(7)
Vectors	$\mathbf{x}\,\mathbf{a} = \mathbf{x} \cdot \mathbf{a} + \mathbf{x} \wedge \mathbf{a}$	(8)
Associativity	$A(BC) = (AB)C$	(9)



2 **Invertibility of the geometric product. The inner product determines a plane, the outer product determines a line, but the geometric product determines a unique vector and is therefore invertible.**

We have thus defined the geometric product in terms of the inner and outer product, but we claimed that it is more fundamental than either. Mathematically, it is more elegant to replace Equation 8 with "the square of a vector **x** is a scalar $Q(\mathbf{x})$." We can then actually interpret this function Q as the metric of the space, the same as the one we used in the inner product, and it gives the same geometric algebra.[3] Our choice for Equation 8 was to define the new product in terms of more familiar quantities, to aid intuitive understanding of it.

Let us show by example how we can use these rules to develop the geometric algebra of 3D Euclidean space. We introduce, for convenience only, an orthonormal basis $\{e_i\}_{i=1}^3$. Because this implies that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, we get the commutation rules

$$\mathbf{e}_i\mathbf{e}_j = \begin{cases} -\mathbf{e}_j\mathbf{e}_i & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{10}$$

In fact, the former is equal to $\mathbf{e}_i \wedge \mathbf{e}_j$, whereas the latter equals $\mathbf{e}_i \cdot \mathbf{e}_i$. Considering the unit 2-blade $\mathbf{e}_i \wedge \mathbf{e}_j$, we find its square:

$$\begin{aligned}\left(\mathbf{e}_i \wedge \mathbf{e}_j\right)^2 &= \left(\mathbf{e}_i \wedge \mathbf{e}_j\right)\left(\mathbf{e}_i \wedge \mathbf{e}_j\right) = \left(\mathbf{e}_i\mathbf{e}_j\right)\left(\mathbf{e}_i\mathbf{e}_j\right) \\ &= \mathbf{e}_i\mathbf{e}_j\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_i\mathbf{e}_i\mathbf{e}_j\mathbf{e}_j = -1\end{aligned} \tag{11}$$

So a unit 2-blade squares to –1. Continued application of Equation 10 gives the full multiplication for all basis elements in the Clifford algebra of 3D space. The resulting multiplication table is in Table 1 (next page). We can express arbitrary elements as a linear combination of these basis elements, so Table 1 determines the full algebra.

Exponential representation. Note that the geometric product is sensitive to the vectors' relative directions. For parallel vectors **a** and **b**, the outer product contribution is zero, and **ab** is a scalar and commutative in its factors. For perpendicular vectors, **ab** is a 2-blade and anticommutative. In general, if the angle between **a** and **b** is ϕ in their common plane with unit 2-blade **I**, we can write (in a Euclidean space)

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} = |\mathbf{a}||\mathbf{b}|\left(\cos\phi + \mathbf{I}\sin\phi\right) \tag{12}$$

Table 1. The multiplication table of the geometric algebra of 3D Euclidean space, on an orthonormal basis. (For shorthand, $e_{12} \equiv e_1 \wedge e_2$ and so forth.)

	1	e_1	e_2	e_3	e_{12}	e_{31}	e_{23}	e_{123}
1	1	e_1	e_2	e_3	e_{12}	e_{31}	e_{23}	e_{123}
e_1	e_1	1	e_{12}	$-e_{31}$	e_2	$-e_3$	e_{123}	e_{23}
e_2	e_2	$-e_{12}$	1	e_{23}	$-e_1$	e_{123}	e_3	e_{31}
e_3	e_3	e_{31}	$-e_{23}$	1	e_{123}	e_1	$-e_2$	e_{12}
e_{12}	e_{12}	$-e_2$	e_1	e_{123}	-1	e_{23}	$-e_{31}$	$-e_3$
e_{31}	e_{31}	e_3	e_{123}	$-e_1$	$-e_{23}$	-1	e_{12}	$-e_2$
e_{23}	e_{23}	e_{123}	$-e_3$	e_2	e_{31}	$-e_{12}$	-1	$-e_1$
e_{123}	e_{123}	e_{23}	e_{31}	e_{12}	$-e_3$	$-e_2$	$-e_1$	-1

using a common rewriting of the inner product and Equation 3. We have already seen that $\mathbf{II} = -1$, and this permits the shorthand of the exponential notation (by the usual definition of the exponential as a converging series of terms):

$$\mathbf{ab} = |\mathbf{a}|\,|\mathbf{b}|\left(\cos\phi + \mathbf{I}\sin\phi\right) = |\mathbf{a}|\,|\mathbf{b}|\,e^{\mathbf{I}\phi} \qquad (13)$$

All this might remind you of complex numbers, but it is different. First, geometric algebra has given a straightforward real geometrical interpretation of all elements occurring in this equation, notably of $\mathbf{I}$ as the unit area element of the common plane of $\mathbf{a}$ and $\mathbf{b}$. Second, the math differs. If $\mathbf{I}$ were a complex scalar, it would have to commute with all elements of the algebra by Equation 7, but instead, it satisfies $\mathbf{aI} = -\mathbf{Ia}$ for vectors $\mathbf{a}$ in the $\mathbf{I}$ plane. We will use the exponential notation a lot when we study rotations in part two.

Many grades in the geometric product. Equation 8 implies that the geometric product of a vector with itself is a scalar. Therefore, when you multiply two blades, the vectors in them may multiply to a scalar (if they are parallel) or to a 2-blade (if they are not). As a consequence, when you multiply two blades of grade k and l using the geometric product, the result potentially contains parts of all grades $(k + l)$, $(k + l - 2)$, …, $(|k - l| + 2)$, $|k - l|$, depending on how their factors align. This series of terms contains all information about the geometrical relationships of the blades—their span, intersection, relative orientation, and so on.

Inner product of blades

In geometric algebra, we can see the standard inner product of two vectors as the symmetrical part of their geometric product:

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}\left(\mathbf{ab} + \mathbf{ba}\right)$$

Just as in the usual definition, this embodies the metric of the vector space, and we can use it to define distances. It also codifies the perpendicularity required in projection operators. Now that we view vectors as representatives of 1D subspaces, we want to extend this metric capability to arbitrary subspaces. We can generalize the inner product to general subspaces in several ways. Lounesto[3] and Dorst[4] explain the mathematically most tidy method. This is the *contraction inner product* (denoted $\rfloor$), which has a clean geometric meaning. In this intuitive introduction, we prefer to give the geometric meaning first:

> $\mathbf{A} \rfloor \mathbf{B}$ is a blade representing the complement (within the subspace $\mathbf{B}$) of the orthogonal projection of $\mathbf{A}$ onto $\mathbf{B}$. It is linear in $\mathbf{A}$ and $\mathbf{B}$, and it coincides with the usual inner product $\mathbf{a} \cdot \mathbf{b}$ of V^m when computed for vectors $\mathbf{a}$ and $\mathbf{b}$.

This statement determines our inner product uniquely. (The resulting contraction inner product differs slightly from the inner product commonly used in the geometric algebra literature. The contraction inner product has a cleaner geometric semantics and more compact mathematical properties, which makes it better suited to computer science. We can express the two inner products in terms of each other, so this is not a severely divisive issue. They codify the same geometric concepts, in just slightly different ways.) The contraction inner product turns out not to be symmetrical or associative. But we do demand linearity to make it computable between any two elements in our linear space (not just blades). Note that earlier on we used only the inner product between vectors $\mathbf{a} \cdot \mathbf{b}$, which we would now write as $\mathbf{a} \rfloor \mathbf{b}$.

Here, we just give the rules for computing the resulting inner product for arbitrary blades, omitting their derivation. In the following, α and β are scalars; $\mathbf{a}$ and $\mathbf{b}$ vectors; and A, B, and C general elements of the algebra:

Scalars

$$\alpha \rfloor \beta = \alpha\,\beta \qquad (14)$$

Vector and scalar

$$\mathbf{a} \rfloor \beta = 0 \qquad (15)$$

Scalar and vector

$$\alpha \rfloor \mathbf{b} = \alpha\,\mathbf{b} \qquad (16)$$

Vectors

$\mathbf{a} \rfloor \mathbf{b}$ is the usual inner product $\mathbf{a} \cdot \mathbf{b}$ in V^m (17)

Vector and element

$$\mathbf{a} \rfloor (\mathbf{b} \wedge B) = (\mathbf{a} \rfloor \mathbf{b}) \wedge B - \mathbf{b} \wedge (\mathbf{a} \rfloor B) \qquad (18)$$

Distribution

$$(A \wedge B) \rfloor C = A \rfloor (B \rfloor C) \qquad (19)$$

As we said, linearity and distributivity over + also hold, but the inner product is not associative. The inner product of two blades is again a blade[5] (as we would hope because they represent subspaces and so should the result). It is easy to see that the grade of this blade is

$$\text{grade}(\mathbf{A} \rfloor \mathbf{B}) = \text{grade}(\mathbf{B}) - \text{grade}(\mathbf{A}) \qquad (20)$$

because the projection of $\mathbf{A}$ onto $\mathbf{B}$ has the same grade as $\mathbf{A}$, and its complement in $\mathbf{B}$ is the codimension of this projection in the subspace spanned by $\mathbf{B}$. Because no subspace has a negative dimension, the contraction $\mathbf{A} \rfloor \mathbf{B}$ is zero when grade ($\mathbf{A}$) > grade ($\mathbf{B}$) (and this is the main difference between the contraction and the other inner product).

When used on blades as $(\mathbf{A} \wedge \mathbf{B}) \rfloor \mathbf{C}) = \mathbf{A} \rfloor (\mathbf{B} \rfloor \mathbf{C})$, Equation 19 gives the inner product its meaning of being the perpendicular part of one subspace inside another. In words, it would read like this: to get the part of **C** perpendicular to the subspace that is the span of **A** and **B**, take the part of **C** perpendicular to **B**; then of that, take the part perpendicular to **A**.

Figure 3 gives this example: the inner product of a vector **a** and a 2-blade **B**, producing the vector $\mathbf{a} \rfloor \mathbf{B}$. Note that the usual inner product for vectors **a** and **b** has the right semantics. The subspace that is the orthogonal complement (in the space spanned by **b**) of the projection of **a** onto **b** contains only the point at their common origin and is therefore represented by a scalar (0-blade) linear in **a** and **b**.

With the definition of the inner product for blades, we can generalize the relationship in Equation 8 between a geometric product and its inner and outer product parts. For a vector **x** and a blade **A**, we have

$$\mathbf{xA} = \mathbf{x} \rfloor \mathbf{A} + \mathbf{x} \wedge \mathbf{A} \qquad (21)$$

Note that if the first argument **x** is not a vector, this formula does not apply. In general, the geometric product of two blades contains many more terms, which we can write as interleavings of the inner and outer products of vectors spanning the blades.

Outer product

Once we have the geometric product, it is better to see the outer product as its antisymmetric part:

$$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}\left(\mathbf{ab} - \mathbf{ba}\right)$$

and, more generally, if the second factor is a blade,

$$\mathbf{a} \wedge \mathbf{B} = \frac{1}{2}\left(\mathbf{aB} + \left(-1\right)^{\mathrm{grade}(\mathbf{B})}\mathbf{Ba}\right) \qquad (22)$$

This leads to the defining properties we saw before:

Scalars
$$\alpha \wedge \beta = \alpha\,\beta \qquad (23)$$

Scalar and vector
$$\alpha \wedge \mathbf{b} = \alpha\,\mathbf{b} \qquad (24)$$

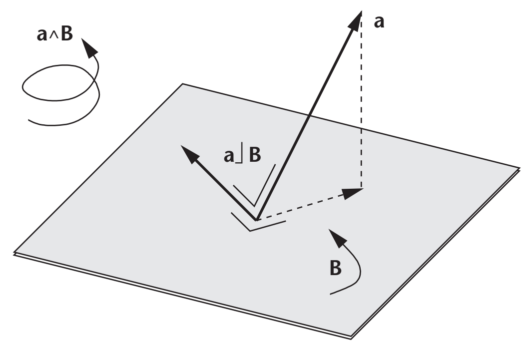
Antisymmetry for vectors
$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a} \qquad (25)$$

Associativity
$$(A \wedge B) \wedge C = A \wedge (B \wedge C) \qquad (26)$$

(As before, α and β are scalars; **a** and **b** are vectors; and *A*, *B*, and *C* are general elements.) Linearity and distributivity over + also hold.The grade of a *k*-blade is the number of vector factors that span it. Therefore, the grade of an outer product of two blades is

$$\mathrm{grade}\,(\mathbf{A} \wedge \mathbf{B}) = \mathrm{grade}\,(\mathbf{A}) + \mathrm{grade}\,(\mathbf{B}) \qquad (27)$$

Of course, the outcome can be 0, so we should view this zero element of the algebra as an element of an arbitrary grade. There is then no need to distinguish separate zero scalars, zero vectors, zero 2-blades, and so forth.



3 **The inner product of blades: the inner product of a vector and a plane is a perpendicular vector in the plane. (The corkscrew denotes the orientation of the space's volume element.)**

Subspace objects without shape. We reiterate that the outer product of *k*-vectors gives a bit of *k*-space, in a manner that includes the space element's attitude, orientation (or handedness), and magnitude. Equation 3 conveys this for a 2-blade $\mathbf{a} \wedge \mathbf{b}$.

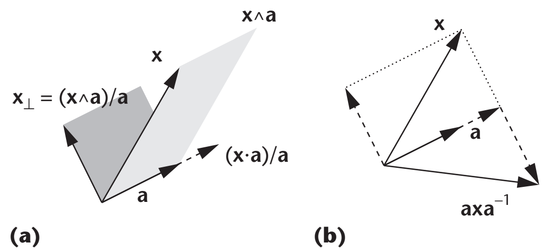
Yet $\mathbf{a} \wedge \mathbf{b}$ is not an area element with well-defined shape, even though we are tempted to draw it as a parallelogram (as in Figure 1c). For instance, by the outer product's properties, $\mathbf{a} \wedge \mathbf{b} = \mathbf{a} \wedge (\mathbf{b} + \lambda\mathbf{a})$, for any λ, so $\mathbf{a} \wedge \mathbf{b}$ is just as much the parallelogram spanned by **a** and $\mathbf{b} + \lambda\mathbf{a}$. Playing around, you find that you can move around pieces of the area elements while still maintaining the same product $\mathbf{a} \wedge \mathbf{b}$. So really, a bivector does not have any fixed shape or position; it is just a chunk of a precisely defined amount of 2D directed area in a well-defined plane. It follows that the 2-blades have an existence of their own, independent of any vectors that we might use to define them.

We will take these nonspecific shapes made by the outer product and force them into shape with carefully chosen geometric products. This will turn out to be a powerful and flexible technique to get closed coordinate-free computational expressions for geometrical constructions.

Linear (in)dependence. Note that if three vectors are linearly dependent, they satisfy

$$\mathbf{a}, \mathbf{b}, \mathbf{c}, \text{ linearly dependent} \Leftrightarrow \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 0$$

We interpret the latter immediately as the geometric statement that the vectors span a zero volume. This makes linear dependence a computational property rather than a predicate—three vectors can be almost linearly dependent. The magnitude of $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ obviously involves the



4 (a) Projecting and rejecting x relative to a. (b) Reflecting x in a.

determinant of the matrix ($\mathbf{a}\ \mathbf{b}\ \mathbf{c}$), so this view corresponds with the usual computation of determinants to check degeneracy.

Solving geometric equations

The geometric product is invertible, so dividing by a vector has a unique meaning. We usually do this through multiplication by the inverse of the vector. Because multiplication is not necessarily commutative, we must be careful; there is a left and right division. As you can verify, the unique inverse of a vector $\mathbf{a}$ is

$$\mathbf{a}^{-1} = \frac{\mathbf{a}}{\mathbf{a} \rfloor \tilde{\mathbf{a}}}$$

because that is the unique element that satisfies $\mathbf{a}^{-1}\mathbf{a} = 1 = \mathbf{a}\mathbf{a}^{-1}$. In general, an element A has the inverse

$$A^{-1} = \frac{A}{A \rfloor \tilde{A}}$$

where $\tilde{A}$ is the reverse of A, obtained by switching its spanning factors for each grade in A. So if A is a k-blade $\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_k$, then $\tilde{A} = \mathbf{a}_k \wedge \ldots \wedge \mathbf{a}_2 \wedge \mathbf{a}_1$. You can verify that $A \rfloor \tilde{A}$ is a scalar (and in Euclidean space, even a positive scalar, which we can consider to be the norm squared of A; if it is zero, the element A has no inverse, but this does not happen for blades in Euclidean spaces).

Invertibility is a great help in solving geometric problems in a closed coordinate-free computational form. The common procedure is as follows: we know certain defining properties of objects in the usual terms of perpendicularity, spanning, rotations, and so on. These give equations typically expressed using the derived products. We combine these equations algebraically, with the goal of finding an expression for the unknown object involving only the geometric product; then division (permitted by the invertibility of the geometric product) should provide the result.

Let us illustrate this with an example. Suppose we want to find the component $\mathbf{x}_\perp$ of a vector $\mathbf{x}$ perpendicular to a vector $\mathbf{a}$. The perpendicularity demand is clearly

$$\mathbf{x}_\perp \rfloor \mathbf{a} = 0$$

A second demand is required to relate the magnitude of $\mathbf{x}_\perp$ to that of $\mathbf{x}$. Some practice in seeing subspaces in geometrical problems reveals that the area spanned by $\mathbf{x}$ and $\mathbf{a}$ is the same as the area spanned by $\mathbf{x}_\perp$ and $\mathbf{a}$ (see Figure 4a). We express this using the outer product: $\mathbf{x}_\perp \wedge \mathbf{a} = \mathbf{x} \wedge \mathbf{a}$.

We combine these two equations to form a geometric product. In this example, just adding the two equations works, yielding

$$\mathbf{x}_\perp \rfloor \mathbf{a} + \mathbf{x}_\perp \wedge \mathbf{a} = \mathbf{x}_\perp \mathbf{a} = \mathbf{x} \wedge \mathbf{a}$$

This one equation contains the full geometric relationship between $\mathbf{x}$, $\mathbf{a}$, and the unknown $\mathbf{x}_\perp$. Geometric algebra solves this equation through division on the right by $\mathbf{a}$:

$$\mathbf{x}_\perp = (\mathbf{x} \wedge \mathbf{a})/\mathbf{a} = (\mathbf{x} \wedge \mathbf{a})\,\mathbf{a}^{-1} \qquad (28)$$

We rewrote the division by $\mathbf{a}$ as multiplication by the subspace $\mathbf{a}^{-1}$ to clearly show that we mean division on the right.

This is an example of how the indefinite shape $\mathbf{x} \wedge \mathbf{a}$ spanned by the outer product is just the right element to generate a perpendicular to a vector $\mathbf{a}$ in its plane, through the geometric product. Note that Equation 28 agrees with the well-known expression $\mathbf{x}_\perp$ using the inner product of vectors:

$$\mathbf{x}_\perp = \left(\mathbf{x} \wedge \mathbf{a}\right)\mathbf{a}^{-1} = \left(\mathbf{x}\mathbf{a} - \mathbf{x}\cdot\mathbf{a}\right)\mathbf{a}^{-1} = \mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} \qquad (29)$$

The geometric algebra expression using outer product and inverse generalizes immediately to arbitrary subspaces $\mathbf{A}$.

Projecting subspaces

We generalize this technique as the decomposition of a vector to an arbitrary blade $\mathbf{A}$, using the geometric product decomposition of Equation 21:

$$\mathbf{x} = (\mathbf{x}\mathbf{A})\mathbf{A}^{-1} = (\mathbf{x} \rfloor \mathbf{A})\mathbf{A}^{-1} + (\mathbf{x} \wedge \mathbf{A})\mathbf{A}^{-1} \qquad (30)$$

We can show that the first term is a blade fully inside $\mathbf{A}$—it is the projection of $\mathbf{x}$ onto $\mathbf{A}$. Likewise, we can show that the second term is a vector perpendicular to $\mathbf{A}$, sometimes called the rejection of $\mathbf{x}$ by $\mathbf{A}$. The projection of a blade $\mathbf{X}$ of arbitrary dimensionality (grade) onto a blade $\mathbf{A}$ is given by

$$\text{projection of } \mathbf{X} \text{ onto } \mathbf{A}: \quad \mathbf{X} \mapsto (\mathbf{X} \rfloor \mathbf{A})\mathbf{A}^{-1}$$

Geometric algebra often allows such a straightforward extension to arbitrary dimensions of subspaces, without additional computational complexity. (We will see why when we treat linear mappings in part two.)

Reflecting subspaces

The reflection of a vector $\mathbf{x}$ relative to a fixed vector $\mathbf{a}$ can be constructed from the decomposition of Equation 30 (used for a vector $\mathbf{a}$) by changing the sign of the rejection (see Figure 4b). We can rewrite this in terms of the geometric product:

$$\begin{aligned}(\mathbf{x} \rfloor \mathbf{a})\mathbf{a}^{-1} - (\mathbf{x} \wedge \mathbf{a})\mathbf{a}^{-1} &= (\mathbf{a} \rfloor \mathbf{x} + \mathbf{a} \wedge \mathbf{x})\,\mathbf{a}^{-1} \\ &= \mathbf{a}\mathbf{x}\mathbf{a}^{-1} \end{aligned} \qquad (31)$$

So the reflection of $\mathbf{x}$ in $\mathbf{a}$ is the expression $\mathbf{a}\mathbf{x}\mathbf{a}^{-1}$ (see Figure 4b), and the reflection in a plane perpendicular to **a** is then $-\mathbf{axa}^{-1}$. (We will see this sandwiching operator in more detail in part two.)

We can extend this formula to the reflection of a blade **X** relative to the vector **a**. This is

$$\text{reflection in vector } \mathbf{a}\text{:} \ \mathbf{X} \mapsto \mathbf{aXa}^{-1}$$

and even to the reflection of a blade **X** in a k-blade **A**, which turns out to be

$$\text{general reflection:} \ \mathbf{X} \mapsto -(-1)^k \mathbf{AXA}^{-1}$$

Note that these formulas let us do reflections of subspaces without first decomposing them into constituent vectors. It gives the possibility of reflecting a polyhedral object by directly using a facet representation, rather than acting on individual vertices.

Vector division

With subspaces as basic computational elements, we can directly solve equations in similarity problems such as Figure 5 suggests:

> Given two vectors **a** and **b**, and a third vector **c**, determine **x** so that **x** is to **c** as **b** is to **a**—that is, solve $\mathbf{x} : \mathbf{c} = \mathbf{b} : \mathbf{a}$.

In geometric algebra the problem reads $\mathbf{x}\,\mathbf{c}^{-1} = \mathbf{b}\,\mathbf{a}^{-1}$, and through right multiplication by **c**, the solution is

$$\mathbf{x} = (\mathbf{ba}^{-1})\,\mathbf{c} \tag{32}$$

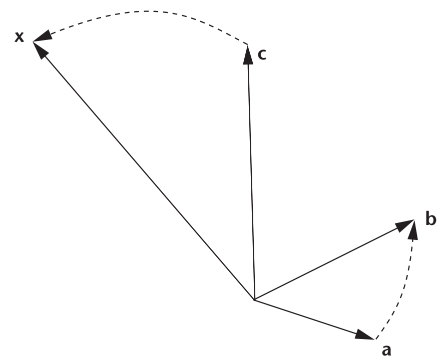
This is a computable expression. For instance, with $\mathbf{a} = \mathbf{e}_1$, $\mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2$, and $\mathbf{c} = \mathbf{e}_2$ in the standard orthonormal basis, we obtain

$$\mathbf{x} = \left(\left(\mathbf{e}_1 + \mathbf{e}_2\right)\mathbf{e}_1^{-1}\right)\mathbf{e}_2 = \left(1 - \mathbf{e}_1\mathbf{e}_2\right)\mathbf{e}_2 = \mathbf{e}_2 - \mathbf{e}_1$$

In part two, we will develop this into a method to handle rotations.

Conclusion

In this article, we've introduced blades and three products of geometric algebra. The geometric product is the most important because it is the only one that is invertible. We hope that this introduction has given readers a hint of geometric algebra's structure. In part two, we will show how to wield these products to construct operations like rotations and look at more advanced topics such as differentiation, linear algebra, and homogeneous representations. ■



5 **The ratio of vectors determines a rotation/dilation operator.**

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