Function approximation

If you need to evaluate a complicated function f very quickly, many times around the same point $x = x_0$, or if you would like to understand the local behavior around that point, then it is convenient to write the function as a polynomial. After all, polynomials can be computed quickly, and are simple to draw and visualize.

A *Taylor approximation* provides the technique to do this. The Taylor series in 1 dimension is:

$$f(x_0 + \epsilon) = f(x_0) + f'(x_0)\epsilon + f''(x_0)\frac{\epsilon^2}{2} + f'''(c_\epsilon)\frac{\epsilon^3}{6}, \quad \text{with } x_0 \le c_\epsilon \le x_0 + \epsilon. \qquad (1)$$

We play with it in this problem, for the function $\sin(x)$, around $x = 0$.

(1) Give the terms to 5th order of the Taylor series development of $\sin(x)$ around the point $x = 0$.

Answer $f(0 + \epsilon) \approx \epsilon - \frac{\epsilon^3}{6} + \frac{\epsilon^5}{120}$.

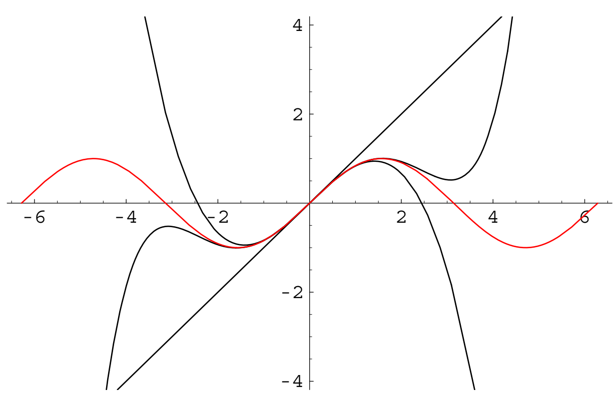
(2) The first (and second!) order approximation is $\sin(\epsilon) \approx \epsilon$. This is reasonable in a small interval. When is the 'error' made smaller than 0.01 ?

Answer: use the third order term to bound the error; $|\frac{\epsilon^3}{6}| < 0.01$, so $|\epsilon| < 0.4$.

(3) The third (and fourth) order approximation is $\sin(\epsilon) \approx \epsilon - \frac{\epsilon^3}{6}$. This is reasonable in quite an interval. When is the 'error' made smaller than 1% ?

Answer: use the fifth order term to bound the error; $|\frac{\epsilon^5}{120}| < 0.01$, so $|\epsilon| < 1.0$.

(4) This figure shows what is going on. In red: $\sin(x)$, and in black: the subsequent Taylor approximations.



(5) How many extrema may a polynomial of order n have? So which curve is which approximation? Using this insight, how many terms would you need at least to make a good approximation of a full period $x \in (-\pi, \pi)$ of the sine function? And is that a good approximation?

(6) The polynomial you get for the Taylor approximation of the sine function depends on where you approximate it. Give the approximation to 5-th order of $\sin(\frac{\pi}{4} + \epsilon)$ and compare to **(1)**. Before doing so, do you expect an ϵ^2 term?

Answer: $sin(\frac{\pi}{4} + \epsilon) \approx \frac{1}{\sqrt{2}}(1 + \epsilon - \frac{1}{2}\epsilon^2 - \frac{1}{6}\epsilon^3 + \frac{1}{24}\epsilon^4 + \frac{1}{120}\epsilon^5)$. And you should have expected an ϵ^2-term since the sine function is not locally straight at $\frac{\pi}{4}$.

(7) Now apply Taylor series in two dimensions to computer vision, in `taylor2.ps`!