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THE RELATIVISTIC ENERGY-MOMENTUM TENSOR IN POLARIZED MEDIA

I. THE ATOMIC ENERGY-MOMENTUM CONSERVATION LAWS

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Synopsis

The relativistic conservation laws of energy and momentum in a substance, which consists of atoms carrying both electric and magnetic dipoles, is derived from the microscopic force laws for point particles in the presence of an electromagnetic field.

The energy-momentum tensor which occurs in these "atomic" conservation laws, consists of a material part and a field part, for which explicit expressions are obtained.

§ 1. *Introduction.* Since the beginning of the century the correct form of the electromagnetic energy-momentum tensor in polarized media has been a controversial issue. Forms, based on electron theory, were obtained by Lorentz[1]) (already before relativity theory existed) and by Einstein and Laub[2]); different expressions were put forward by Minkowski[3]), Abraham[4]) and others[5]). None of these authors gave explicit expressions for the material part of the energy-momentum tensor. As a result of this the field part of the energy-momentum tensor could not be determined in a unique way.

A solution of the problem can only be given if one starts from the microscopic energy-momentum laws for a system of charged point particles (electrons and nuclei). The point particles are grouped into atoms (or other stable groups such as molecules, ions etc.), which in the present treatment are supposed to carry electric and magnetic dipole moments. The derivation proceeds in two steps: first (in this paper) the energy-momentum laws on the "atomic (kinetic) level" are derived; then by covariant statistical averaging one obtains the macroscopic energy-momentum conservation laws, which contain the material and field parts of the macroscopic energy-momentum tensor[6]).

§ 2. *The microscopic force law.* The stable groups of electrons and nuclei of which the system consists, will be labelled by an index k and referred to as "atoms". The constituent particles of the "atoms" are numbered with a second index i. The microscopic force law for the constituent particle

ki with mass m_{ki}, charge e_{ki} and time-space coordinates $R^\alpha_{ki} = (c\,t_{ki}, \boldsymbol{R}_{ki})$ reads:

$$cm_{ki}\frac{\mathrm{d}}{\mathrm{d}\tau_{ki}}\left(\frac{\mathrm{d}R^\alpha_{ki}}{\mathrm{d}\tau_{ki}}\right) = e_{ki}\,f^{\alpha\beta}_{(t)}(R_{ki})\,\frac{\mathrm{d}R_{ki\beta}}{\mathrm{d}\tau_{ki}} \quad (\alpha = 0, 1, 2, 3), \tag{1}$$

where τ_{ki} is the proper time of ki (given by $c^2\,\mathrm{d}\tau^2_{ki} = -\mathrm{d}R^\alpha_{ki}\,\mathrm{d}R_{ki\alpha}$; we use the metric $g^{00} = -1$, $g^{ii} = 1$ for $i = 1, 2, 3$, $g^{\alpha\beta} = 0$ for $\alpha \neq \beta$) and $f^{\alpha\beta}_{(t)}$ the total electromagnetic field which contains contributions due to the other particles of the system and possibly an external field acting on the system. The components of the (antisymmetric) field tensor are $(f^{01}_{(t)}, f^{02}_{(t)}, f^{03}_{(t)}) = \boldsymbol{e}_{(t)}$ and $(f^{23}_{(t)}, f^{31}_{(t)}, f^{12}_{(t)}) = \boldsymbol{b}_{(t)}$.

Let us introduce a privileged point R^α_k of atom k, characterizing the motion of the atom as a whole (with $\mathrm{d}R^\alpha_k$ supposed to be time-like). The proper time s along the trajectory R^α_k is given by $c^2\,\mathrm{d}s^2 = -\mathrm{d}R^\alpha_k\,\mathrm{d}R_{k\alpha}$. We now want to introduce a parametrization s along the trajectories of the constituent particles ki through a "perpendicular" projection from the trajectory of the privileged point of atom k. This can be done with the help of the relation:

$$\{R_{ki\alpha}(s) - R_{k\alpha}(s)\}\frac{\mathrm{d}R^\alpha_k(s)}{\mathrm{d}s} = 0. \tag{2}$$

We shall call $R_{ki\alpha}(s) - R_{k\alpha}(s)$ the internal coordinate $r_{ki\alpha}(s)$ of ki with respect to k. Then (2) reads:

$$r_{ki\alpha}(s)\frac{\mathrm{d}R^\alpha_k(s)}{\mathrm{d}s} = 0. \tag{3}$$

This relation expresses the fact that in the (momentary) rest frame of point k the internal coordinate $r_{ki\alpha}$ is purely space-like.

The equation (1) can be written with the help of a four-dimensional delta-function:

$$\sum_i c\int m_{ki}\frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{\mathrm{d}R^\alpha_{ki}}{\mathrm{d}s}\left(\frac{\mathrm{d}\tau_{ki}}{\mathrm{d}s}\right)^{-1}\right\}\delta(R_k - R)\,\mathrm{d}s =$$

$$= \sum_i\int e_{ki}\,f^{\alpha\beta}_{(t)}(R_{ki})\frac{\mathrm{d}R_{ki\beta}}{\mathrm{d}s}\,\delta(R_k - R)\,\mathrm{d}s. \tag{4}$$

By a Taylor-expansion of $(\mathrm{d}\tau_{ki}/\mathrm{d}s)^{-1}$ in powers of r_{ki} one obtains up to second order in r_{ki}:

$$\left(\frac{\mathrm{d}\tau_{ki}}{\mathrm{d}s}\right)^{-1} = 1 + \frac{1}{c^2}\frac{\mathrm{d}r_{ki\alpha}}{\mathrm{d}s}\frac{\mathrm{d}R^\alpha_k}{\mathrm{d}s} +$$

$$+ \frac{1}{2c^2}\frac{\mathrm{d}r_{ki\alpha}}{\mathrm{d}s}\frac{\mathrm{d}r^\alpha_{ki}}{\mathrm{d}s} + \frac{3}{2c^4}\frac{\mathrm{d}r_{ki\alpha}}{\mathrm{d}s}\frac{\mathrm{d}R^\alpha_k}{\mathrm{d}s}\frac{\mathrm{d}r_{ki\beta}}{\mathrm{d}s}\frac{\mathrm{d}R^\beta_k}{\mathrm{d}s}. \tag{5}$$

With the help of this relation equation (4) becomes up to second order in r_{ki}:

$$\sum_i c \int m_{ki} \frac{\mathrm{d}^2 R_{ki}^\alpha}{\mathrm{d}s^2} \delta(R_k - R)\,\mathrm{d}s =$$

$$= \sum_i \int e_{ki} f_{(i)}^{\alpha\beta}(R_{ki}) \frac{\mathrm{d}R_{ki\beta}}{\mathrm{d}s} \delta(R_k - R)\,\mathrm{d}s$$

$$- \sum_i \frac{1}{c} \int m_{ki} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}R_k^\alpha}{\mathrm{d}s} \frac{\mathrm{d}r_{ki\beta}}{\mathrm{d}s} \frac{\mathrm{d}R_k^\beta}{\mathrm{d}s}\right)\delta(R_k - R)\,\mathrm{d}s$$

$$- \sum_i \frac{1}{c} \int m_{ki} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}r_{ki}^\alpha}{\mathrm{d}s} \frac{\mathrm{d}r_{ki\beta}}{\mathrm{d}s} \frac{\mathrm{d}R_k^\beta}{\mathrm{d}s}\right)\delta(R_k - R)\,\mathrm{d}s$$

$$- \sum_i \frac{1}{2c} \int m_{ki} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}R_k^\alpha}{\mathrm{d}s} \frac{\mathrm{d}r_{ki\beta}}{\mathrm{d}s} \frac{\mathrm{d}r_{ki}^\beta}{\mathrm{d}s}\right)\delta(R_k - R)\,\mathrm{d}s$$

$$- \sum_i \frac{3}{2c^3} \int m_{ki} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}R_k^\alpha}{\mathrm{d}s} \frac{\mathrm{d}r_{ki\beta}}{\mathrm{d}s} \frac{\mathrm{d}R_k^\beta}{\mathrm{d}s} \frac{\mathrm{d}r_{ki\gamma}}{\mathrm{d}s} \frac{\mathrm{d}R_k^\gamma}{\mathrm{d}s}\right)\delta(R_k - R)\,\mathrm{d}s. \quad (6)$$

Now the privileged point k of atom k can be specified by defining it as the centre of gravity of atom k in the momentary rest frame of one of the constituent particles ki say kj (with fixed j). Since this can be done at every moment, it determines in a unique way the path of point k. In fact, if s_j is defined by

$$\{R_{kj\alpha}(s_j) - R_{k\alpha}(s)\}\left(\frac{\mathrm{d}R_{kj}^\alpha}{\mathrm{d}s}\right)_{s=s_j} = 0, \quad (7)$$

and then s_i (with $i \neq j$) by

$$\{R_{ki\alpha}(s_i) - R_{kj\alpha}(s_j)\}\left(\frac{\mathrm{d}R_{kj}^\alpha}{\mathrm{d}s}\right)_{s=s_j} = 0, \quad (8)$$

the centre of gravity in the momentary rest frame of kj is given by:

$$m_k R_k(s) \equiv \sum_i m_{ki} R_k(s) = \sum_i m_{ki} R_{ki}(s_i). \quad (9)$$

By a Taylor expansion of $R_{ki}(s_i)$ around $R_{ki}(s)$ one obtains from (7) for $i = j$ and from (8) for $i \neq j$, retaining terms up to second order in r_{ki}:

$$s_i - s = \frac{1}{c^2} r_{ki\alpha} \frac{\mathrm{d}r_{kj}^\alpha}{\mathrm{d}s}. \quad (10)$$

Expanding now $R_{ki}(s_i)$ around $R_{ki}(s)$ in (9) one gets with (10):

$$\sum_i m_{ki} R_k(s) = \sum_i m_{ki} R_{ki}(s) + O(3),$$

where $O(3)$ is of third order in r_{ki}. Thus

$$\sum_i m_{ki}\, r_{ki}(s) = 0, \tag{11}$$

if terms of third and higher order are neglected.

§ 3. *The force equation up to second order in the internal coordinates.* Let us write $R_k + r_{ki}$ instead of R_{ki} in (6), except in the field $f_{(t)}(R_{ki})$, which will be considered in the next section. Then with the help of (3) (and its derivative with respect to s), (11) and the equation of motion (1), which in the calculation is only needed up to first order:

$$cm_{ki}\frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{\mathrm{d}R_k^\alpha}{\mathrm{d}s}\left(1+\frac{1}{c^2}\frac{\mathrm{d}r_{ki\beta}}{\mathrm{d}s}\frac{\mathrm{d}R_k^\beta}{\mathrm{d}s}\right)+\frac{\mathrm{d}r_{ki}^\alpha}{\mathrm{d}s}\right\} = e_{ki}\, f_{(t)}^{\alpha\beta}(R_{ki})\frac{\mathrm{d}R_{ki\beta}}{\mathrm{d}s}, \tag{12}$$

or even zeroth order:

$$cm_{ki}\frac{\mathrm{d}^2R_k^\alpha}{\mathrm{d}s^2} = e_{ki}\, f_{(t)}^{\alpha\beta}(R_{ki})\frac{\mathrm{d}R_{k\beta}}{\mathrm{d}s}, \tag{13}$$

the force law (6) may be written, retaining terms up to second order in r_{ki} only:

$$\begin{aligned}
&c\int m_k\frac{\mathrm{d}^2R_k^\alpha}{\mathrm{d}s^2}\,\delta(R_k-R)\,\mathrm{d}s\\
&+\sum_i\frac{1}{2c}\int m_{ki}\frac{\mathrm{d}^2R_k^\alpha}{\mathrm{d}s^2}\frac{\mathrm{d}r_{ki\beta}}{\mathrm{d}s}\frac{\mathrm{d}r_{ki}^\gamma}{\mathrm{d}s}\,\Delta_{k\gamma}^\beta\,\delta(R_k-R)\,\mathrm{d}s\\
&+\sum_i\frac{1}{c}\int m_{ki}\frac{\mathrm{d}}{\mathrm{d}s}\left\{\Delta_{k\beta}^\alpha\left(r_{ki}^\beta\frac{\mathrm{d}r_{ki}^\gamma}{\mathrm{d}s}-r_{ki}^\gamma\frac{\mathrm{d}r_{ki}^\beta}{\mathrm{d}s}\right)\frac{\mathrm{d}^2R_{k\gamma}}{\mathrm{d}s^2}\right\}\delta(R_k-R)\,\mathrm{d}s =\\
&=\sum_i\int e_{ki}\, f_{(t)}^{\alpha\beta}(R_{ki})\frac{\mathrm{d}R_{k\beta}}{\mathrm{d}s}\,\delta(R_k-R)\,\mathrm{d}s\\
&+\sum_i\int e_{ki}\,\Delta_{k\beta}^\alpha\, f_{(t)}^{\beta\gamma}(R_{ki})\frac{\mathrm{d}r_{ki\gamma}}{\mathrm{d}s}\,\delta(R_k-R)\,\mathrm{d}s\\
&+\sum_i\frac{1}{c^2}\int e_{ki}\frac{\mathrm{d}}{\mathrm{d}s}\left(r_{ki}^\alpha\frac{\mathrm{d}r_{ki}^\gamma}{\mathrm{d}s}\frac{\mathrm{d}R_k^\varepsilon}{\mathrm{d}s}\right) f_{(t)\gamma\varepsilon}(R_{ki})\,\delta(R_k-R)\,\mathrm{d}s\\
&+\sum_i\frac{1}{c^2}\int e_{ki}\,\Delta_{k\beta}^\alpha\, r_{ki}^\beta\frac{\mathrm{d}r_{ki}^\gamma}{\mathrm{d}s}\frac{\mathrm{d}R_k^\varepsilon}{\mathrm{d}s}\frac{\mathrm{d}R_k^\zeta}{\mathrm{d}s}\,\partial_\zeta f_{(t)\gamma\varepsilon}(R_{ki})\,\delta(R_k-R)\,\mathrm{d}s\\
&-\sum_i\frac{1}{c^4}\int e_{ki}\frac{\mathrm{d}R_k^\alpha}{\mathrm{d}s}\frac{\mathrm{d}^2R_{k\beta}}{\mathrm{d}s^2}\, r_{ki}^\beta\frac{\mathrm{d}r_{ki}^\gamma}{\mathrm{d}s}\frac{\mathrm{d}R_k^\varepsilon}{\mathrm{d}s}\, f_{(t)\gamma\varepsilon}(R_{ki})\,\delta(R_k-R)\,\mathrm{d}s,
\end{aligned} \tag{14}$$

where

$$\Delta^{\beta}_{k\alpha} = \delta^{\beta}_{\alpha} + \frac{1}{c^2}\,\frac{\mathrm{d}R_{k\alpha}}{\mathrm{d}s}\,\frac{\mathrm{d}R^{\beta}_{k}}{\mathrm{d}s}. \tag{15}$$

It may be remarked that the internal product of both sides of (14) with $\mathrm{d}R_{k\alpha}/\mathrm{d}s$ vanishes.

§ 4. *The intra-atomic electromagnetic field.* The field $f^{\alpha\beta}_{(t)}(R_{ki})$ may be split into two parts

$$f^{\alpha\beta}_{(t)}(R_{ki}) = f^{\alpha\beta}(R_{ki}) + f^{\alpha\beta}_{(in)}(R_{ki}), \tag{16}$$

where $f^{\alpha\beta}$ contains the fields produced by the atoms $l \neq k$ and possibly a field acting from outside the system; $f^{\alpha\beta}_{(in)}$ is the intra-atomic field, i.e. the field produced by the constituent particles of atom k itself.

This splitting is performed because, while the field $f^{\alpha\beta}$ shows only small variations over distances of the atomic dimensions (and can thus be expanded in powers of r_{ki}), the intra-atomic field varies rapidly inside the atom. The latter will therefore be considered separately. We shall assume that in the momentary rest frame of the atom the motion of the constituent particles is such that it is not needed to include terms of higher order than c^{-2} in the intra-atomic field $f^{\alpha\beta}_{(in)}$. This is also the approximation which follows from the Darwin potentials. In the theory of atomic (and molecular) structure often the cruder nonrelativistic approximation, in which the intra-atomic fields contain only terms up to order c^{-1} is considered to be sufficient. The components of the field $f^{\alpha\beta}_{(in)}$ up to order c^{-2} are

$$\begin{aligned}
\boldsymbol{e}^{(0)}_{(in)}(R^{(0)}_{ki}) = \sum_{j(\neq i)} e_{kj}\Bigg[& \frac{\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj}}{4\pi\,|\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj}|^3} \\
& + \frac{\dot{\boldsymbol{r}}^{(0)2}_{kj}}{2c^2}\,\frac{\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj}}{4\pi\,|\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj}|^3} \\
& - \frac{3}{2c^2}\,\{\dot{\boldsymbol{r}}^{(0)}_{kj}\cdot(\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj})\}^2\,\frac{\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj}}{4\pi\,|\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj}|^5} \\
& - \frac{1}{2c^2}\,\frac{(\ddot{\boldsymbol{r}}^{(0)}_{kj} + \ddot{\boldsymbol{R}}^{(0)}_{k})\cdot\mathbf{T}(\boldsymbol{r}^{(0)}_{ki}, \boldsymbol{r}^{(0)}_{kj})}{4\pi\,|\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj}|}\Bigg],
\end{aligned} \tag{17}$$

$$\boldsymbol{b}^{(0)}_{(in)}(R^{(0)}_{ki}) = \frac{1}{c}\sum_{j(\neq i)} e_{kj}\,\frac{\dot{\boldsymbol{r}}^{(0)}_{kj} \wedge (\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj})}{4\pi\,|\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj}|^3}, \tag{18}$$

with

$$\mathbf{T}(\boldsymbol{r}^{(0)}_{ki}, \boldsymbol{r}^{(0)}_{kj}) = \mathbf{U} + \frac{(\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj})(\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj})}{|\boldsymbol{r}^{(0)}_{ki} - \boldsymbol{r}^{(0)}_{kj}|^2}. \tag{19}$$

Here the dot in $\dot{\boldsymbol{r}}^{(0)}_{kj}$, $\ddot{\boldsymbol{r}}^{(0)}_{kj}$ and $\ddot{\boldsymbol{R}}^{(0)}_{k}$ means a differentiation with respect to

time, while the suffix $^{(0)}$ indicates the momentary Lorentz rest frame; $\mathbf{U}$ is the unit tensor. Substituting (17) and (18) one gets for (14)

$$
\begin{aligned}
& c\int (m_k + \delta m_k^I + \delta m_k^{II})\,\frac{\mathrm{d}^2 R_k^\alpha}{\mathrm{d}s^2}\,\delta(R_k - R)\,\mathrm{d}s \\
& + \sum_i c\int \Delta_{k\beta}^\alpha\,\frac{\mathrm{d}^2}{\mathrm{d}s^2}\,(\delta m_{ki}^I\, r_{ki}^\beta)\,\delta(R_k - R)\,\mathrm{d}s \\
& + \sum_i \frac{1}{c}\int m_{ki}\,\frac{\mathrm{d}}{\mathrm{d}s}\left\{\Delta_{k\beta}^\alpha\left(r_{ki}^\beta\,\frac{\mathrm{d}r_{ki}^\gamma}{\mathrm{d}s} - r_{ki}^\gamma\,\frac{\mathrm{d}r_{ki}^\beta}{\mathrm{d}s}\right)\frac{\mathrm{d}^2 R_{k\gamma}}{\mathrm{d}s^2}\right\}\delta(R_k - R)\,\mathrm{d}s = \\
& = \sum_i \int e_{ki}\, f^{\alpha\beta}(R_{ki})\,\frac{\mathrm{d}R_{k\beta}}{\mathrm{d}s}\,\delta(R_k - R)\,\mathrm{d}s \\
& + \sum_i \int e_{ki}\,\Delta_{k\beta}^\alpha\, f^{\beta\gamma}(R_{ki})\,\frac{\mathrm{d}r_{ki\gamma}}{\mathrm{d}s}\,\delta(R_k - R)\,\mathrm{d}s \\
& + \sum_i \frac{1}{c^2}\int e_{ki}\,\frac{\mathrm{d}}{\mathrm{d}s}\left(r_{ki}^\alpha\,\frac{\mathrm{d}r_{ki}^\gamma}{\mathrm{d}s}\,\frac{\mathrm{d}R_k^\varepsilon}{\mathrm{d}s}\right) f_{\gamma\varepsilon}(R_{ki})\,\delta(R_k - R)\,\mathrm{d}s \\
& + \sum_i \frac{1}{c^2}\int e_{ki}\, r_{ki}^\alpha\,\frac{\mathrm{d}r_{ki}^\gamma}{\mathrm{d}s}\,\frac{\mathrm{d}R_k^\varepsilon}{\mathrm{d}s}\,\frac{\mathrm{d}R_k^\zeta}{\mathrm{d}s}\,\partial_\zeta f_{\gamma\varepsilon}(R_{ki})\,\delta(R_k - R)\,\mathrm{d}s \\
& - \sum_i \frac{1}{c^4}\int e_{ki}\,\frac{\mathrm{d}R_k^\alpha}{\mathrm{d}s}\,\frac{\mathrm{d}^2 R_{k\beta}}{\mathrm{d}s^2}\, r_{ki}^\beta\,\frac{\mathrm{d}r_{ki}^\gamma}{\mathrm{d}s}\,\frac{\mathrm{d}R_k^\varepsilon}{\mathrm{d}s}\, f_{\gamma\varepsilon}(R_{ki})\,\delta(R_k - R)\,\mathrm{d}s, \qquad (20)
\end{aligned}
$$

where

$$\delta m_{ki}^I = \frac{1}{c^2}\sum_{j(\neq i)}\frac{e_{ki}\, e_{kj}}{8\pi\,|\boldsymbol{r}_{ki}^{(0)} - \boldsymbol{r}_{kj}^{(0)}|}, \qquad (21a)$$

$$\delta m_k^I = \sum_i \delta m_{ki}^I = \frac{1}{c^2}\sum_{i,j,i\neq j}\frac{e_{ki}\, e_{kj}}{8\pi\,|\boldsymbol{r}_{ki}^{(0)} - \boldsymbol{r}_{kj}^{(0)}|} \qquad (21b)$$

represent masses equivalent to internal Coulomb energies of atom k, and

$$\delta m_k^{II} = \frac{1}{2c^2}\sum_i m_{ki}\,\frac{\mathrm{d}r_{ki\alpha}}{\mathrm{d}s}\,\frac{\mathrm{d}r_{ki}^\beta}{\mathrm{d}s}\,\Delta_{k\beta}^\alpha \qquad (22)$$

is a mass corresponding to the internal kinetic energy of atom k. We note that in (20) the terms of intra-atomic origin (i.e. the terms with δm_{ki}^I and δm_k^I) are both of order c^{-2}; this means that nonrelativistically these terms would disappear.

In (20) one recognizes besides the ordinary rest mass density

$$\rho_k' = c\int m_k\,\delta(R_k - R)\,\mathrm{d}s \qquad (23)$$

the mass densities

$$\delta\rho_{ki}^{I'} = c \int \delta m_{ki}^{I}\, \delta(R_k - R)\, \mathrm{d}s, \tag{24a}$$

$$\delta\rho_{k}^{I'} = \sum_i \delta\rho_{ki}^{I'} = c \int \delta m_{k}^{I}\, \delta(R_k - R)\, \mathrm{d}s, \tag{24b}$$

$$\delta\rho_{k}^{II'} = c \int \delta m_{k}^{II}\, \delta(R_k - R)\, \mathrm{d}s. \tag{25}$$

Furthermore one has at the left-hand side a term containing

$$\sigma_{k\alpha\beta} = \sum_i c \int m_{ki}\, \Delta_{k\alpha}^{\gamma}\, \Delta_{k\beta}^{\varepsilon} \left(r_{ki\gamma} \frac{\mathrm{d}r_{ki\varepsilon}}{\mathrm{d}s} - r_{ki\varepsilon} \frac{\mathrm{d}r_{ki\gamma}}{\mathrm{d}s} \right) \delta(R_k - R)\, \mathrm{d}s, \tag{26}$$

which is the "atomic angular momentum" density (in the rest frame it reduces to the angular momentum density of the atom).

§ 5. *Expansion of the interatomic fields. The dipole case.* The interatomic field $f^{\alpha\beta}(R_{ki})$, which occurs in (20) can be expanded in powers of r_{ki}. Thus one obtains for the right-hand side of (20), retaining terms up to second order

$$\begin{aligned}
&\sum_i \int e_{ki}\, f^{\alpha\beta}(R) \frac{\mathrm{d}R_{k\beta}}{\mathrm{d}s}\, \delta(R_k - R)\, \mathrm{d}s \\
&+ \sum_i \int e_{ki}\, r_{ki}^{\gamma}\, \partial_\gamma f^{\alpha\beta}(R) \frac{\mathrm{d}R_{k\beta}}{\mathrm{d}s}\, \delta(R_k - R)\, \mathrm{d}s \\
&+ \sum_i \int e_{ki}\, \Delta_{k\beta}^{\alpha}\, f^{\beta\gamma}(R) \frac{\mathrm{d}r_{ki\gamma}}{\mathrm{d}s}\, \delta(R_k - R)\, \mathrm{d}s \\
&+ \sum_i \tfrac{1}{2} \int e_{ki}\, r_{ki}^{\gamma}\, r_{ki}^{\varepsilon}\, \partial_\gamma \partial_\varepsilon f^{\alpha\beta}(R) \frac{\mathrm{d}R_{k\beta}}{\mathrm{d}s}\, \delta(R_k - R)\, \mathrm{d}s \\
&+ \sum_i \int e_{ki}\, \Delta_{k\beta}^{\alpha}\, r_{ki}^{\varepsilon}\, \partial_\varepsilon f^{\beta\gamma}(R) \frac{\mathrm{d}r_{ki\gamma}}{\mathrm{d}s}\, \delta(R_k - R)\, \mathrm{d}s \\
&+ \sum_i \frac{1}{c^2} \int e_{ki} \frac{\mathrm{d}}{\mathrm{d}s} \left(r_{ki}^{\alpha} \frac{\mathrm{d}r_{ki}^{\gamma}}{\mathrm{d}s} \frac{\mathrm{d}R_k^{\varepsilon}}{\mathrm{d}s} \right) f_{\gamma\varepsilon}(R)\, \delta(R_k - R)\, \mathrm{d}s \\
&+ \sum_i \frac{1}{c^2} \int e_{ki}\, r_{ki}^{\alpha} \frac{\mathrm{d}r_{ki}^{\gamma}}{\mathrm{d}s} \frac{\mathrm{d}R_k^{\varepsilon}}{\mathrm{d}s} \frac{\mathrm{d}R_k^{\zeta}}{\mathrm{d}s}\, \partial_\zeta f_{\gamma\varepsilon}(R)\, \delta(R_k - R)\, \mathrm{d}s \\
&- \sum_i \frac{1}{c^4} \int e_{ki} \frac{\mathrm{d}R_k^{\alpha}}{\mathrm{d}s} \frac{\mathrm{d}^2 R_{k\beta}}{\mathrm{d}s^2}\, r_{ki}^{\beta} \frac{\mathrm{d}r_{ki}^{\gamma}}{\mathrm{d}s} \frac{\mathrm{d}R_k^{\varepsilon}}{\mathrm{d}s}\, f_{\gamma\varepsilon}(R)\, \delta(R_k - R)\, \mathrm{d}s.
\end{aligned} \tag{27}$$

The field acting on atom k is a sum of partial fields $f_l^{\alpha\beta}$ due to atoms $l\,(\neq k)$:

$$f^{\alpha\beta} = \sum_{l(\neq k)} f_l^{\alpha\beta}, \tag{28}$$

where external fields acting from outside the system are omitted. The partial fields satisfy the atomic field equations[7])

$$\partial_\alpha f_{l\beta\gamma} + \partial_\beta f_{l\gamma\alpha} + \partial_\gamma f_{l\alpha\beta} = 0, \tag{29}$$

$$\partial_\beta f_l^{\alpha\beta} = c^{-1} j_l^\alpha + \partial_\beta m_l^{\alpha\beta}, \tag{30}$$

with the atomic four-current density:

$$c^{-1} j_l^\alpha = \int e_l (\mathrm{d}R_l^\alpha/\mathrm{ds})\, \delta(R_l - R)\, \mathrm{ds}, \tag{31}$$

and the atomic polarization tensor

$$m_l^{\alpha\beta} = m_l^{(1)\alpha\beta} + m_l^{(2)\alpha\beta}. \tag{32}$$

Here only first and second order contributions are considered:

$$m_l^{(1)\alpha\beta} = \sum_i \int e_{li} \left(r_{li}^\alpha \frac{\mathrm{d}R_l^\beta}{\mathrm{ds}} - r_{li}^\beta \frac{\mathrm{d}R_l^\alpha}{\mathrm{ds}} \right) \delta(R_l - R)\, \mathrm{ds}, \tag{33}$$

$$m_l^{(2)\alpha\beta} = \sum_i \tfrac{1}{2} \int e_{li} \left(r_{li}^\alpha \frac{\mathrm{d}r_{li}^\beta}{\mathrm{ds}} - r_{li}^\beta \frac{\mathrm{d}r_{li}^\alpha}{\mathrm{ds}} \right) \delta(R_l - R)\, \mathrm{ds}. \tag{34}$$

From now on we shall limit ourselves to the consideration of electric and magnetic dipole moments only. Whereas (33) contains only electric dipoles, expression (34) contains both magnetic dipoles and electric quadrupoles. The latter are discarded if, instead of (34), we now write

$$m_l^{(2)\alpha\beta} = \sum_i \tfrac{1}{2} \int e_{li}\, \Delta_{l\gamma}^\alpha\, \Delta_{l\varepsilon}^\beta \left(r_{li}^\gamma \frac{\mathrm{d}r_{li}^\varepsilon}{\mathrm{ds}} - r_{li}^\varepsilon \frac{\mathrm{d}r_{li}^\gamma}{\mathrm{ds}} \right) \delta(R_l - R)\, \mathrm{ds}. \tag{35}$$

From (33) and (35) follow the properties:

$$\Delta_{l\alpha}^\gamma\, \Delta_{l\beta}^\varepsilon\, m_{l\gamma\varepsilon}^{(1)} = 0, \tag{36}$$

$$u_l^\alpha\, m_{l\alpha\beta}^{(2)} = 0, \tag{37}$$

where

$$u_l^\alpha(R) = \frac{\mathrm{d}R_l^\alpha}{\mathrm{ds}} \quad (s \text{ from } R_l(s) = R). \tag{38}$$

With these relations one deduces from (32):

$$m_l^{(1)\alpha\beta} = -c^{-2}\, u_l^\alpha\, u_{l\gamma}\, m_l^{\gamma\beta} - c^{-2}\, u_l^\beta\, u_{l\gamma}\, m_l^{\alpha\gamma}, \tag{39}$$

$$m_l^{(2)\alpha\beta} = \Delta_{l\gamma}^\alpha\, \Delta_{l\varepsilon}^\beta\, m_l^{\gamma\varepsilon}. \tag{40}$$

Expression (27) becomes with the help of (3), (28), (29), (31)–(40), and

discarding electric quadrupoles:

$$\sum_{l(\neq k)} f_l^{\alpha\beta}(c^{-1}\, j_{k\beta} + \partial^\gamma m_{k\beta\gamma})$$
$$+ \sum_{l(\neq k)} \partial_\beta\{f_l^{\alpha\gamma} m_{k\cdot\gamma}^{\beta} + c^{-2}(m_k^{\alpha\gamma} f_{l\gamma\varepsilon} - f_l^{\alpha\gamma} m_{k\gamma\varepsilon})\, u_k^\varepsilon u_k^\beta + \tfrac{1}{2}c^{-2}\, u_k^\alpha u_k^\beta f_l^{\gamma\varepsilon} m_{k\gamma\varepsilon}^{(1)}\}$$
$$- \sum_{l(\neq k)} u_k^\alpha\{\tfrac{1}{2}c^{-2} f_{l\beta\gamma}\, \partial_\varepsilon(u_k^\varepsilon m_k^{(1)\beta\gamma}) + c^{-4} f_{l\beta\gamma}(D_k u_k^\beta)\, u_{k\varepsilon} m_k^{(1)\varepsilon\gamma}$$
$$- \tfrac{1}{2}c^{-2} m_k^{(2)\beta\gamma} u_k^\varepsilon\, \partial_\varepsilon f_{l\beta\gamma} - c^{-4} f_{l\beta\gamma}(D_k u_{k\varepsilon})\, u_k^\beta m_k^{(2)\varepsilon\gamma}\}. \tag{41}$$

Here use has been made of the operator

$$D_k = u_k^\alpha\, \partial_\alpha, \tag{42}$$

which permitted us to write

$$\frac{\mathrm{d}^2 R_k^\alpha}{\mathrm{d}s^2}\ (s \text{ from } R_k(s) = R) = u_k^\beta(R)\, \partial_\beta u_k^\alpha(R) = D_k u_k^\alpha. \tag{43}$$

From the definition (23) follows the conservation of proper mass

$$\partial_\alpha(\rho_k' u_k^\alpha) = 0. \tag{44}$$

Using (42) and (44) the last bracket of (41) becomes:

$$\tfrac{1}{2}c^{-2} f_{l\beta\gamma}\, \rho_k' D_k(v_k' m_k^{(1)\beta\gamma}) + c^{-4} f_{l\beta\gamma}(D_k u_k^\beta)\, u_{k\varepsilon} m_k^{(1)\varepsilon\gamma}$$
$$- \tfrac{1}{2}c^{-2} m_k^{(2)\beta\gamma} D_k f_{l\beta\gamma} - c^{-4} f_{l\beta\gamma}(D_k u_{k\varepsilon})\, u_k^\beta m_k^{(2)\varepsilon\gamma}, \tag{45}$$

where $v_k' = (\rho_k')^{-1}$ is the specific volume. We now wish to express the invariant quantity (45) completely in terms of quantities defined in the rest frame, in which the atom is at rest all the time. This frame, which we indicate by dashes is a succession of Lorentz frames, not a Lorentz frame itself. The Lorentz transformation of an antisymmetric tensor $A_{\alpha\beta}$ with components $A_{i0} = \boldsymbol{X}_i$; $A_{ij} = \boldsymbol{Y}_k$ (cycl.; $i, j, k = 1, 2, 3$) from the dashed frame to the momentary rest (Lorentz) frame yields:

$$D\boldsymbol{X}' = D\boldsymbol{X}^{(0)} + c^{-1}\, D\boldsymbol{v}^{(0)} \wedge \boldsymbol{Y}^{(0)}, \tag{46}$$

$$D\boldsymbol{Y}' = D\boldsymbol{Y}^{(0)} - c^{-1}\, D\boldsymbol{v}^{(0)} \wedge \boldsymbol{X}^{(0)}, \tag{47}$$

where D denotes a differentiation and $\boldsymbol{v}^{(0)}$ the transformation velocity. From this follows the relation (with transformation four-velocity $U^{\alpha(0)}$):

$$DA_{\alpha\beta}' = DA_{\alpha\beta}^{(0)} + c^{-2}(U_\alpha^{(0)} A_{\beta\gamma}^{(0)} DU^{\gamma(0)} - U_\beta^{(0)} A_{\alpha\gamma}^{(0)} DU^{\gamma(0)})$$
$$- c^{-2}(U^{\gamma(0)} A_{\gamma\alpha}^{(0)} DU_\beta^{(0)} - U^{\gamma(0)} A_{\gamma\beta}^{(0)} DU_\alpha^{(0)}), \tag{48}$$

as may be checked for its components separately. Then by means of a contraction with a different antisymmetric tensor $B_{\alpha\beta}$ one gets:

$$B'^{\alpha\beta} DA_{\alpha\beta}' = B^{\alpha\beta} DA_{\alpha\beta} + 2c^{-2}\, U_\alpha(B^{\alpha\beta} A_{\beta\gamma} - A^{\alpha\beta} B_{\beta\gamma})\, DU^\gamma. \tag{49}$$

Using this relation and (36), (37) the expression (45) becomes

$$\tfrac{1}{2}c^{-2}\, f'_{l\beta\gamma}\, \rho'_k\, D_k(v'_k\, m_k^{(1)\beta\gamma\prime}) - \tfrac{1}{2}c^{-2}\, m_k^{(2)\beta\gamma\prime}\, D_k\, f'_{l\beta\gamma}. \tag{50}$$

Now we can rewrite the energy-momentum balance equation (20), using (23)–(26), (29), (30), (38), (39), (41)–(44) and (50) in the form (with a summation over k):

$$\begin{aligned}
&\sum_k \partial_\beta(\rho'_k\, u_k^\alpha\, u_k^\beta) + \sum_k \partial_\beta(c^{-2}\, u_k^\beta\, \sigma_k^{\alpha\gamma}\, D_k u_{k\gamma}) \\
&\quad + \sum_{ki} \partial_\beta\{u_k^\beta\, \partial_\gamma(u_k^\gamma\, \delta\rho_{ki}^{I\prime}\, r_{ki}^\alpha)\} + \partial_\beta t_{(f)}^{\alpha\beta} + \sum_k (\delta\rho_k^{I\prime} + \delta\rho_k^{II\prime})\, D_k u_k^\alpha \\
&\quad + \sum_{k,l,k\neq l} \tfrac{1}{2}c^{-2}\, u_k^\alpha\{f'_{l\beta\gamma}\, \rho'_k\, D_k(v'_k m_k^{(1)\beta\gamma\prime}) - m_k^{(2)\beta\gamma\prime}\, D_k f'_{l\beta\gamma}\} \\
&\quad + \sum_{ki} c^{-2}\, u_k^\alpha u_{k\varepsilon}\, \partial_\beta\{u_k^\beta\, \partial_\gamma(u_k^\gamma\, \delta\rho_{ki}^{I\prime}\, r_{ki}^\varepsilon)\} = 0,
\end{aligned} \tag{51}$$

where we have introduced the atomic energy-momentum tensor of the field:

$$\begin{aligned}
t_{(f)\alpha}^{\cdot\beta} = \sum_{k,l,k\neq l} \{&f_{l\alpha\gamma}\, h_k^{\beta\gamma} - \tfrac{1}{4} f_{l\gamma\varepsilon}\, f_k^{\gamma\varepsilon}\, \delta_\alpha^\beta + c^{-2}\, u_k^\beta (f_{l\alpha\gamma}\, m_k^{\gamma\varepsilon} - m_{k\alpha\gamma}\, f_l^{\gamma\varepsilon})\, u_{k\varepsilon} \\
&- c^{-4}\, u_{k\alpha}\, u_k^\beta\, u_k^\gamma\, f_{l\gamma\varepsilon}\, m_k^{\varepsilon\zeta}\, u_{k\zeta}\},
\end{aligned} \tag{52}$$

with $h_k^{\alpha\beta} = f_k^{\alpha\beta} - m_k^{\alpha\beta}$. The last three terms in (51) are not in the form of a divergence: in other words (51) has still the form of a balance equation, not yet of a conservation law. Consideration of the internal atomic energy balance will allow us to write the three terms concerned as a divergence as well.

§ 6. *The internal atomic energy balance.* In order to obtain the balance equation of internal atomic energy the equation of motion (1) is multiplied by $\mathrm{d}r_{ki\alpha}/\mathrm{d}s$:

$$cm_{ki}\, \frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{\mathrm{d}R_{ki}^\alpha}{\mathrm{d}s}\left(\frac{\mathrm{d}\tau_{ki}}{\mathrm{d}s}\right)^{-1}\right\}\frac{\mathrm{d}r_{ki\alpha}}{\mathrm{d}s} = e_{ki} f_{(t)}^{\alpha\beta}(R_{ki})\, \frac{\mathrm{d}R_{ki\beta}}{\mathrm{d}s}\, \frac{\mathrm{d}r_{ki\alpha}}{\mathrm{d}s}. \tag{53}$$

Expanding up to second order in r_{ki}^α one gets:

$$\begin{aligned}
\sum_i c \int m_{ki}\, \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}R_k^\alpha}{\mathrm{d}s} + \frac{\mathrm{d}r_{ki}^\alpha}{\mathrm{d}s} + \frac{1}{c^2}\, \frac{\mathrm{d}R_k^\alpha}{\mathrm{d}s}\, \frac{\mathrm{d}r_{ki\beta}}{\mathrm{d}s}\, \frac{\mathrm{d}R_k^\beta}{\mathrm{d}s}\right)\frac{\mathrm{d}r_{ki\alpha}}{\mathrm{d}s}\, \delta(R_k - R)\, \mathrm{d}s = \\
= \sum_i \int e_{ki}\, f_{(t)}^{\alpha\beta}(R_{ki})\, \frac{\mathrm{d}R_{k\beta}}{\mathrm{d}s}\, \frac{\mathrm{d}r_{ki\alpha}}{\mathrm{d}s}\, \delta(R_k - R)\, \mathrm{d}s,
\end{aligned} \tag{54}$$

where an integration over a delta-function has been added. With the help of (3) and (11) one obtains:

$$\begin{aligned}
\sum_i \frac{c}{2} \int m_{ki}\, \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}r_{ki\alpha}}{\mathrm{d}s}\, \frac{\mathrm{d}r_{ki}^\beta}{\mathrm{d}s}\, \Delta_{k\beta}^\alpha\right)\delta(R_k - R)\, \mathrm{d}s = \\
= \sum_i \int e_{ki}\, f_{(t)}^{\alpha\beta}(R_{ki})\, \frac{\mathrm{d}R_{k\beta}}{\mathrm{d}s}\, \frac{\mathrm{d}r_{ki\alpha}}{\mathrm{d}s}\, \delta(R_k - R)\, \mathrm{d}s \\
+ \sum_i \frac{1}{c} \int m_{ki}\, \frac{\mathrm{d}^2 R_k^\alpha}{\mathrm{d}s^2}\, \frac{\mathrm{d}^2 R_k^\beta}{\mathrm{d}s^2}\, \frac{\mathrm{d}r_{ki\alpha}}{\mathrm{d}s}\, r_{ki\beta}\, \delta(R_k - R)\, \mathrm{d}s.
\end{aligned} \tag{55}$$

With the use of the equation of motion in zeroth order (13) the right-hand side of equation (55) becomes:

$$\sum_i \int e_{ki} f^{\alpha\beta}_{(i)}(R_{ki}) \frac{\mathrm{d}R_{k\beta}}{\mathrm{d}s} \frac{\mathrm{d}r_{ki\alpha}}{\mathrm{d}s} \delta(R_k - R)\,\mathrm{d}s$$

$$+ \sum_i \frac{1}{c^2} \int e_{ki} f^{\alpha\gamma}_{(i)}(R_{ki}) \frac{\mathrm{d}R_{k\gamma}}{\mathrm{d}s} \frac{\mathrm{d}^2 R^{\beta}_k}{\mathrm{d}s^2} \frac{\mathrm{d}r_{ki\alpha}}{\mathrm{d}s} r_{ki\beta}\, \delta(R_k - R)\,\mathrm{d}s. \quad (56)$$

Let us consider first this expression for the intra-atomic fields only. Since the internal energy balance, deduced here, will be employed in the last two terms of (51), which contain already a factor c^{-2}, we may confine ourselves here to the Coulomb expression for the intra-atomic field in the momentary rest frame of the atom. In this way (56) becomes for the intra-atomic fields

$$-c^3 \int \frac{\mathrm{d}}{\mathrm{d}s} (\delta m^I_k)\, \delta(R_k - R)\,\mathrm{d}s, \quad (57)$$

where δm^I_k is defined in (21).

The interatomic field part of expression (56) is considered up to second order in r^{α}_{ki} by expanding the fields: discarding the electric quadrupole moments one gets with the help of (3), (28), (29), (33), (35)–(38), (42)–(44) and (49):

$$\sum_{l(\neq k)} \tfrac{1}{2}\{f'_{l\alpha\beta}\, \rho'_k\, D_k(v'_k\, m^{(1)\alpha\beta\prime}_k) - m^{(2)\alpha\beta\prime}_k\, D_k(f'_{l\alpha\beta})\}. \quad (58)$$

The balance equation (55) becomes with (22), (57) and (58):

$$c^3 \int \frac{\mathrm{d}}{\mathrm{d}s} (\delta m^{II}_k)\, \delta(R_k - R)\,\mathrm{d}s = -c^3 \int \frac{\mathrm{d}}{\mathrm{d}s} (\delta m^I_k)\, \delta(R_k - R)\,\mathrm{d}s$$

$$+ \sum_{l(\neq k)} \tfrac{1}{2}\{f'_{l\alpha\beta}\, \rho'_k\, D_k(v'_k\, m^{(1)\alpha\beta\prime}_k) - m^{(2)\alpha\beta\prime}_k\, D_k(f'_{l\alpha\beta})\}. \quad (59)$$

Finally, after a partial integration of two mass terms and using (24), (25) and (38) the equation (59) gets the form (with a summation over k):

$$\sum_k c^2\, \partial_\beta\{(\delta p^{I\prime}_k + \delta p^{II\prime}_k)\, u^{\beta}_k\} =$$

$$= \sum_{k,l,k\neq l} \tfrac{1}{2}\{f'_{l\alpha\beta}\, \rho'_k\, D_k(v'_k\, m^{(1)\alpha\beta\prime}_k) - m^{(2)\alpha\beta\prime}_k\, D_k(f'_{l\alpha\beta})\}, \quad (60)$$

which is the internal atomic energy balance. The right-hand side of this equation appears in the penultimate term of the first member of (51). The last term of the first member of (51) may now be neglected since it is of order c^{-2} as compared to the penultimate term, if (60) is used.

§ 7. *The atomic energy-momentum balance in the form of a conservation law.* If the balance equations of energy-momentum (51) and of internal atomic

energy (60) are combined, one obtains the conservation laws:

$$\partial_\beta t^{\alpha\beta} = 0, \qquad t^{\alpha\beta} = t^{\alpha\beta}_{(f)} + t^{\alpha\beta}_{(m)}, \tag{61}$$

for energy ($\alpha = 0$) and momentum ($\alpha = 1, 2, 3$). Here t^{00} is the energy density, $c\, t^{0i}$ the energy flow, $c^{-1}t^{i0}$ the momentum density and t^{ij} the momentum flow.

The atomic field energy-momentum tensor is given by (52) and the atomic material energy-momentum tensor by

$$t^{\alpha\beta}_{(m)} = \sum_k (\rho'_k + \delta\rho_k^{I'} + \delta\rho_k^{II'})\, u_k^\alpha u_k^\beta + \sum_k c^{-2}\, \sigma_k^{\alpha\gamma}(D_k u_{k\gamma})\, u_k^\beta + \sum_k \partial_\gamma\{u_k^\gamma(\sum_i \delta\rho_{ki}^{I'}\, r_{ki}^\alpha)\}\, u_k^\beta. \tag{62}$$

Here the first term contains besides the rest mass density ρ'_k the mass densities corresponding to the internal Coulomb and kinetic energy of the atoms. The contribution of atom k to this term is purely time-time-like in the rest frame of k. The contribution of atom k to the second term is different from zero only if the atom, which carries the angular momentum σ_k, is accelerated. In the rest frame of k it yields a contribution to the momentum density only, which is of the form $\partial_0 \boldsymbol{\beta}_k \wedge \boldsymbol{\sigma}_k$. The third term is a very small relativistic correction related to the internal Coulomb energy of the atoms.

(It should be remarked that in view of the form of (61) the energy-momentum tensor is determined up to divergence-free part.)

The field energy-momentum tensor (52) consists of a sum of terms for each atom

$$t^{\alpha\beta}_{(f)} = \sum_k t^{\alpha\beta}_{(f)k}. \tag{63}$$

It is interesting to consider the partial tensor $t^{\alpha\beta}_{(f)k}$ in the rest frame of atom k:

$$\begin{pmatrix} t^{00}_{(f)k} & t^{0i}_{(f)k} \\ t^{i0}_{(f)k} & t^{ij}_{(f)k} \end{pmatrix} = \sum_{l(\neq k)} \begin{pmatrix} \frac{1}{2}\boldsymbol{e}_k\cdot\boldsymbol{e}_l + \frac{1}{2}\boldsymbol{b}_k\cdot\boldsymbol{b}_l & (\boldsymbol{e}_l \wedge \boldsymbol{h}_k)^i \\ (\boldsymbol{e}_l \wedge \boldsymbol{h}_k)^i & -\boldsymbol{e}_l^i \boldsymbol{d}_k^i - \boldsymbol{h}_k^i \boldsymbol{b}_l^i + (\frac{1}{2}\boldsymbol{e}_k\cdot\boldsymbol{e}_l + \frac{1}{2}\boldsymbol{b}_k\cdot\boldsymbol{b}_l - \boldsymbol{m}_k\cdot\boldsymbol{b}_l) g^{ij} \end{pmatrix}. \tag{64}$$

The energy-momentum conservation laws on the atomic ("kinetic") level are thus derived. In a subsequent paper they will serve as a basis for the macroscopic equations. In fact these are obtained by averaging (61). Then a macroscopic material energy-momentum tensor is obtained which is the average of (62) combined with the correlation and velocity fluctuation part of (52). The remaining part of the average of (52) constitutes the macroscopic field energy-momentum tensor. Its form in the local rest frame is

analogous to (64):

$$T^{\alpha\beta}_{(f)} = \begin{pmatrix} \frac{1}{2}\boldsymbol{E}^2 + \frac{1}{2}\boldsymbol{B}^2 & (\boldsymbol{E} \wedge \boldsymbol{H})^i \\ (\boldsymbol{E} \wedge \boldsymbol{H})^i & -\boldsymbol{E}^i\boldsymbol{D}^j - \boldsymbol{H}^i\boldsymbol{B}^j + (\frac{1}{2}\boldsymbol{E}^2 + \frac{1}{2}\boldsymbol{B}^2 - \boldsymbol{M}\cdot\boldsymbol{B})g^{ij} \end{pmatrix}, \quad (65)$$

and gives the expressions for the energy density $T^{00}_{(f)}$, the Poynting vector $c\,T^{0i}_{(f)}$, the momentum density $c^{-1}\,T^{i0}_{(f)}$, and the Maxwell pressure $T^{ij}_{(f)}$ of the macroscopic field.

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