Arithmetical Measure

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Abstract. We develop arithmetical measure theory along the lines of Lutz [10]. This yields the same notion of “measure 0 set” as considered before by Martin-Löf, Schnorr, and others. We prove that the class of sets constructible by r.e.-constructors, a direct analogue of the classes Lutz devised his resource bounded measures for in [10], is not equal to RE, the class of r.e. sets, and we locate this class exactly in terms of the common recursion-theoretic reducibilities below K. We note that the class of sets that bounded truth-table reduce to K has r.e.-measure 0, and show that this cannot be improved to “truth-table.” For Δ2-measure the borderline between measure zero and measure nonzero lies between weak truth-table reducibility and Turing reducibility to K. It follows that there exists a Martin-Löf random set that is tt-reducible to K, and that no such set is btt-reducible to K. In fact, by a result of Kautz, a much more general result holds.

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1 Introduction

Restricted versions of the classical theory of Lebesgue measure have been used by a large number of authors for the study of the notion of “random infinite sequence.” More recently, Lutz [10] systematically developed the theory of “resource bounded measure.” His measure theory allows one to make quantitative assertions about countable classes, in the style of the “almost all” statements referring to Lebesgue measure. In particular, Lutz developed a theory of measure in exponential time computable classes, using polynomial time computable
martingales. This measure theory quickly developed into a full-blown research area within structural complexity theory with many interesting results (see [11], [13] for a survey).

Following the approach of Lutz, in Section 3 we define the notion of r.e.-measure 0. An extension of the work of Schnorr quickly shows that this yields the same notion of measure studied before by Martin-Löf [12], Schnorr [15], and others. In his paper [10], Lutz uses the concept of constructor to define the classes that his various measure theories are for. Recursively enumerable constructors give rise to a class $R(\text{r.e.})$ of sets which lies between $\text{RE}$ and $\Delta_2$ in the arithmetical hierarchy (we show that both inclusions are proper). The r.e.-measure of $\text{RE}$ is 0, hence r.e.-measure cannot be used to make quantitative assertions about $\text{RE}$. We prove that $R(\text{r.e.})$ does not have r.e.-measure 0. Of course, being countable, $R(\text{r.e.})$ is of Lebesgue measure 0. This holds for all the classes that are considered in this paper. From the fact that $R(\text{r.e.})$ does not have measure 0 it follows that the class $\leq^u(K)$ of sets that truth-table reduce to the universal r.e. set $K$ does not have r.e.-measure 0. On the other hand, $\leq^\text{tt}(K)$, the class of sets that bounded truth-table reduce to $K$, does have r.e.-measure 0.

In Section 4 we generalize the notions of Section 3. We thus obtain measures $\mu_{\Sigma_n}$, $\mu_{\Pi_n}$, and $\mu_{\Delta_n}$ for every level of the arithmetical hierarchy. We indicate an exact border between “$\Delta_n$-measure zero” and “$\Delta_n$-measure nonzero” by proving that the class of sets that weakly truth-table reduce to $\emptyset^{(n-1)}$ has $\Delta_n$-measure 0 and that $\Delta_n = \leq^\tau(\emptyset^{(n-1)})$ itself does not have $\Delta_n$-measure 0. We prove that $\mu_{\Pi_n}$ and $\mu_{\Delta_n}$ are equal and that they are different from $\mu_{\Sigma_n}$. This at first sight surprising situation is caused by the asymmetry of the supermartingale property.

Finally we make some notes on generalization of the results from Section 3 and on related results. It follows from the results in the previous sections that there is a Martin-Löf random set in $\leq^u(K)$, but not in $\leq^\text{tt}(K)$. We have thus obtained an optimal placement of the Martin-Löf random sets in $\Delta_2$ in terms of the well-known reducibilities below $K$. In fact, using a result by Kautz we have that for any set $C$ there exists a $C$-random set in $\leq^u(C^{(n)})$, but not in $\leq^\text{tt}(C^{(n)})$.

## 2 Preliminaries

We assume that the reader is familiar with the basic notions of computability theory. Our basic recursion-theoretic notation is as in Soare [16]. So $\omega$ is the set of natural numbers, $2^\omega$ is Cantor space, the power set of $\omega$, and $2^{<\omega}$ is the set of finite strings of zero’s and one’s, viewed as the set of initial segments of
characteristic strings of sets in $2^\omega$. We fix a recursive pairing function $(\cdot,\cdot) : \omega \times \omega \rightarrow \omega$ and denote the set $\{(x,n) : x \in \omega\}$ by $\omega^{[n]}$ (the $n$-th “row” of $\omega$). The empty set is denoted by $\emptyset$, and the empty string in $2^{<\omega}$ is denoted by $\varepsilon$.

For $A \in 2^\omega$, $\overline{A}$ denotes the complement, $\omega - A$, of $A$. Sets are identified with their characteristic strings, and $A^{\upharpoonright n}$ denotes the initial segment of length $n$ of $A$. For elements $v, w \in 2^{<\omega}$, $vw$ denotes the concatenation of $v$ and $w$, and $v \sqsubseteq w$, $v \sqsubseteq w$ denotes that $v$ is a (proper) prefix of $w$. Subsets of $2^\omega$ are called classes, and for a class $A$, $\text{co-}A = \{ \overline{A} : A \in A \}$. $K$ is the universal r.e. set. The jump of a set $C$ is denoted by $C'$, and the $n$-th iterated jump of $C$ by $C^{(n)}$.

The $e$-th partial recursive function is $\varphi_e$, and $\varphi_{e,s}(x)$ is the result of running $\varphi_e$ on $x$ using less than $s$ computation steps. We denote by $\varphi_e(x) \downarrow (\varphi_e(x) \uparrow)$ that $\varphi_e(x)$ is defined (undefined). For a function $f : \omega \rightarrow \omega$, $f^{\upharpoonright n}$ denotes the string $f(0)f(1)\ldots f(n-1)$, and if $f$ is partial then $(f^{\upharpoonright n}) \downarrow$ denotes that this string is defined. The sets of nonzero rationals and reals are denoted by $\mathbb{Q}^+$ and $\mathbb{R}^+$, respectively. We will use the standard recursion-theoretic reducibilities $\leq_m$, $\leq_{btt}$, $\leq_{tt}$, $\leq_{wtt}$, and $\leq_T$. We will make use of the following characterization of $\leq_T$.

**Lemma 2.1** (Carstens [3]) For every $A \in 2^\omega$, $A \leq_T K$ if and only if there exist recursive functions $g$ and $h$ such that for every $x \in \omega$, $\lim_s g(s,x) = A(x)$ with $|\{s : g(s,x) \neq g(s+1,x)\}| \leq h(x)$.

For $r \in \{m, btt, tt, wtt, T\}$, $\preceq_r (A)$ denotes the class of sets that are $r$-reducible to the set $A$.

Throughout this paper, the standard notion of a recursively enumerable function will play an important role:

**Definition 2.2** Let $A$ be a countable set recursively isomorphic to $\omega$. Let $B$ be a countable set totally ordered by $\leq_B$ such that $(B, \leq_B)$ is recursively isomorphic to $(\omega, \leq)$. A function $f : A \rightarrow B$ is recursively enumerable (r.e.) if the set $\{(x,y) : y \leq_B f(x)\}$ is an r.e. set. Similarly, $f$ is co-r.e. or $\Pi_1$ if $\{(x,y) : f(x) \leq_B y\}$ is r.e. The class of (total) r.e.-functions is denoted by r.e.

Note that a function $f$ is r.e. if and only if there exists a recursive approximation $f_s$ such that for all $x$, $f_s(x) \leq f_{s+1}(x)$ and $(\exists s)(\forall t \geq s)[f_t(x) = f(x)]$. We will often make use of this last fact, namely that the recursive approximation $f_s$ actually attains its limit (see Section 3 for further discussion on this point). Clearly a function which is both r.e. and co-r.e. is recursive. The class of (total) recursive functions is denoted by rec.

$\Sigma_n$, $\Pi_n$, and $\Delta_n$ denote the classes from Kleene’s arithmetical hierarchy. They are also used to denote the corresponding function classes, defined in
Section 4. We also write REC for $\Delta_1$ and RE for $\Sigma_1$, the classes of recursive and recursively enumerable sets, respectively. For a measurable class $A$ the Lebesgue measure of $A$ is denoted by $\lambda(A)$.

**Definition 2.3** A supermartingale $d$ is a function from $2^{<\omega}$ to $\mathcal{R}^+$ with the property

$$d(w0) + d(w1) \leq 2d(w)$$

for every $w \in 2^{<\omega}$.

Functions with this property are called supermartingales, as opposed to martingales with the property $d(w0) + d(w1) = 2d(w)$. Lutz [10, p239] remarks that for the classes that he considers it makes no difference whether one uses supermartingales or martingales. In our setting it will make a huge difference. We will consider supermartingales which are (approximable by) r.e.-functions, and one can easily check that r.e.-martingales with the property $d(w0) + d(w1) = 2d(w)$ are always recursive. The difference then is clear from Corollary 3.8.

A supermartingale $d$ succeeds on a set $A$ if $\limsup_n d(A\upharpoonright n) = \infty$. The class of all sets on which $d$ succeeds is denoted by $S[d]$. A supermartingale succeeds on a class if and only if it succeeds on every member of it. One can prove that a class has Lebesgue measure 0 if and only if there is a (super)martingale that succeeds on it. By imposing restrictions on the complexity of the martingales Lutz obtained his resource bounded generalization of the classical theory of Lebesgue measure.

Since supermartingales are real-valued functions, we need a notion of computability for them.

**Definition 2.4** A supermartingale $d$ is an r.e.-supermartingale if there is an r.e.-function $\hat{d} : \omega \times 2^{<\omega} \rightarrow \mathbb{Q}^+$ such that

$$\left( \forall k \in \omega \right) \left( \forall w \in 2^{<\omega} \right) \left| d(w) - \hat{d}(k, w) \right| \leq 2^{-k}.$$

The function $\hat{d}$ is called an r.e.-computation of $d$.

Now a class $A$ has r.e.-measure $0$, denoted $\mu_{r.e.}(A) = 0$, if there exists an r.e.-supermartingale that succeeds on $A$. Similarly, $A$ has rec-measure $0$ ($\mu_{rec}(A) = 0$) if there is a rec-martingale (i.e. a martingale with a recursive computation) that succeeds on $A$. $A$ has r.e.-rec-measure $1$ if $A^c = \{ X : X \not\in A \}$ has r.e.-rec-measure $0$.

We will also use the notion of constructor. A constructor is a function $\delta : 2^{<\omega} \rightarrow 2^{<\omega}$ with the property that $\delta(x) \supsetneq x$ for every $x \in 2^{<\omega}$. The set constructed by $\delta$ is the unique set $R(\delta)$ with $\delta^0(e) \supsetneq R(\delta)$, where $\delta^n$ denotes the $n$-th iterate of $\delta$. 

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\textbf{Definition 2.5} For a class of functions $\Delta$, $R(\Delta)$ denotes the class
\[ \{ R(\delta) : \delta \text{ is a constructor and } \delta \in \Delta \} . \]

3 R.e.-measure

We will use the following fact, which is a version of a folklore fact obtained by many people independently (e.g. [1, Lemma 2.1], [13]).

\textbf{Lemma 3.1} Let $d$ be an r.e.-supermartingale. Then there is a supermartingale $d' : 2^{\omega} \to Q^+$ which is r.e. such that $S[d'] \supseteq S[d]$.

\textbf{Proof.} Let $f : \omega \times 2^{\omega} \to Q^+$ be an r.e.-computation of $d$:
\[ (\forall k \in \omega)(\forall w \in 2^{\omega}) |d(w) - f_k(w)| \leq 2^{-k} . \]

Define $\tilde{d}(w) = f_{|w|}(w) + 4 \cdot 2^{-|w|}$. Then $d(w) \geq \tilde{d}(w) + 3 \cdot 2^{-|w|}$ and $\tilde{d}(w) \leq d(w) + 5 \cdot 2^{-|w|}$. Furthermore,
\[
\begin{align*}
\tilde{d}(w0) + \tilde{d}(w1) & \leq d(w0) + 5 \cdot 2^{-|w|-1} + d(w1) + 5 \cdot 2^{-|w|-1} \\
& \leq 2(d(w) + 5/2 \cdot 2^{-|w|}) \\
& \leq 2(d(w) + 3 \cdot 2^{-|w|}) \\
& \leq 2\tilde{d}(w),
\end{align*}
\]
so $\tilde{d}$ is a supermartingale, and $\tilde{d}$ is r.e. because $f$ is. Finally, $S[\tilde{d}] \supseteq S[d]$ since $\tilde{d}(w) \geq d(w)$. \hfill \qed

Having defined r.e.-measure using the approach of Lutz [10] we now prove that this yields the same notion of measure as considered before by Martin-Löf, Schnorr, and others. The proof is a simple extension of the work of Schnorr [15].

\textbf{Definition 3.2} (Martin-Löf [12], Kautz [5]) A class $A$ of Lebesgue measure 0 is $\Sigma^C_n$-approximable if there is a recursive sequence of $\Sigma^C_n$-classes $\{ S_i \}_{i \in \omega}$ with $\lambda(S_i) \leq 2^{-i}$ and $A \subseteq \bigcap_i S_i$. A set $A$ is $n$-random if $A$ is not $\Sigma^C_n$-approximable. The $1$-random sets are also called Martin-Löf random.

Schnorr [15, Satz 5.3] has shown that class $A$ is $\Sigma_1$-approximable if and only if there is a subcomputable (“subberechenbare”) martingale that succeeds on $A$. Here a martingale $g$ is subcomputable if it has a recursive approximation $g_s$ satisfying $g_s(w) \leq g_{s+1}(w)$ for every $s \in \omega$ and $w \in 2^{\omega}$, and such that $\lim_s g_s(w) = g(w)$ (note that in this definition it is not required that there is an $s \in \omega$ such that $g_s(w) = g(w)$). Using Lemma 3.1 it is easy to see that a class
\( A \) has r.e.-measure 0 if an only if there is a subcomputable martingale that succeeds on \( A \) with the additional requirement that the recursive approximation \( g_s \) of \( g \) reaches its limit. So clearly every class of r.e.-measure 0 has subcomputable measure 0. But the converse holds too: Given a subcomputable martingale \( g \) with recursive approximation \( g_s \), we define an r.e.-supermartingale \( d \) with \( S[d] \supseteq S[g] \) as follows. Define \( d \) through a recursive approximation \( d_s \): For every \( w \in 2^{<\omega} \), \( d_0(w) = 0 \), and if \( g_{s+1}(w) > d_s(w) - 2^{-|w|-1} \) then put \( d_{s+1}(w) = g_{s+1}(w) + 2^{-|w|} \), and put \( d_{s+1}(w) = d_s(w) \) otherwise. It is immediate from the definition of \( d_s \) that

\[
(\forall w \in 2^{<\omega})(\exists s)(\forall t \geq s)[d_t(w) = d_s(w)],
\]

hence \( d_s(w) \) reaches a limit \( d(w) \) after a finite number of steps. It holds for every \( w \in 2^{<\omega} \) that \( g(w) + 2^{-|w|-1} \leq d(w) \leq g(w) + 2^{-|w|} \), hence it follows from the martingale property of \( g \) that

\[
\begin{align*}
d(w0) + d(w1) &\leq g(w0) + g(w1) + 2 \cdot 2^{-(|w|+1)} \\
&\leq 2(g(w) + 2^{-|w|-1}) \\
&\leq 2d(w),
\end{align*}
\]

whence \( d \) is indeed an r.e.-supermartingale. So we have arrived at

**Theorem 3.3** For every class \( A \subseteq 2^{\omega} \), \( A \) is \( \Sigma_1 \)-approximable if and only if 

\[ \mu_{r.e.}(A) = 0. \]

**Corollary 3.4** A class \( A \) has r.e.-measure 0 if and only if \( A \) does not contain a Martin-Löf random set.

**Proof.** From the existence of a universal Martin-Löf-test (Martin-Löf [12]) it follows that

\[ \mu_{r.e.}(\{A \in 2^{\omega} : A \text{ is Martin-Löf random}\}) = 1. \]

Note that this is stronger than merely saying that the class of Martin-Löf random sets has Lebesgue measure 1 (Schnorr [15, Korollar 4.7]). Hence if \( A \) contains no Martin-Löf random set then \( \mu_{r.e.}(A) = 0 \). The converse is true by the definition of Martin-Löf random set.

In particular, we can use all the known facts about Martin-Löf randomness in the study of r.e.-measure.

Recall the definition of \( R(\Delta) \) from Definition 2.5. Lutz [10] observed that

\[ R(\text{rec}) = \text{REC} \]

\[ \mu_{\text{rec}}(\text{REC}) \neq 0. \]
$R(\text{r.e.})$ is the set of all $R(\delta)$ for $\delta$ r.e., the results of constructors $\delta : 2^{<\omega} \to 2^{<\omega}$ for which the set $\{ (x, y) : y \leq \delta(x) \}$ is r.e., where $\leq$ denotes the usual lexicographic ordering on $2^{<\omega}$.

**Definition 3.5** We say that $A \in 2^\omega$ is right-limit of the infinite set of initial segments $X \subseteq 2^{<\omega}$ if $(\forall \sigma \in X)[\sigma \leq A\upharpoonright |\sigma|]$ and $(\forall n)(\exists m \geq n)[A\upharpoonright m \in X].$

We say that $\sigma \in 2^{<\omega}$ is right-limit of the (possibly infinite) set of initial segments $X \subseteq 2^{<\omega}$ if $(\forall \tau \in X)[\tau \subseteq \sigma \lor (\exists n)(\tau\upharpoonright n < \sigma\upharpoonright n)].$

It will be useful to have the following characterization. $R(\text{r.e.})$ is the class of sets that are right-limits of r.e. sets of initial segments in $2^{<\omega}$. Equivalently, $A \in R(\text{r.e.})$ iff there is a recursive function $\phi : \omega \times \omega \to \{0, 1\}$ such that $\lim_k \phi(k, n) = A(n)$, and for every $k, n \in \omega$, $\phi(k)\downarrow n \leq \phi(k+1)\downarrow n$ and

$$(\forall m < n)[\phi(k, m) = A(m)] \land \phi(k, n) = 1 \Rightarrow A(n) = 1$$

(1)

(if $A$ is the right-limit of an r.e. set $X \subseteq 2^{<\omega}$, with recursive enumeration $\{X_s\}_{s \in \omega}$, we get $\phi$ as in (1) by putting $\phi(k, n) = \tau_k(n)$, where $\tau_k$ is the right-limit of $X_s$).

From this characterization it follows immediately that $\text{RE} \subseteq R(\text{r.e.}) \subseteq \Delta_2$ (the second inclusion follows from the Limit Lemma [16, III.3.3]). In the next theorem we locate the class $R(\text{r.e.})$ more precisely. We denote by DRE the class of differences $W_e - W_d$ of r.e. sets (the d.r.e. sets, see [16, p57]). This is precisely the class of sets with a recursive approximation that changes at most two values for every argument (starting with 0).

**Theorem 3.6** The inclusions $\text{RE} \subseteq R(\text{r.e.}) \subseteq \leq^m(K)$ are both proper. Furthermore, $R(\text{r.e.}) \not\subseteq \leq^m(K)$ and DRE $\not\subseteq R(\text{r.e.})$.

$$
\begin{array}{cccccc}
\quad & \quad & \quad & \quad & \quad & \quad \\
\text{DRE} & \subset & \subseteq^m(K) & \quad & \subseteq^m(K) & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\text{RE} & \subset & \subseteq^m(K) & \quad & \subseteq^m(K) & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad & \quad \\
\text{R(\text{r.e.})} & \quad & \quad & \quad & \quad & \quad \\
\end{array}
$$

**Proof.** Note that indeed we have the inclusion $\text{R(\text{r.e.})} \subseteq \leq^m(K)$ by the characterization in Lemma 2.1, where we can take $h(x) = 2^x$. That the two inclusions $\text{RE} \subseteq \text{R(\text{r.e.})} \subseteq \leq^m(K)$ are both proper will follow from $\text{R(\text{r.e.})} \not\subseteq \leq^m(K)$ and DRE $\not\subseteq \text{R(\text{r.e.})}$, since $\text{RE} \subseteq \text{DRE} \subseteq \leq^m(K) \subseteq \leq^m(K)$. 

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One can prove \( R(\text{r.e.}) \not\subseteq \mathsf{Set}(K) \) directly by a diagonalization construction, but the result will also follow from Theorem 3.7 (iii). So we do not give a proof here.

That \( \text{DRE} \not\subseteq R(\text{r.e.}) \) can be proved directly using a finite injury argument, but, as pointed out to us by an anonymous referee, it also follows from results in [16]: There exist d.r.e. sets that do not have r.e. degree [16, VII.2.4] (this also requires a finite injury priority argument), but on the other hand every set in \( R(\text{r.e.}) \) does have r.e. degree. This last fact follows from the Limit Lemma, see [16, Corollary 3.4]. Hence there exist sets in DRE that are not in \( R(\text{r.e.}) \). □

One may now wonder what the r.e.-measure is of classes such as \( \text{REC}, \text{RE}, \) and \( R(\text{r.e.}) \). One can easily prove that \( \mu_{\text{re}}(\text{REC}) \neq 0 \) by constructing for each recursive martingale \( d \) a recursive set \( A \) such that \( d \) does not succeed on \( A \) Given \( A \upharpoonright n \) define \( A(n) = 1 \iff d((A \upharpoonright n)1) \leq d((A \upharpoonright n)0) \). However, this argument fails if \( d \) is an r.e.-supermartingale since the set \( A \) constructed as above will only be \( \Delta_2 \). The following theorem shows that indeed the analogous result is not true at all.

Let \( B \) be the smallest Boolean algebra containing all the r.e. sets, i.e. \( B \) is the closure of the class of r.e. sets under complementation, union, and intersection (\( B \) is Ershov’s Boolean, or difference hierarchy, see Odifreddi [14]). \( B \) is exactly the “cone” of sets that are bounded truth-table reducible to \( K \) ([14, Prop. III.8.7]).

**Theorem 3.7**

(i) \( \mu_{\text{re}}(\text{RE}) = 0 \).

(ii) \( \mu_{\text{re}}(\mathsf{Set}(K)) = 0 \).

(iii) \( \mu_{\text{re}}(R(\text{r.e.})) \neq 0 \).

**Proof.** (i) follows from the well-known fact that no Martin-Löf random set is r.e. (In fact, every Martin-Löf random set is bi-immune [6].) (ii) follows from (i) and the fact that for every set \( A \) in \( B \) either \( A \) or \( \overline{A} \) contains an infinite r.e. set (Jockusch and others [16, III.3.10]).

(iii) Let \( d \) be any r.e.-supermartingale. We have to show that there is an element of \( R(\text{r.e.}) \) on which \( d \) does not succeed. Let \( A \) be the leftmost path in \( 2^\omega \) such that \( d(A \upharpoonright n) \leq 1 \) for every \( n \). Note that \( A \) exists since for any (super)martingale \( d \) with \( d(e) = 1, \{ B \subseteq \omega : \forall n(d(B \upharpoonright n) \leq 1) \} \) is nonempty, in fact, has positive Lebesgue measure. It is easy to see, using that \( d \) is r.e., that \( A \) is the right-limit of an r.e. set of initial segments, and hence that \( A \in R(\text{r.e.}) \). □

**Corollary 3.8** The measures \( \mu_{\text{re}} \) and \( \mu_{\text{r.e.}} \) are unequal.

**Proof.** \( \mu_{\text{r.e.}}(\text{REC}) = 0 \) by Theorem 3.7 (i), but Lutz proved that \( \mu_{\text{re}}(\text{REC}) \neq 0 \), cf. the discussion preceding Theorem 3.7. □
Theorem 3.7 shows that r.e.-measure is not suited for the quantitative study of RE, hence Lutz’s approach [10], that worked for classes like $2^\omega$, REC, and the linear and polynomial deterministic exponential time and space classes, does not work here.

It follows from Theorem 3.7 that if $A$ is r.e. then $\mu_{r.e.}(\{A\}) = 0$. The converse is certainly not true: it is easy to construct a martingale which succeeds on all nondense sets (i.e. sets with a characteristic string that contains significantly more zero’s than one’s), and among those are sets of arbitrary high complexity.

Corollary 3.8 shows that the approach using martingales $d$ with the property $d(w0) + d(w1) = 2d(w)$ instead of our supermartingales with the weaker property $d(w0) + d(w1) \leq 2d(w)$ does indeed make a difference (cf. the discussion after Definition 2.3).

It follows from Theorem 4.4 that $\mu_{r.e.}(\Delta_2) \neq 0$. $\Delta_2$ is exactly the “cone” of sets that are Turing reducible to $K$, the halting set. In our notation: $\Delta_2 = \leq^T(K)$. Results on the measure of cones form a classical topic in the intersection of measure theory and computability theory. As a corollary to Theorem 3.7 we have a stronger result than $\mu_{r.e.}(\leq^u(K)) \neq 0$, saying that for truth-table reducibility the cone $\leq^u(K)$ does not have r.e.-measure 0.

**Corollary 3.9** $\mu_{r.e.}(\leq^u(K)) \neq 0$.

**Proof.** In Theorem 3.6 we saw that $R(\text{r.e.}) \subset \leq^u(K)$, so the result follows from Theorem 3.7 (iii). \qed

Theorem 3.7 (ii), in conjunction with Corollary 3.9, gives a precise border between r.e.-measure zero and r.e.-measure nonzero in terms of the common recursion-theoretic reducibilities below $K$.

**4 Arithmetical measure**

The results from Section 3 generalize to all the levels of the arithmetical hierarchy. We have the following function classes corresponding to the various levels of the arithmetical hierarchy.

**Definition 4.1** The class of (total) $\Sigma_n$-functions, $n \geq 1$, is defined to be

$$\{f : 2^{\leq \omega} \to \mathbb{Q}^+ : \{(x,y) : f(x) \geq y\} \in \Sigma_n\}.$$  

Similarly, the class of (total) $\Pi_n$-functions, $n \geq 1$, consists of $\{f : 2^{\leq \omega} \to \mathbb{Q}^+ : \{(x,y) : f(x) \leq y\} \in \Sigma_n\}$. The function classes $\Delta_n$ are defined as $\Delta_n = \Sigma_n \cap \Pi_n$.
Note that the $\Sigma_n$-functions are those that are $\Delta_n$-approximable from below, by a $\Delta_n$-function that attains its limit value. Similarly, $\Pi_n$-functions can be approximated from above in the same manner. Therefore, the $\Delta_n$-functions coincide with the functions computable recursively in $\theta^{(n)}$.

The measures $\mu_\Sigma_n$, $\mu_\Pi_n$, and $\mu_\Delta_n$, with $n \geq 1$, are defined exactly as the measures $\mu_{r.e.}$ and $\mu_{rec}$. So, for example, $\mu_\Pi_n(A) = 0$ if there is no supermartingale with a computation in $\Pi_n$ that succeeds on $A$. As in the case of rec-measure, in the definition of $\mu_\Delta_n$ we may use martingales instead of supermartingales.

Now Lemma 3.1 is proved exactly as before, and the proof of Theorem 3.7 relativizes.

**Theorem 4.2** For all $n \geq 1$, $\mu_\Sigma_n(\leq^{\text{ett}}(\theta^{(n)})) = 0$.

**Proof.** Relativize the proof of Theorem 3.7 (ii) to the oracle $\theta^{(n)}$. $\square$

**Corollary 4.3** For all $n \geq 1$, $\mu_\Sigma_n \neq \mu_\Delta_n$.

In the next theorem we find an exact border between $\Delta_2$-measure zero and $\Delta_2$-measure nonzero in terms of the reducibilities $\leq_{wu}$ and $\leq_T$ below $K$.

**Theorem 4.4** For every $n \geq 2$, $\mu_\Delta_n(\leq^{\text{ett}}(\theta^{(n-1)})) = 0$, and for every $n \geq 1$, $\mu_\Delta_n(\Delta_n) \neq 0$.

**Proof.** For the first part, we prove that $\mu_{\Delta_2}(\leq^{\text{ett}}(K)) = 0$, and note that the proof relativizes. By [14, Ex. III.8.14], $\leq^{\text{ett}}(K)$ equals $\leq^u(K)$, the class of sets that truth-table reduce to $K$. Whence it suffices to prove that $\mu_{\Delta_2}(\leq^u(K)) = 0$. Given codes $e$ and $d$ for the recursive functions $\varphi_e$, $\varphi_d$, we assume that these represent a tt-reduction of a set $A$ to $K$ as in Lemma 2.1. We construct a $\Delta_2$-martingale $d_{(e,d)}$ that succeeds on $A$ if this guess is correct. Put $d_{(e,d)}(e) = 1$. Given $d_{(e,d)}(w)$, use the oracle $\theta'$ to compute whether $\varphi_d(|w|) \downarrow$ and $\varphi_e(s,|w|)$ changes at most $\varphi_d(|w|)$ times. Note that in this case we can compute with $\theta'$ the “limit” $c \in \omega$, i.e. we can $\theta'$-compute $s$ and $c$ in $\omega$ such that for all $t \geq s$ either $\varphi_e(t,|w|) \downarrow = c$ or $\varphi_e(t,|w|) \uparrow$. If this limit exists and is $i \in \{0,1\}$, put $d_{(e,d)}(w) = 2 \cdot d_{(e,d)}(w)$ and $d_{(e,d)}(w(1-i)) = 0$. Otherwise set $d_{(e,d)}(w(0)) = d_{(e,d)}(w(1)) = 0$. It is easy to check that if the pair $\varphi_e, \varphi_d$ forms a tt-reduction, then $d_{(e,d)}$ succeeds on the set that is reduced. The definition of $d_{(e,d)}$ is uniform in $e$ and $d$, so $\leq^u(K)$ is a $\Delta_2$-union of $\Delta_2$-measure 0 classes, hence has $\Delta_2$-measure 0.

The second part, $\mu_\Delta_n(\Delta_n) \neq 0$, is proved exactly as $\mu_{rec}(\text{REC}) \neq 0$ (see the discussion preceding Theorem 3.7). $\square$
Note that analogous to Theorem 3.6 we have that \( \Sigma_n \subseteq R(\Sigma_n) \subseteq \Delta_{n+1} \) and \( \Pi_n \subseteq R(\Pi_n) \subseteq \Delta_{n+1} \). From Theorem 4.4 it follows that for every \( n \geq 1 \), \( \mu_{\Delta_{n+1}}(R(\Sigma_n)) = 0 \) and \( \mu_{\Delta_{n+1}}(R(\Pi_n)) = 0 \). In contrast to this result we have the generalized version of Theorem 3.7 (iii) (the proof of the second part is completely symmetric):

**Theorem 4.5** For all \( n \geq 1 \), \( \mu_{\Sigma_n}(R(\Sigma_n)) \neq 0 \) and \( \mu_{\Sigma_n}(R(\Pi_n)) \neq 0 \).

To complete the picture of inclusions, note that \( \Delta_n = R(\Delta_n) = R(\Sigma_n) \cap R(\Pi_n) \). It follows that \( \Sigma_n \not\subseteq R(\Pi_n) \) since otherwise \( \Sigma_n \subseteq R(\Sigma_n) \cap R(\Pi_n) = \Delta_n \), a contradiction. In particular \( R(\Sigma_n) \not\subseteq R(\Pi_n) \). So the only inclusion relations are \( \Delta_n \subseteq \Sigma_n \subseteq R(\Sigma_n) \subseteq \Delta_{n+1} \), \( \Delta_n \subseteq \Pi_n \subseteq R(\Pi_n) \subseteq \Delta_{n+1} \), and no other inclusions hold.

In Corollary 4.3 we have seen that the measure induced by the function class \( \Sigma_n \) differs from the measure induced by \( \Delta_n \). We now prove that, surprisingly, the measure induced by \( \Pi_n \) equals the latter. Whence the measure \( \mu_{\Sigma_n} \) is stronger (more sets have measure 0) than the measure \( \mu_{\Pi_n} \). The reason for this asymmetry lies in the asymmetry of the supermartingale property \( d(w0) + d(w1) \leq 2d(w) \), which makes \( \Sigma_n \)-supermartingales more powerful than \( \Pi_n \)-supermartingales.

**Theorem 4.6** For all \( n \geq 1 \), \( \mu_{\Pi_n} = \mu_{\Delta_n} \).

**Proof.** We give the proof for \( n = 1 \). The proof for arbitrary \( n \in \omega \) is obtained by relativizing the following proof to the oracle \( \emptyset^{(n)} \). It suffices to prove that if a class has \( \Pi_1 \)-measure 0 then it has rec-measure 0. Let \( d \) be a \( \Pi_1 \)-supermartingale, with nonincreasing recursive approximation \( d_s \) say. We prove that there exists a rec-supermartingale \( \check{d} \) with \( S[\check{d}] \supseteq S[d] \). W.l.o.g. \( d(\epsilon) = 1 \). Define \( \check{d}(\epsilon) = 1 \). Suppose now that \( \check{d}(w) \) has been defined and that \( \check{d}(w) \geq d(w) \). Choose the least \( s \in \omega \) such that \( d_s(w0) + d_s(w1) \leq 2d_s(w) \). Note that \( s \) exists since \( d \) is a martingale and \( (\exists s)(\forall t \geq s)[d_t(w) = d(w)] \). Define \( \check{d}(wi) = d_s(wi) \), for \( i \in \{0,1\} \). Then \( \check{d}(wi) \geq d(wi) \) because \( d_s(wi) \) is non-increasing in \( s \). For \( \check{d} \) thus defined we have that \( \check{d}(w0) + \check{d}(w1) \leq 2d_s(w) \leq 2\check{d}(w) \), so \( \check{d} \) is a supermartingale. Clearly \( \check{d} \) is recursive, and \( S[\check{d}] \supseteq S[d] \) because for every \( w \in 2^{<\omega} \), \( \check{d}(w) \geq d(w) \). \( \square \)

5 Notes

Kautz [5, p26] has shown that a class \( A \) is \( \Sigma_n^C \)-approximable if and only if there is a \( C(n-1) \)-recursive sequence (rather than just recursive) of \( \Sigma_n^C \)-classes \( \{S_i\}_{i \in \omega} \)
with \( \lambda(S_i) \leq 2^{-i} \) and \( A \subseteq \bigcap_i S_i \). It follows that we may relativize the result of Theorem 3.3 to the oracle \( \emptyset^{(n-1)} \) to obtain

**Theorem 5.1** For every class \( A \subseteq 2^{\omega} \) and every \( n \geq 1 \), \( A \) is \( \Sigma_n \)-approximable if and only if \( \mu_{\Sigma_n}(A) = 0 \).

Schnorr [15, Satz 7.6] proves that there is a 1-random (Martin-Löf random) set \( A \) in \( \Delta_2 \). A natural example of such a set is Chaitin’s Halting Probability \( \Omega \) [9, p187]. For more on Martin-Löf random sets in \( \Delta_2 \) see M. van Lambalgen [8]. The results from the previous sections show

**Theorem 5.2** (i) There is a Martin-Löf random set in \( \leq_t(K) \).

(ii) There is no Martin-Löf random set in \( \leq_{tt}(K) \).

**Proof.** Immediate from Corollary 3.4, Corollary 3.9 and Theorem 3.7 (ii). \( \square \)

With a similar proof one can actually show, using Theorem 5.1, that for any \( C \in 2^{\omega} \), there exists a \( C \)-random set in \( \leq_t(C^{(n)}) \) (even in \( R(\Sigma_n^C) \)) but not in \( \leq_{tt}(C^{(n)}) \).

It is known that the Martin-Löf random set of Theorem 5.2 (i) cannot have the same tt-degree as \( K \) (Bennett [2], Juedes, Lathrop, and Lutz [4]).

Note that there is not an analogue of the result \( \mu_{t.e}((A \in 2^{\omega} : A \text{ is Martin-Löf random}) = 1 \) for the case of \( \Delta_n \)-measure. In fact, it is easy to see that \( \mu_{\Delta_n}((A \in 2^{\omega} : A \text{ is } \Delta_n \text{-random}) \neq 1 \). Namely, for every \( \Delta_n \)-martingale \( d \) there exists \( A \in \Delta_n \) such that \( A \not\in \mathcal{S}[d] \). Hence \( d \) does not succeed on all the non-\( \Delta_n \)-random sets. However, the class of \( \Delta_n \)-random sets does of course have Lebesgue measure 1.

It follows from results of Arslanov and Kučera [6] that if \( A \) is an r.e. set that is Turing-incomplete then \( \mu_{t.e.}(\leq^*_\tau(A)) = 0 \).

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**References**


