The Origins of Modern Modal Logic

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Today's diversity of research topics in modal logic is almost bewildering: modal logic appears as

- a logic of necessity and possibility;
- a language for studying provability and expressibility in various formal theories;
- a language for talking about relational (and topological) structures and their uses in computer science, cognitive science, computational linguistics, ...;
- a knowledge representation formalism;
- a language for talking about the behavior of programs;
- a fragment of first-order language balancing expressivity and complexity;
- a formalism for representing linguistic meaning.

In the face of such multifariousness it is natural to ask, what is this thing called *modal logic*? In this introduction we will give our answer to that question by pointing to the roots of modal logic. In our view, modal logic — hereafter: ML — has grown out of three areas: philosophy, foundations of mathematics and computer science (CS), including artificial intelligence (AI). We will explain this view in the next three sections.

To say that ML has its roots in a certain area must not be taken to imply that most workers in that area consider ML a good thing. On the contrary! Mathematicians are often convinced that ML is part of philosophy or philosophical logic and that it has little to do with 'real' mathematics. Philosophers for their part often feel that ML has

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become too mathematical to have much claim to philosophical interest. And the interest in ML of researchers in computer science is usually of a pragmatic nature, even though their discipline has given rise to many systems of ML. We hope that the articles in this book will go some way towards dispelling negative attitudes of this kind.

1 Philosophy

If we confine ourselves to formal logic (and ignore important predecessors like Aristotle, Ockham, and Leibniz), then the first recognized systems of ML were constructed by the philosopher C.I. Lewis in 1918 (see Lewis 1918, Lewis and Langford 1932). But the philosopher-logician Hugh McColl already developed a formal system of ML in 1906 (see McColl 1906).

The beginning of the twentieth century was a very productive time for logic. The foundations of mathematics were hit by the crisis of set-theoretic paradoxes, and as a result they were exposed to revision, not excluding the underlying classical logic. Compared to actual human reasoning, the model of reasoning provided by classical logic is rather stereotypical; it is not difficult to find problematic features. One such feature that Lewis did not like was that classical logic gives rise to the so-called 'paradoxes of material implication.' Why on Earth do we have to regard propositions like

If the Moon is made of green cheese then $2 \times 2 = 4$

as true? For we do have to regard it as true if we consider that $2 \times 2 = 4$ and accept the classical laws

$$\varphi \to (\psi \to \varphi), \quad \varphi \to \psi, \varphi / \psi.$$

Lewis's idea was to consider the *strict implication*

'it is necessary that' $(\varphi \to \psi)$

instead of the usual material implication $\varphi \to \psi$. To axiomatize the necessity operator, Lewis constructed first one modal calculus and then four more, calling them simply **S1**–**S5**. Of these, **S4** and **S5** have become famous; **S4** — which was also discovered, independently, by Orlov (1928)[1] — may be defined as classical logic (**Cl**) plus four postulates:

$$\begin{aligned}\mathbf{S4} = \mathbf{Cl} \quad &+ \quad \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)\\ &+ \quad \Box\varphi \to \Box\Box\varphi\\ &+ \quad \Box\varphi \to \varphi\\ &+ \quad \varphi/\Box\varphi.\end{aligned}$$

[1]More precisely, Orlov based his system on a logic weaker than classical logic, but the modal axioms and inference rules intended to axiomatize the operator 'it is provable' (see below) were those of **S4**.

Strict implication $\Box(\varphi \to \psi)$ behaves better than material implication. At least,

$$\Box(\varphi \to \Box(\psi \to \varphi))$$

is not provable in Lewis's systems. However, Lewis's solution was not completely satisfactory: he could not get rid of all paradoxical formulas. E.g.,

$$\Box(\Box\varphi \to \Box(\psi \to \varphi))$$

is a theorem of **S4**. We will not go deeper into this branch of research in philosophical logic; the reader can find more information in (Anderson and Belnap 1975).

Lewis never tried to describe his semantic intuitions in a formal way. The first semantics for modal logic — except for Łukasiewicz's many-valued logics — was the modal algebra developed by McKinsey and Tarski (1944, 1946, 1948). While the philosophers found algebra useful as a technical tool, their philosophy was not influenced by it. Yet, without formal semantics, ML would never have flourished. The first philosophically important semantics was that of Carnap (1947), whose distinction between extension and intension was inspired by Frege's (1892) distinction between *Bedeutung* and *Sinn*. (*Meaning* is a possible translation of both those German nouns, but English-speaking philosophers have more or less settled for using *reference* for *Bedeutung* and *sense* for *Sinn*.)

But the semantics that has been most important philosophically is of course the possible worlds semantics. While the origins of the latter are still discussed, there is no doubt that the enormous impact of the possible worlds semantics is due to Kripke (1959, 1963); it is only fitting that 'Kripke semantics' has become synonymous with 'possible worlds semantics.' However, historians point out that several other logicians had similar ideas, independently, at about the same time: Kanger (1957a, 1957b), Hintikka (1962). The discussion about priority loses some of its importance when it is recognized that, from a mathematical point of view, the possible worlds semantics can be retrieved from the Stone representation theorem for Boolean algebras with operators that is given in a much earlier paper by Jónsson and Tarski (1951): there you find — though not by name — both possible worlds and accessibility relations. That logic is not discussed in their paper, and that neither Jónsson nor Tarski seems to have realized the philosophical significance of their work is another matter.

Von Wright's (1951) deontic logic and Prior's (1957) tense logic are prime examples of ML in the 1950s, as is Hintikka's (1962) epistemic logic in the 1960s. Montague grammar (Montague 1974) is a much less orthodox example of ML. Whereas previous philosophers had been engaged in conceptual analysis, at least primarily, Montague's later work fused conceptual analysis with analysis of natural language. While the scope of Montague's theory goes far beyond that of previous modal logicians, it should still be possible to fit all more traditional MLs into his system. In that sense one might see Montague grammar as a development, extreme but natural, of the Frege-Carnap-Kripke line.

2 Foundations of Mathematics

After Lewis, hundreds, if not thousands, of other systems of philosophical logic were constructed to represent and analyze various kinds of necessity and related notions. However, they attracted little attention from mathematicians and mathematical logicians. (As George Boolos (1993) said, "Because of the metaphysical character of the notions of necessity and possibility, their remoteness from sensory experience, and uncertain application of the terms 'necessary' and 'possible', modal logic has always been a subject more or less on the periphery of logic.")

In 1933, however, Gödel, in a two-page abstract (Gödel 1933), presented work on modal logic whose importance was realized only much later. While Lewis had been interested in logical necessity, Gödel was interested in necessity as provability. To understand why, let us return once again to the beginning of the twentieth century. Another very serious and ambitious attempt to revise classical logic and mathematics was undertaken by the Dutch mathematician Brouwer (1907, 1908). His criticism was aimed at the non-constructive character of classical logic. Here is a well-known example of a non-constructive proof.

Theorem. *There exists an irrational number x such that $x^{\sqrt{2}}$ is rational.*

Proof. The number $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational. If it is rational then let $x = \sqrt{2}$. Otherwise, let $x = \sqrt{2}^{\sqrt{2}}$. ⊣

This proof neither provides us with a number x as required, nor does it show us a way of constructing one. Brouwer didn't like such proofs, regarding them as dangerous games with infinity that might lead to paradox — or even worse: as meaningless. To avoid this, we need another logic, one that singles out and describes the laws of 'constructive' reasoning in the sense that if we want to prove that something exists, then we have to provide an algorithm constructing this something. He called this logic *intuitionistic* (in reference to Kant's notion of 'pure intuition').

The main principle of Brouwer's intuitionism asserts that the truth of a mathematical statement can be established only by producing a constructive proof of the statement. So the intended meaning of the intuitionistic logical connectives is defined in terms of (canonical) *proofs* and *constructions*:

- A proof of $\varphi \wedge \psi$ consists of a proof of φ and a proof of ψ.
- A proof of $\varphi \vee \psi$ is given by presenting either a proof of φ or a proof of ψ.
- A proof of $\varphi \to \psi$ is a construction which, given a proof of φ, returns a proof of ψ.
- $\bot$ has no proof.

This interpretation — given by Brouwer's younger colleague Heyting and also, earlier and independently, by Kolmogorov — can hardly be regarded as a precise semantic definition suitable for constructing intuitionistic logic. Nevertheless, the Brouwer-Heyting-Kolmogorov interpretation calls into question the validity of the Law of Excluded Middle $\varphi \vee \neg\varphi$, and, indeed, the intuitionists reject this classically valid principle.

Intuitionistic logic was first constructed in the form of a calculus by Heyting (1930). Actually, from a formal point of view, Heyting's logic may be viewed as a subsystem of classical logic, collapsing into the latter if the Law of Excluded Middle is added. But the problem of finding a good semantics for intuitionistic logic remained open. This was an attractive problem and became connected with the notion of algorithm that was being developed in those days. Prominent mathematicians like Kolmogorov (1925, 1932), Gödel (1932, 1933), Tarski (see McKinsey and Tarski, 1948), and Stone (1937) were interested in intuitionistic logic. But this introduction is not about intuitionism; it is about modal logic.

Gödel (1933) proposed to interpret intuitionistic logic in classical logic by means of extending the latter with an explicit modal operator 'it is provable' and formalizing the informal interpretation above. What are the logical laws describing provability? Gödel formulated them precisely:

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi),\ \ \Box\varphi \to \Box\Box\varphi,\ \ \Box\varphi \to \varphi,\ \ \varphi/\Box\varphi.$$

In other words, the resulting provability logic was none other than **S4**. Then Gödel defined a translation T taking intuitionistic formulas into modal ones by putting

- $\mathsf{T}(p) = \Box p$, where p is a proposition letter;
- $\mathsf{T}(\bot) = \Box\bot$;
- $\mathsf{T}(\varphi \wedge \psi) = \mathsf{T}(\varphi) \wedge \mathsf{T}(\psi)$;
- $\mathsf{T}(\varphi \vee \psi) = \mathsf{T}(\varphi) \vee \mathsf{T}(\psi)$;
- $\mathsf{T}(\varphi \to \psi) = \Box(\mathsf{T}(\varphi) \to \mathsf{T}(\psi))$.

The intuitionistic connectives are transformed by T into the corresponding classical ones, but they are understood now in the context of 'provability.' This translation turns out to be an embedding of intuitionistic logic into **S4** in the sense that

$$\mathbf{Int} \vdash \varphi \text{ iff } \mathbf{S4} \vdash \mathsf{T}(\varphi),$$

and so we really get a classical proof-interpretation of intuitionistic logic. Yet, that was not precisely what Gödel wanted. What kind of provability is axiomatized by **S4**? Some vague informal notion of provability? No, he wanted a notion of formal provability, within a decent logical system, like Peano Arithmetic **PA**. Let us recall some basic facts about **PA**.

All syntactical constructions of the arithmetic language (terms, formulas, proofs, etc.) can be effectively codified by natural numbers; the code $\ulcorner\phi\urcorner$ of an arithmetic formula ϕ is called the *Gödel number* of ϕ.[2] Gödel constructed a formula $Bew(x)$ (from *beweisbar* — provable) with a single free variable x such that, for every natural number n,

$$\mathbf{PA} \vdash Bew(n) \text{ iff } n = \ulcorner\phi\urcorner \text{ and } \mathbf{PA} \vdash \phi \text{ for some arithmetic formula } \phi.$$

In other words, $Bew(\ulcorner\phi\urcorner)$ asserts that the formula ϕ is provable in **PA**.

Now, what if we interpret the provability operator $\Box$ of **S4** as the provability predicate Bew of **PA**? To be more precise, we consider an arbitrary map * from modal formulas to arithmetic sentences such that

- $\bot^*$ is $0 = 1$;
- $(\varphi \odot \psi)^* = \varphi^* \odot \psi^*$, for $\odot \in \{\wedge, \vee, \rightarrow\}$;
- $(\Box\varphi)^* = Bew(\ulcorner\varphi^*\urcorner)$;

and ask: what modal formulas reflect the laws of arithmetic provability? But then we have got a problem: $\Box\bot \rightarrow \bot$ is an axiom of **S4**, and so by necessitation we have $\mathbf{S4} \vdash \Box(\Box\bot \rightarrow \bot)$. If we read this in **PA** then we'll get:

$$Bew(\ulcorner\neg Bew(\ulcorner 0 = 1\urcorner)\urcorner),$$

i.e., it is provable in **PA** that **PA** is consistent. But this contradicts Gödel's Second Theorem!

So what is the true modal logic of provability? Is there any reasonable provability interpretation of **S4**? These questions raised by Gödel attracted serious mathematicians to modal logic and brought a number of interesting results. Here we will briefly sketch only one of them: a result obtained by Solovay (1976). He studied a modal calculus which is

[2] For simplicity we do not distinguish here between natural numbers and the arithmetic terms representing them.

now known as the Gödel–Löb logic:

$$\begin{aligned}\mathbf{GL} = \mathbf{Cl} \quad &+ \quad \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)\\ &+ \quad \Box\varphi \to \Box\Box\varphi\\ &+ \quad \Box(\Box\varphi \to \varphi) \to \Box\varphi\\ &+ \quad \varphi/\Box\varphi.\end{aligned}$$

Solovay proved that this modal propositional calculus adequately describes the properties of the predicate $Bew(x)$ which are provable in **PA** in the sense that, for every modal formula φ, $\mathbf{GL} \vdash \varphi$ iff, for all arithmetic interpretations *, $\mathbf{PA} \vdash \varphi^*$. This discovery has led to an interesting analysis of provability in formal arithmetic and other theories (see e.g. Boolos 1993). One quotation from a paper by Albert Visser is particularly apposite (Visser 1998):

> "A miracle happens. In one hand we have a class of marvelously complex theories in predicate logic, ... like **PA** or **ZF**. In the other we have certain propositional modal theories of striking simplicity. We translate the modal *operators* of the modal theories to certain specific, fixed, defined *predicates* of the predicate logical theories. These special predicates generally contain an astronomical number of symbols. We interpret the propositional variables by arbitrary predicate logical sentences. And see: the modal theories are sound and complete for this interpretation. They codify precisely the schematic principles in their scope. Miracles do happen ..."

Thus, the second root of ML is in the foundations of mathematics, where ML proved to be a suitable tool for studying the notions of provability, interpretability of one system in another, and others. Moreover, the tool is so powerful that it can produce new arithmetic results like the following recent characterization of primitive recursive functions found by Beklemishev (1997). Consider a theory formulated as **PA**, but with the induction schema only applicable to Π_2 formulas without parameters, i.e., formulas with the prefix $\forall\exists$. If this theory proves that some program terminates for all inputs, then the function computed by that program has to be primitive recursive. In other words: provably total computable functions coincide with the primitive recursive ones.

We conclude this section by mentioning some very recent work:

- Visser's, de Jongh's and their students' and collaborators' investigations into provability in intuitionistic arithmetic (see, for instance, Iemhoff's paper in this volume);

- Artemov's proof-interpretation of **S4** (see his paper below);
- Barwise and Moss's use of ML for the analysis of the foundations of set-theory (see the paper of their PhD student Baltag in this volume).

3 Computer Science

The third root of ML grew in CS and AI. After the discovery of the possible-worlds semantics in the late 1950s it didn't take long to realize that the language of propositional modal logic provides a good alternative to that of first-order logic if we want to talk about various relational structures, for instance labelled transition systems representing the behavior of computer programs, or relational structures like $\langle \mathbb{N}, < \rangle$ representing flows of time. Temporal logics are perhaps the most popular kind of modal logics used in program specification and verification (see e.g. Manna and Pnueli 1992, 1995), temporal databases (see e.g. Abiteboul *et al.* 1996, Chomicki 1994), distributed and multi-agent systems (see e.g. Fagin *et al.* 1995), and other fields. A less known example is the use of **S4** for qualitative spatial representation and reasoning (see e.g. Bennett 1996, Renz and Nebel 1999), which is possible due to another remarkable feature of **S4** discovered by McKinsey and Tarski (1946): the **S4**-box can be interpreted as the interior operator of topological spaces, and **S4** is sound and complete with respect to this interpretation.

Dynamic logics were a new type of modal logics, the beginnings of which took place in the depths of theoretical computer science. In 1967, Robert W. Floyd had written a paper attempting "to provide an adequate basis for formal definitions of the meanings of programs in appropriately defined programming languages, in such a way that a rigorous standard is established for proofs about computer programs, including proofs of correctness, equivalence, and termination." In the same year, C.A.R. Hoare took this idea further. He noted that computer programming "is an exact science in that all the properties of a program and all the consequences of executing it in any given environment can, in principle, be found out from the text of the program itself by means of purely deductive reasoning." He also noted the difficulty caused by the proliferation of programming languages: the "exact choice of axioms will to some extent depend on the choice of programming language." Hoare (1967) aims for as much generality as seemed possible to him.

The fundamental new primitive in Hoare's paper is a three-place operator $\cdot\{\cdot\}\cdot$. The idea is that $\varphi\{\alpha\}\psi$ obtains if φ is a precondition, α is a program and ψ is a description of the result of executing α in a situation when φ obtains. In Hoare's words (notation modified), the notation $\varphi\{\alpha\}\psi$ is interpreted, "If the assertion φ is true before initiation of a program α, then the assertion ψ will be true on its completion ... provided that the program successfully terminates". Hoare proposes a small number of rules appropriate for his new primitive, among them the following:

If $\vdash \varphi\{\alpha\}\psi$ and $\vdash \psi \to \theta$, then $\vdash \varphi\{\alpha\}\theta$.
If $\vdash \varphi\{\alpha\}\psi$ and $\vdash \theta \to \varphi$, then $\vdash \theta\{\alpha\}\psi$.
If $\vdash \varphi\{\alpha\}\psi$ and $\vdash \psi\{\beta\}\theta$, then $\vdash \varphi\{\alpha\,;\beta\}\theta$.
If $\vdash (\varphi \wedge \theta)\{\alpha\}\psi$, then $\vdash \varphi\{\textbf{while}\ \theta\ \textbf{do}\ \alpha\}(\neg\theta \wedge \psi)$.

Here, $\alpha\,;\beta$ is the program composed of the programs α and β, in that order (α immediately followed by β), while **while** θ **do** α is an iterative program that tests the condition θ, omits α if θ is false but, if θ is true, executes α and then again tests the condition θ, and so on. Thus an execution of **while** θ **do** α, always repetitive, terminates when θ is false, if it terminates at all. In other words, a terminating execution of **while** θ **do** α consists of a finite number (zero or more) of executions of α with θ being false on termination.

Dynamic logic began when Vaughan Pratt saw that Hoare's ideas cried out for a modal logical analysis. Pratt gave a more formal treatment than that of Hoare, and he also widened the scope to include nondeterministic automata. It is not quite clear how sharply Hoare wanted to distinguish between propositions and programs, but in Pratt's theory they belong to distinct categories. To this semantic distinction corresponded a syntactic distinction between formulas and terms. The operations on formulas included the Boolean connectives, and the operations on terms included Kleene's regular operations (sum, concatenation and the Kleene star):

$\alpha + \beta$	'α or β,'
$\alpha\,;\beta$	'first α, then β,'
α^*	'α some number of times.'

But there were also two more complex kinds of operators. First, if φ represents a proposition, then $\varphi?$ represents the test program with respect to φ: if it is executed, the machine terminates if φ obtains and fails otherwise. Thus the test program checks *that* rather than *whether*:

$\varphi?$ 'check that φ'.

Second, if α represents any program, then $[\alpha]$ and $\langle\alpha\rangle$ are unary propositional operators:

$[\alpha]\varphi$	'after every terminating execution of α, φ',
$\langle\alpha\rangle\varphi$	'after some terminating execution of α, φ'.

It is easy to give a formal semantics that reflects, in a precise yet natural manner, the informal intuition just presented. What is interesting from the point of view of modal logic is that each term is interpreted as an accessibility relation. This means that ordinary modal logic, like ordinary tense logic, can be seen as a very special case of dynamic logic. Note that Hoare's rules are readily translated into Pratt's language by putting

$$\varphi\{\alpha\}\psi \;=_{def}\; \varphi \wedge [\alpha]\psi$$

in addition to requiring

$$\langle\alpha\rangle\varphi \to [\alpha]\varphi$$

to be generally valid (the latter because of Hoare's determinism). Also more standard concepts can be simulated in this language, for example, **if** φ **then do** α, **else do** β by

$$(\varphi? \,;\alpha) \;+\; ((\neg\varphi)? \,;\beta)$$

and **while** ϕ **do** α by

$$(\varphi? \,;\alpha)^* \,;(\neg\varphi)?.$$

One philosophically interesting fact about Pratt's modelling is that it gives a coherent, perfectly respectable, set-theoretical representation of action, a concept many philosophers wish to explain away.

In this connection it may be appropriate to mention the logic of common knowledge (see Fagin *et al.* 1995). If there are n agents $0, \ldots, n-1$, for some positive number n, then, for each $i < n$, let K_i be a box operator with the intuitive reading 'agent i knows that.' It is easy to define the notion 'Everybody knows that': $E\varphi =_{def} K_0\varphi \vee \cdots \vee K_{n-1}\varphi$. But common knowledge is not so easily pinned down: when everybody knows, and everybody knows that everybody knows, and everybody knows that everybody knows that everybody knows, and so on. So let C be a new primitive operator bearing this intuitive interpretation. It turns out that the characteristic axiom schemata of the logic of common knowledge are

$$\begin{array}{ll} C\varphi \to \varphi & C\varphi \to E\varphi \\ C\varphi \to CC\varphi & (\varphi \wedge C(\varphi \to E\varphi)) \to C\varphi. \end{array}$$

If this is compared to the characteristic axiom schemata of dynamic logic, the resemblance is obvious:

$$\begin{array}{ll} [\alpha^*]\varphi \to \varphi & [\alpha^*]\varphi \to [\alpha]\varphi \\ [\alpha^*]\varphi \to [\alpha^*][\alpha^*]\varphi & (\varphi \wedge [\alpha^*](\varphi \to [\alpha]\varphi)) \to [\alpha^*]\varphi. \end{array}$$

Epistemic logics with modal operators of the form 'agent i knows' or 'agent i believes' are also applied for analyzing multi-agent systems. But these are to a certain extent applications and further developments of a known formalism rather than a genuine root. So maybe we had better tell another story about knowledge representation in AI.

The idea of using first-order logic as the language for representing knowledge and reasoning about it appeared to be extremely promising in the 1950s and 1960s. This is no surprise, because the language is

- very expressive;
- formal (suitable for encoding);
- it has a semantics;
- there are sound and complete proof systems (means for formal reasoning).

The only weak point:

- it is not effective (in fact, undecidable).

In the 1960s and 1970s, numerous attempts were made to use first-order logic in systems of AI, verification of programs, and automated theorem proving. One can argue to which extent and in which fields these attempts were successful. However, many researchers in AI were deeply disappointed in the logical approach and tried to find other ways of treating knowledge, more structural, visual, object-oriented, and without using logic.

One of them — semantic networks — was proposed by Quillian and Raphael in 1968 (see Brachman and Levesque 1985). Roughly, a semantic network is a directed graph in which nodes represent entities and arcs represent binary relations between entities like in the simple example in Figure 1.

Of course, researchers in AI familiar with logic understood that this formalism can be embedded into the language of first-order logic. For instance, Deliyanni and Kowalski (1979) defined an extended form of semantic networks which can be regarded as a syntactic variant of the clausal form of logic. Anyway, semantic networks became popular, probably for three reasons. First, they are intuitive and simple. Second, they can be used to represent structured knowledge exhibiting a hierarchical order (as in the above figure). And third, they correspond to only a fragment of first-order logic, which can be more effectively implemented. The only disadvantage was that semantic networks did not have any reasonable formal semantics, a feature for which they were severely criticized. (In the depicted network, it is not clear, for instance, whether *all* members of the class Child are children of Eve or only *some* of them.) A solution was found by the end of the 1970s when Brachman constructed a system called KL-ONE and defined a knowledge representation language that was very close to semantic networks, on the one hand, and possessed a Tarski-type semantics, on the other (see Brachman and Schmolze 1985). A variant of this language, proposed by Schmidt-Schauß

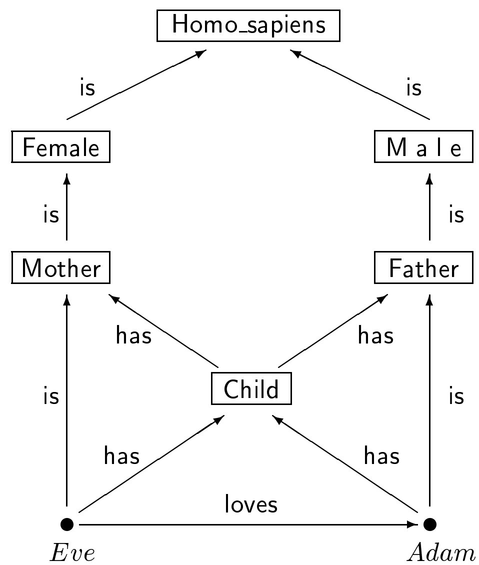

FIGURE 1 A semantic network.

and Smolka (1991) and known as $\mathcal{ALC}$, has the following primitive symbols:

- concept names: $C_0, C_1, \dots$ (for sets or concepts);
- role names: $R_0, R_1, \dots$ (for binary relations between objects);
- object names: $a_0, a_1, \dots$ (for objects);
- the booleans: $\wedge$, $\neg$, $\top$,
- quantifiers over roles $\exists R_i$, $\forall R_i$.

Complex concepts are defined inductively as follows: all concept names as well as $\top$ are (*atomic*) *concepts*, and if C, D are concepts, R is a role name, then $C \wedge D$, $\neg C$, $\exists R.C$ and $\forall R.C$ are *concepts*. If C and D are concepts, R a role name and a, b object names, then expressions of the form $C = D$, aRb, $a : C$ are *formulas*. Languages of this sort were called KL-ONE type languages, terminological logics, concept description languages or description logics. Here are some examples of what you can express in $\mathcal{ALC}$.

$$(\mathsf{Child} \rightarrow \exists\mathsf{has.Mother} \wedge \exists\mathsf{has.Father}) = \top$$
$$\mathsf{Fortune\_hunter} = \mathsf{Male} \wedge \forall\mathsf{loves.}(\neg\mathsf{Female} \vee \mathsf{Rich})$$
$$Eve \ \mathsf{loves} \ Adam$$
$$Eve : \exists\mathsf{has.Child}$$

A *model* of $\mathcal{ALC}$ is a structure of the form

$$I = \langle \Delta, R_0^I, \dots, C_0^I, \dots, a_0^I, \dots \rangle ,$$

where Δ is a non-empty set of objects, the *domain* of I, R_i^I are binary relations on Δ interpreting the role names, C_i^I subsets of Δ interpreting the concept names, and a_i^I are objects in Δ interpreting the object names. The *value* C^I of a concept C and the *truth-relation* $I \models \varphi$ are defined inductively in the following way:

1. $\top^I = \Delta$ and $C^I = C_i^I$, for $C = C_i$;
2. $(C \wedge D)^I = C^I \cap D^I$;
3. $(\neg C)^I = \Delta - C^I$;
4. $x \in (\exists R_i.C)^I$ iff $\exists y\ (xR_i^I y \wedge y \in C^I)$;
5. $x \in (\forall R_i.C)^I$ iff $\forall y\ (xR_i^I y \rightarrow y \in C^I)$;
6. $I \models C = D$ iff $C^I = D^I$;
7. $I \models a : C$ iff $a^I \in C^I$;
8. $I \models aR_ib$ iff $(a^I, b^I) \in R_i^I$.

A lot of real working knowledge representation systems based on description logics of this type and supplied with rather effective sound and complete reasoning procedures were constructed in the 1980s and 1990s. Description logic has become one of the main topics of each big conference in AI. But what is the connection with ML? Let us have a closer look at the definition of models and think of I as a Kripke frame with worlds Δ and accessibility relations R_i; let us think of concept names as propositional variables, of $\exists R_i$ and $\forall R_i$ as $\Diamond_i$ and $\Box_i$ interpreted by the relation R_i. Then we immediately understand that what we have is simply a terminological variant of a poly-modal language interpreted on the usual poly-modal Kripke frames. For

$$x \models \Diamond_i p \text{ iff } \exists y\ (xR_iy \wedge y \models p);$$

$$x \models \Box_i p \text{ iff } \forall y\ (xR_iy \rightarrow y \models p).$$

This fact was noticed first by K. Schild (1991). Trying to find a formalism that is better than first-order logic we come to the language of ML!

In Sections 1 and 2, ML was explicitly called for: there was a need for a language with modal operators ‘it is necessary’ or ‘it is provable.’ In this section (and in other applications in CS and AI) ML emerges as a sort of compromise between expressive power and complexity when we are talking about relational structures. On the one hand, the choice of ML as a knowledge representation formalism shows that it has enough expressive power when relational structures are concerned. And on the other hand, standard MLs turn out to be decidable. For instance, the satisfiability problem for the basic systems of ML is PSPACE-complete.

The decidability of standard modal logics is an interesting phenomenon. Researchers in computational logic in general and theorem proving in particular always wanted to single out natural, sufficiently expressive and decidable fragments of first-order logic. But not many fragments of this sort are known. Quite recently, Andréka, van Benthem and Németi (1998) started a new line of investigations aimed to single out well-behaved 'modal' features of first-order logic.

Let us first look at ML from the point of view of first-order logic. It is well-known that there is a translation of modal formulas into first-order ones:

$$\begin{aligned} ST(p_i) &= P_i(x) \\ ST(\bot) &= \bot \\ ST(\psi \odot \chi) &= ST(\psi) \odot ST(\chi), \text{ for } \odot \in \{\wedge, \vee, \rightarrow\} \\ ST(\Box\psi) &= \forall y\ (xRy \rightarrow ST(\psi)\{y/x\}), \end{aligned}$$

where y is an individual variable not occurring in $ST(\psi)$. The first-order formula $ST(\varphi)$ is called the *standard translation* of φ. Starting from this translation, Andréka, van Benthem and Németi defined the following class of first-order formulas:

$$\exists \overline{y}\, (G(\overline{x}, \overline{y}) \wedge \phi(\overline{x}, \overline{y})).$$

Here, $\overline{x}$, $\overline{y}$ stand for finite sequences of individual variables and $G(\overline{x}, \overline{y})$ is an atomic formula, called a *guard*, containing all free variables of $\phi(\overline{x}, \overline{y})$. The fragment containing such formulas, called the *guarded fragment* of first-order logic, turned out to be decidable. Andréka, van Benthem and Németi explain the good computational behavior of the guarded fragment in terms of its special syntactic features, and especially in terms of restricted quantification. Vardi (1997) and Grädel (1999) provide an alternative explanation in terms of the so-called tree model property.

Interestingly, the proof of the decidability of the guarded fragment is based upon the idea of viewing first-order logic as a (multi-dimensional) modal logic: not only the modal operators can be regarded as (bounded) quantifiers, but also the first-order quantifiers themselves can be treated as modal operators. Briefly, first-order models are represented as Kripke models of the form $\mathfrak{M} = \langle A, \{R_x\}_{x \in var}, I\rangle$, where 'worlds' in A are all possible assignments to the variables in var, aR_xb holds iff assignments a and b differ only on variable x, and I gives truth-values to predicates at every world. Then $\mathfrak{M} \models \exists x\varphi[a]$ iff there is $b \in A$ such that aR_xb and $\mathfrak{M} \models \varphi[b]$. This semantical approach was developed by Tarski, Halmos, Quine and their followers (see e.g. Henkin *et al.* 1971, 1985; Halmos 1962; Quine 1972; Németi 1991). Thus we have been brought to one more important connection of ML — this time with algebraic logic.

4 Conclusion

We have tried to elucidate the question what ML is by pointing to its roots. Here are some partial answers that came up during our exposition:

- ML is a branch of mathematical and philosophical logic studying formal models of correct reasoning which involves various concepts of a modal kind such as 'necessity' and 'possibility'.
- ML is a tool for investigating natural language semantics in general and, in particular, the meaning of philosophically, linguistically and cognitively relevant notions and constructions.
- ML constructs special purpose languages suitable for talking about relational and topological structures (transition systems, membership in set theories, relations between objects in knowledge bases, topological spaces, etc.) and is concerned with the balance between expressive power and complexity of such languages.

Obviously, looking at the roots of ML is at most half of the story. To fully appreciate the field in all its diversity one should also consider current developments as well as hints of what's to come. The current volume certainly provides a broad view of the state of the art in ML, while glimpses of upcoming themes can also be discerned: ML and game theory, a renewed interest in proof methods, more attention to applications and experiments, ... Above all, what this volume demonstrates is that ML is very much alive. No doubt ML will continue to develop independently of any effort to try to determine what it 'really' is.

This Volume

The first two Advances in Modal Logic (AiML) workshops in Berlin (1996) and Uppsala (1998) attracted 100 contributed and 15 invited papers. This volume contains a selection of papers presented at Advances in Modal Logic 1998 (AiML'98), the second installment of the only international workshop series aimed at presenting an up-to-date picture of the state of the art in modal logic (and modal-like logics) and its applications. AiML'98 was held at Uppsala University, Uppsala, Sweden, on October 16–18, 1998.

The contributions to this volume were selected after a two-stage refereeing process, with one round occurring before the workshop and the second after the workshop. In each round each paper was refereed by at least two referees.

There is one contribution to this volume that we would like to single out. During AiML'98 the Best Paper Award was given to Alexandru Baltag for his paper *STS: A Structural Theory of Sets*; the paper is included as the first contribution to this volume.

Many people provided invaluable help in bringing about AiML'98 and this volume... you know who you are. We want to thank you all for doing a wonderful job! Finally, for all things modal, we would like to refer you to `www.aiml.net`.

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Krister Segerberg
Maarten de Rijke
Heinrich Wansing

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