The logic of Peirce algebras

M. de Rijke

Computer Science/Department of Software Technology

Report CS-R9467 December 1994
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The Logic of Peirce Algebras

Maarten de Rijke
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

Abstract

Peirce algebras combine sets, relations and various operations linking the two in a unifying setting. This paper offers a modal perspective on Peirce algebras. Using modal logic a characterization of the full Peirce algebras is given, as well as a finite axiomatization of their equational theory that uses so-called unorthodox derivation rules. In addition, the expressive power of Peirce algebras is analyzed through their connection with first-order logic, and the fragment of first-order logic corresponding to Peirce algebras is described in terms of bisimulations.

AMS Subject Classification (1991): 03B45, 03G15, 03G25, 08A70.
CR Subject Classification (1991): F.3.0, F.3.1, I.2.4.
Keywords & Phrases: Modal logic, algebraic logic, relation algebras, logics of programs, knowledge representation.

Note: Part of this report will appear in a special issue of the Journal of Logic, Language and Information on Decompositions of First-order Logic, guest-edited by Jerry Seligman.

1. INTRODUCTION

This paper is part of an enterprise to relate modal languages, algebraic languages, and fragments of first-order logic. We will take a fragment of first-order logic for reasoning about binary relations, sets and certain interactions between them, and consider the algebraic framework of Peirce algebras that has recently been designed to capture this fragment (Brink [7]). We will show how Peirce algebras arise as algebraic counterparts of a two-sorted modal language $\mathcal{ML}_2$; this language extends the modal formalism $CG\delta$ that was designed by Venema [26] to reason about binary relations. In $\mathcal{ML}_2$ one can characterize the 'concrete' modal frames corresponding to full Peirce algebras (see De Rijke [22]). Using this characterization we obtain a completeness result for 'concrete' frames. The reason why we work in the modal language $\mathcal{ML}_2$ rather than in its algebraic or first-order counterpart is that it allows us to reason with simple pictures and diagrams, and that powerful techniques for proving modal completeness in rich modal languages have recently become available through results of Venema [27].

Peirce algebras have emerged as the common mathematical structures underlying many phenomena being studied in program semantics, AI and natural language analysis; they are also the modal algebras underlying the dynamic modal logic studied in [19]. Peirce algebras are two-sorted algebras in which sets and relations co-exist together with operations between them that model their interaction. The most important such operations considered here are the Peirce product: that takes a relation and a set, and returns a set

Report CS-R9467
ISSN 0169-118X
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2. Preliminaries

\[ R : A = \{ x \mid \exists y ((x, y) \in R \land y \in A) \} , \]

and right cylindrification \( c \) which takes a set and returns a relation

\[ A^c = \{ (x, y) \mid x \in A \} . \]

In the paper we move back and forth between algebraic logic, modal logic and first-order logic. Although we focus on Peirce algebras as our starting point, we hope that the general message will become clear: to understand fragments of first-order logic it may be very fruitful to view them as modal or algebraic languages, and to understand these one can often exploit known results from each of the vertices in the following triangle:

First-order logic

Algebraic logic

Modal logic

The paper is organized as follows. The next section quickly reviews basic algebraic definitions; it also describes areas where Peirce algebras emerge. §3 briefly discusses the relation between relation algebra and Peirce algebras. §4 introduces the modal language \( \mathcal{ML}_2 \) for describing the modal counterparts of Peirce algebras. §5 presents a characterization of the 'real' or 'concrete' modal frames (corresponding to full Peirce algebras), and §6 builds on this characterization to give a finite axiomatization of these concrete frames. §7 examines the relation between first-order logic the modal language \( \mathcal{ML}_2 \), and §8 concludes with some questions.

2. Preliminaries

In this section we introduce Peirce algebras, and list some application areas where they arise.

Let \( U \) be a set; \( Re(U) \) is \( \{ R \mid R \subseteq U \times U \} \). \( R, S \) typically denote elements of \( Re(U) \), while \( A, B \) typically denote elements of \( 2^U \), the power set of \( U \).

Recall the following operations on elements of \( Re(U) \).

<table>
<thead>
<tr>
<th>Operation</th>
<th>( \nabla )</th>
<th>( { (r, s) \in (U \times U) \mid r, s \in U } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>complement</td>
<td>(- R )</td>
<td>( { (r, s) \in (U \times U) \mid (r, s) \notin R } )</td>
</tr>
<tr>
<td>converse</td>
<td>( R^{-1} )</td>
<td>( { (r, s) \in (U \times U) \mid (s, r) \in R } )</td>
</tr>
<tr>
<td>diagonal</td>
<td>( \text{id} )</td>
<td>( { (r, s) \in (U \times U) \mid r = s } )</td>
</tr>
<tr>
<td>composition</td>
<td>( R \Join S )</td>
<td>( { (r, s) \in (U \times U) \mid \exists u ((r, u) \in R \land (u, s) \in S) } )</td>
</tr>
</tbody>
</table>

We also consider the following operations from \( Re(U) \) and \( Re(U) \times 2^U \) to \( 2^U \)

<table>
<thead>
<tr>
<th>Operation</th>
<th>( \text{do}(R) )</th>
<th>( { x \in U \mid \exists y \in U ((x, y) \in R) } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>range</td>
<td>( \text{ra}(R) )</td>
<td>( { x \in U \mid \exists y \in U ((y, x) \in R) } )</td>
</tr>
<tr>
<td>Peirce product</td>
<td>( R : A )</td>
<td>( { x \in U \mid \exists y \in U ((x, y) \in R \land y \in A) } ),</td>
</tr>
</tbody>
</table>

as well as the following operations going from \( 2^U \) to \( Re(U) \)

<table>
<thead>
<tr>
<th>Operation</th>
<th>( A^? )</th>
<th>( { (x, y) \in (U \times U) \mid x = y \land x \in A } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>right</td>
<td>( A^c )</td>
<td>( { (x, y) \in (U \times U) \mid x \in A } ).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Operation</th>
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<tbody>
<tr>
<td>right</td>
<td>( A^c )</td>
<td>( { (x, y) \in (U \times U) \mid x \in A } ).</td>
</tr>
</tbody>
</table>
A relation type algebra is a Boolean algebra with a binary operation $\;\wedge\;$, a unary operation $^{\vee}$, and a constant $1'$. The class FRA of full relation algebras consists of all relation type algebras isomorphic to an algebra of the form $\mathfrak{A}(U) = (\text{Re}(U), \cup, -, 1, \cdot, 1, \cdot, 1')$. RRA is the class of representable relation algebras, that is, $\text{RRA} = \text{SP}(\text{FRA})$ ($= \text{HSP}(\text{FRA})$ by a result due to Birkhoff). RA is the class of relation algebras, that is, of relation type algebras $\mathfrak{A} = (A, +, -, \cdot, ^\vee, 1')$ satisfying the axioms

\begin{align*}
\text{(R0)} \quad & (A, +, -, \cdot) \text{ is a Boolean algebra} & \text{(R5)} \quad & x; 1' = x = 1; x \\
\text{(R1)} \quad & (x + y); z = x; z + y; z & \text{(R6)} \quad & (x^\vee)^\vee = x \\
\text{(R2)} \quad & (x + y)^\vee = x^\vee + y^\vee & \text{(R7)} \quad & (x; y)^\vee = y^\vee; x^\vee \\
\text{(R4)} \quad & (x; y); z = x; (y; z) & \text{(R8)} \quad & x^\vee; -(x; y) \leq -y.
\end{align*}

The reader is referred to Jónsson [13, 14] for the essentials on relation algebra.

A Peirce type algebra is a two-sorted algebra $(\mathfrak{B}, \mathfrak{R}, \cdot, ^c)$, where $\mathfrak{B}$ is a Boolean algebra, $\mathfrak{R}$ is a relation type algebra, $\cdot$ is a function from $\mathfrak{R} \times \mathfrak{B}$ to $\mathfrak{B}$, and $^c : \mathfrak{B} \rightarrow \mathfrak{R}$. The class FPA of full Peirce algebras consists of all Peirce type algebras isomorphic to an algebra of the form $\mathfrak{B}(U) = ((2^U, \cup, -, \emptyset), (\text{Re}(U), \cup, -, 1, \cdot, 1, \cdot, 1, \cdot, 1'))$.

The class RPA of representable Peirce algebras is defined as $\text{RPA} = \text{HSP}(\text{FPA})$, the variety generated by FPA. PA is the class of Peirce algebras, that is, of all Peirce type algebras $\mathfrak{A} = (\mathfrak{B}, \mathfrak{R}, \cdot, ^c)$ where $\mathfrak{B}$ is a Boolean algebra, $\mathfrak{R}$ is a relation algebra, $\cdot$ is a mapping $\mathfrak{R} \times \mathfrak{B} \rightarrow \mathfrak{B}$ such that

\begin{align*}
\text{(P1)} \quad & r : (a + b) = (r : a) + (r : b) & \text{(P4)} \quad & 1' : a = a \\
\text{(P2)} \quad & (r + s) : a = (r : a) + (s : a) & \text{(P5)} \quad & 0 : a = 0 \\
\text{(P3)} \quad & r : (s : a) = (r; s) : a & \text{(P6)} \quad & r^\vee : -(r : a) \leq -a,
\end{align*}

while $^c$ is a mapping $\mathfrak{B} \rightarrow \mathfrak{R}$ such that

\begin{align*}
\text{(P7)} \quad & a^c : 1 = a & \text{(P8)} \quad & (r : 1)^c = r; 1.
\end{align*}

Algebras of the form $(\mathfrak{B}, \mathfrak{R}, :)$ were introduced by Brink [6] as Boolean modules. Sources for Peirce algebras are Brink, Britz and Schmidt [7] and Schmidt [24].

Unlike the one-sorted language of relation algebras, the algebraic language of Peirce algebras has two sorts of terms: one interpreted in $\mathfrak{B}$, the other in $\mathfrak{R}$. Terms of the first sort are called set terms, terms of the second sort relation terms. Identities between set terms are called set identities; identities between relation terms are relation identities.

Brink et al [7] link Peirce algebras to dynamic algebras. Like Peirce algebras these are two-sorted algebras of sets and relations, but their relations are organized in a Kleene algebra, not in a relation algebra. It may be shown that any join-complete Peirce algebra gives rise to a dynamic algebra.

Another class of algebras closely related to Peirce algebras, is the class of extended relation algebras studied by Suppes [25]. Roughly, these are term-definably equivalent with Peirce algebras in which the sortal distinctions have been dropped.

Where Peirce algebras emerge

In a number of areas frameworks are studied that have Peirce algebras in common as their underlying mathematical structures: modal logic, arrow logic, knowledge representation, natural language analysis, and weakest pre specifications.
In the setting of information processing Van Benthem [4] and De Rijke [19] study a system of dynamic modal logic called DML. DML is similar to propositional dynamic logic (PDL) in that it has formulas and procedures. The formulas $\phi$ and programs $\alpha$ of DML are built up as follows:

$$
\phi ::= p | \bot \phi | \phi \land \phi | \text{do}(\alpha) | \text{ra}(\alpha) | \text{fix}(\alpha) \quad \text{and}
$$

$$
\alpha ::= \exp(\phi) | -\alpha | \alpha^+ | \alpha \land \alpha | \alpha ; \alpha | \phi ?.
$$

Here $\exp(\phi)$ is the special relation of ‘expanding one’s information with $\phi’$, and $\text{fix}(\alpha)$ is a formula that is true precisely at fixed points for $\alpha$. Like PDL, DML only allows equational reasoning with formulas — not with programs. The modal algebras for DML are Peirce algebras over a single relation, the information order underlying the $\exp$ construct. To obtain a proper match one has to allow multiple $\exp$ constructs, each with its own underlying information order. The corresponding structures give rise to full Peirce algebras, and conversely. Moreover, the (extended) DML-operators are definable in full Peirce algebras, and the operators of full Peirce algebras are definable on DML-models:

$$
\begin{array}{|c|c|c|c|c|}
\hline
DML & \text{do}(\alpha) & \exp(\phi) & \phi ? & \text{FPA} \\
\hline
\text{FPA} & \langle \bigvee : \alpha \rangle & \langle \bigwedge : \phi \rangle & \phi ^c \cap \text{Id} & \alpha : \phi \\
\hline
\text{FPA} & \langle \bigwedge : \phi \rangle & \phi ^c \cap \text{Id} & \text{DML} & \text{do}(\alpha; \phi ?) \\
\hline
\end{array}
$$

This implies that the complete axiomatization of DML structures presented in [21] also generates the ‘set equations’ valid in FPA.

Arrow logic arises as an effort to do transition logic without the computational complexity that comes with transition logics based on the identification of transitions as ordered pairs. Instead, arrow logic as developed by Van Benthem [3] takes transitions seriously as dynamic objects in their own right. The general program proposes a re-design of systems of transition logic to isolate the genuine computational aspects from the mathematical modeling aspects. Van Benthem [4] contains samples of this program; in particular, it discusses a two-sorted arrow logic whose models have both states and arrows, and whose formulas are sorted accordingly. The models of this (decidable) arrow logic may be viewed as an ‘arrow-ized’ version of our Peirce algebras; the decidability result is obtained by abstracting away from any set-theoretical assumptions concerning objects and operations of Peirce algebras. For instance, a test $\phi ?$ is successfully performed at an arrow $x_a$ if there exists a state $y_s$ that is ‘test-related’ to $x_a$ and which satisfies $x_a \models \phi ?$ iff for some state $y_t$, $T x_a y_s$ and $y_s \models \phi$.

In terminological languages one expresses information about hierarchies of concepts. They allow the definition of concepts and roles built out of primitive concepts and roles using various language constructs. Concepts are interpreted as sets, and roles as binary relations. Brink et al. [7] propose a terminological language $U^-$ whose operations are a notational variant of the operations of (full) Peirce algebras. For instance, $U^-$ has an operation restrict that takes a relation and a set and returns a relation: $\text{restrict} R C = \{(x, y) | (x, y) \in R \land y \in C\}$. As $U^-$ and (full) Peirce algebras share the same ontology, and the same operations, Peirce algebras supply a semantic interpretation for the terminological language $U^-$, in which the basic terminological concerns, viz. subsumption and satisfiability problems, re-appear as derivability issues in equational logic.

The next example involves the extended relation algebras mentioned earlier; as [25, 5] show, those structures arise in attempts to equip fragments of natural language with variable free
semantics. I will illustrate the main point with an example from [24]. Consider a natural language fragment described by a phrase structure grammar $G$ as in the left-hand side of (2.1), where S, NP, VP, TV, PN have their usual meaning: 'sentence,' 'noun phrase,' 'verb phrase,' 'transitive verb' and 'proper noun.'

\[
\begin{align*}
S & \rightarrow \text{NP + VP} \\
\text{VP} & \rightarrow \text{TV + NP} \\
\text{NP} & \rightarrow \text{PN}
\end{align*}
\]

\[\text{[NP]} \subseteq [\text{VP}] \quad \text{[TV]} : [\text{NP}] \quad [\text{PN}]. \tag{2.1}\]

Production rules in the grammar are associated with a semantic function $\cdot$ in a compositional way as indicated in the right-hand side of (2.1). In other words, semantic trees are construed in parallel with syntactic derivation trees. The semantic trees are linked to extended relation algebras via a valuation that maps terminal symbols of $G$ onto an element of the algebra, where nouns are mapped onto sets and transitive verbs onto binary relations, thus equipping our natural language fragment with a variable free semantics.

The use of relation algebra in proving properties of programs goes back at least to De Bakker and De Roever [1]. The calculus of weakest pre specifications of [10] is used as a formal tool in program specification. In this calculus programs are binary relations that may be combined using relation algebraic connectives. A special class of relations is called conditions; they express conditional statements, and are defined as the right ideal elements, that is, elements $R$ for which $R = R; \nabla$. As the right ideal elements form a Boolean algebra, the natural algebraic setting for the calculus of weakest pre specifications is a Peirce algebra with programs living in a relation algebra, conditions living in a separate Boolean algebra, and $\cdot$ and $;$ being used to move across from one to the other, cf. [7].

3. Peirce algebras and relation algebras

Brink et al [7] show that the equivalence $\langle r : a = b \iff r; a^c = b^c \rangle$ holds for all Peirce algebras; they also show that the boolean elements in a Peirce algebra are precisely the right ideal elements, that is the elements satisfying $r = r; 1$:

**Theorem 3.1 (Brink et al [7])** Let $\langle \mathfrak{B}, \mathfrak{R}, ;, \cdot \rangle$ be a Peirce algebra. Then $\mathfrak{B}$ and $\langle \{r \in R \mid r = r; 1\}, +, , , - , 0, 1 \rangle$ are isomorphic.

This result has a number of consequences. It says that, mathematically, Peirce algebras aren't required to study interactions between sets and relations: relation algebras suffice. However, following Brink et al [7] we argue that for application purposes, Peirce algebras are the more natural framework for modeling such interactions. Mathematically, Peirce algebras also have certain advantages over relation algebras: as we will show in §5, they allow for a very natural representation result.

A further consequence of Theorem 3.1 is the non-finite axiomatizability of representable Peirce algebras. The idea is that any finite axiomatization of representable Peirce algebras would yield a finite axiomatization of representable relation algebras, and by [15] this is impossible. Assume that $\Sigma$ is a finite axiomatization of RPA. $\Sigma$ may contain set identities $a = b$; replace these by relation identities $a^c = b^c$ — by Brink et al [7] the first identity is valid (and hence derivable) iff the latter is. So we may assume that $\Sigma$ contains only relation identities. These may still contain occurrences of the Peirce product or of the cylindrification
operator; to obtain purely relational axioms we need to get rid of such occurrences. By Brink et al [7], cylindrification commutes with all operations; in particular, \((r : a)^c = r ; a^c\). Further, as all identities in \(\Sigma\) are assumed to be relational, every occurrence of \(;\) is in the scope of a \(c\). Using this we can remove all occurrences of \(;\), and push all occurrences of \(c\) down to the atomic level.

Let \(s = t\) be a relation identity thus transformed. Using the fact that the boolean elements are precisely the right ideal elements (Theorem 3.1), we translate \(s = t\) into a quasi-identity of the form

\[
\tau_{a_1} = \tau_{a_1}; 1 \& \ldots \& \tau_{a_n} = \tau_{a_n}; 1 \rightarrow [\tau_{a_1}/a_1^c, \ldots, \tau_{a_n}/a_n^c](s = t),
\]

where \(a_1, \ldots, a_n\) are all the boolean terms occurring in \(s = t\), and \(\tau_{a_1}, \ldots, \tau_{a_n}\) are fresh relation terms.

This results in a finite axiomatization of RRA by means of quasi-identity. Now RRA is a discriminator variety, and by general results from universal algebra, in discriminator varieties every quasi-identity is equivalent to an identity ([8]). Hence, we have obtained a finite axiomatization of RRA by means of identity — the desired contradiction. Thus, we conclude that RPA is not finitely axiomatizable.

4. MODALIZING PEIRCE ALGEBRAS

In this section we introduce a modal language for Peirce algebras. To start, Table 1 lists the notation we adopt.

<table>
<thead>
<tr>
<th>relations</th>
<th>Full version</th>
<th>Abstract version</th>
<th>Modal version</th>
</tr>
</thead>
<tbody>
<tr>
<td>top</td>
<td>(\top)</td>
<td>(1)</td>
<td>(\alpha)</td>
</tr>
<tr>
<td>bottom</td>
<td>(\bot)</td>
<td>(0)</td>
<td>(\beta)</td>
</tr>
<tr>
<td>diagonal</td>
<td>(\text{Id})</td>
<td>(1')</td>
<td>(\delta)</td>
</tr>
<tr>
<td>complement</td>
<td>(-)</td>
<td>(-)</td>
<td>(\otimes)</td>
</tr>
<tr>
<td>converse</td>
<td>(\rightarrow)</td>
<td>(\rightarrow)</td>
<td>(\rightarrow)</td>
</tr>
<tr>
<td>union</td>
<td>(U)</td>
<td>(\land)</td>
<td>(\cup)</td>
</tr>
<tr>
<td>implication</td>
<td>(\rightarrow)</td>
<td>(\rightarrow)</td>
<td>(\rightarrow)</td>
</tr>
<tr>
<td>composition</td>
<td>(\cdot)</td>
<td>(\cdot)</td>
<td>(\cdot)</td>
</tr>
</tbody>
</table>

| sets            | \(A, B\)     | \(a, b\)         | \(\phi, \psi\) |
| top             | \(\top\)     | \(\top\)         | \(\top\)      |
| bottom          | \(\bot\)     | \(\bot\)         | \(\bot\)      |
| complement      | \(-\)         | \(-\)             | \(\neg\)      |
| union           | \(U\)         | \(\lor\)          | \(\lor\)      |
| implication     | \(\rightarrow\) | \(\rightarrow\) | \(\rightarrow\) |

| right cylindrification | \(c\) | \(c^c\) | \(\exists\) |
| Peirce produc: | \(\cdot\) | \(\cdot\) | \(\cdot\) |

Table 1: A plethora of notations.

**Definition 4.1** Let \(\Phi = \{p_0, p_1, \ldots\}\) be a countable set of propositional variables. Let \(\Omega\) be a countable set of atomic relation symbols. The formulas of the two-sorted language \(\mathcal{ML}_2(\delta, \otimes, \circ, (\cdot); \Phi; \Omega)\), or \(\mathcal{ML}_2\) for short, are generated by the rules
\[ \phi ::= \bot \mid T \mid p \mid \neg \phi \mid \phi_1 \land \phi_2 \mid (\alpha)\phi \text{ and} \]
\[ \alpha ::= 0 \mid 1 \mid \delta \mid a \mid -\alpha \mid \alpha_1 \land \alpha_2 \mid \otimes \alpha \mid \alpha_1 \circ \alpha_2 \mid \uparrow \phi. \]

The first sort of formulas will be interpreted as sets and called \textit{set formulas}; formulas of the second sort will be interpreted as relations and called \textit{relation formulas}.

**Definition 4.2** A \textit{two-sorted frame} is a tuple \( \mathfrak{F} = (W_s, W_r, I, R, C, F, P) \), where \( W_s \cap W_r = \emptyset \), \( I \subseteq W_r \), \( R \subseteq W_r \times W_r \), \( C \subseteq W_s \times W_s \), \( F \subseteq W_s \times W_s \), and \( P \subseteq W_s \times W_r \times W_s \).

Given a set \( U \), a two-sorted frame is called the \textit{two-sorted Peirce frame} over \( U \) if, for some base set \( U, W_s = U \) and \( W_r = U \times U \), and

\[
I = \{(u,v) \in U \times U \mid u = v\} \\
R = \{((u_1,v_1),(u_2,v_2)) \in (U \times U) \mid u_1 = v_2 \land u_2 = v_1\} \\
C = \{((u_1,v_1),(u_2,v_2),(u_3,v_3)) \in (U \times U) \mid u_1 = u_2 \land v_1 = v_3 \land v_2 = u_3\} \\
F = \{((u_1,v_1),u_2) \in (U \times U) \times U \mid u_1 = v_2\} \\
P = \{((u_1,u_2,v_2),u_3) \in U \times (U \times U) \times U \mid u_1 = u_2 \land v_2 = u_3\}.
\]

The class of two-sorted Peirce frames is denoted by \( \text{TPF} \).

A model for \( \mathcal{ML}_2 \) is a \textit{model based on a two-sorted frame}, that is, a structure \( \mathfrak{M} = (\mathfrak{F}, V) \) where \( \mathfrak{F} \) is a two-sorted frame, and \( V \) is a \textit{two-sorted valuation}, a function assigning subsets of \( W_s \) to set variables, and subsets of \( W_r \) to relation variables. Truth of a formula at a state is defined inductively, with the interesting clauses being

\[
\mathfrak{M}, x_r \models \delta \iff x_r \in I \\
\mathfrak{M}, x_r \models \otimes \alpha \iff \exists y_r (R x_r y_r \land y_r = \alpha) \\
\mathfrak{M}, x_r \models \alpha \circ \beta \iff \exists y_r z_r (C x_r y_r z_r \land y_r = \alpha \land z_r = \beta) \\
\mathfrak{M}, x_s \models (\alpha)\phi \iff \exists y_s (P x_s y_s z_s \land y_s = \alpha \land z_s = \phi) \\
\mathfrak{M}, x_r \models \uparrow \phi \iff \exists y_s (F x_r y_s \land y_s = \phi).
\]

Here \( x_s, y_s, \ldots \) are taken from \( W_s \); \( x_r, y_r, \ldots \) are taken from \( W_r \).

In models based on Peirce frames all the modal connectives receive their intended interpretation. That is, one has \( (u,v) \models \delta \iff u = v \); \( (u,v) \models \otimes \alpha \iff (v,u) \models \alpha \); \( (u,v) \models \alpha \circ \beta \iff \exists w ((u,w) \models \alpha \land (w,v) \models \beta) \); \( u \models (\alpha)\phi \iff \exists v ((u,v) \models \alpha \land v \models \phi) \); and \( (u,v) \models \uparrow \phi \iff u \models \phi \).

**Remark 4.3** We adopt a \textit{local} perspective on satisfiability and consequence. The two-sorted setting of this paper calls for some comments. To avoid messy complications we define consequence only for \textit{one-sorted} sets of formulas \( \Sigma \), and formulas \( \xi \) of the same sort: (compare§6.1). For \( K \) a class of frames we put \( \Sigma \models_K \xi \) iff for all models \( (\mathfrak{F}, V) \) with \( \mathfrak{F} \in K \), and for every element \( x \) in \( \mathfrak{F} \) of the appropriate sort:

\[
(\mathfrak{F}, V), x \models \Sigma \text{ implies } (\mathfrak{F}, V), x \models \xi.
\]

For one-sorted sets of formulas, notions like satisfiability are defined in the usual way.
To be able to state the connection between two-sorted Peirce frames and Peirce algebras, we recall that the complex algebra \( \mathcal{Cm}(\mathcal{F}) \) of a two-sorted frame \( \mathcal{F} \) is given as \( \mathcal{A} = (2^W, \neg, \cap, \emptyset, W) \), \((2^W, \neg, \cap, m_\emptyset, m_\neg, m_\emptyset, m, W)\), where, for \( \# \) an \( n \)-ary modal operator, \( m_\# \) is an \( n \)-ary operator on the power set(s) of the appropriate domain(s) of \( \mathcal{F} \). To be precise

\[
\begin{align*}
m_\emptyset &= \{ x_r | x_r \in I \} \\
m_\neg(X) &= \{ x_r | \exists y_r (R x_r y_r \land y_r \in X) \} \\
m_\cap(X, Y) &= \{ x_r | \exists y_r z_r (C x_r y_r z_r \land y_r \in X \land z_r \in Y) \} \\
m_\emptyset(X, Y) &= \{ x_s | \exists y_r z_s (P x_s y_r z_s \land y_r \in X \land z_s \in Y) \} \\
m_\neg(X) &= \{ x_r | \exists y_s (P x_r y_s \land y_s \in X) \}.
\end{align*}
\]

For \( K \) a class of frames \( \mathcal{Cm}(K) \) is the class of complex algebras of frames in \( K \).

The following justifies our introduction of TPF and the modal language \( \mathcal{ML}_2 \) as tools for understanding full Peirce algebras: if \( \mathcal{F} \) is a two-sorted frame, then \( \mathcal{F} \) is a Peirce frame (or: in TFF) iff \( \mathcal{Cm}(\mathcal{F}) \) is (isomorphic) to a full Peirce algebra. In other words: \( \mathcal{Cm}(\mathcal{TPF}) = \mathcal{FPA} \). Thus, instead of studying full Peirce algebras by algebraic means we can as well study two-sorted Peirce frames by modal means.

However, Peirce frames can’t be characterized in \( \mathcal{ML}_2 \); the reason is that \( \mathcal{FPA} = \mathcal{Cm}(\mathcal{TPF}) \) is not a variety as it is not closed under products or subalgebras. However, if we are willing to extend the modal language, a characterization can be obtained.

More precisely, to characterize the Peirce frames we will use special modal operators called difference operators; their special feature is that they are interpreted using the diversity relation \( \neq \), one for each domain in a two-sorted frame. We use \( D_\neq \) and \( D_\neq \) to denote them:

\[
\begin{align*}
x_s &\models D_\neq \phi \text{ iff for some } y_s \neq x_s, \ y_s &\models \phi \text{ where } x_s, \ y_s \in W_s \\
x_r &\models D_\neq \alpha \text{ iff for some } y_r \neq x_r, \ y_r &\models \alpha \text{ where } x_r, \ y_r \in W_r.
\end{align*}
\]

Using the difference operators we can define other useful operators such as \( E \), where \( E \xi := \xi \lor D_\neq \xi \) (there exists an object with \( \xi \)), and \( O \), where \( O \xi = E(\xi \land \neg D_\neq \xi) \) (there is only one object with \( \xi \)). These defined operators will be indexed with an \( s \) or an \( r \). The reader is referred to De Rijke [18] for details about logics with difference operators.

Observe that on Peirce frames the difference operators can be defined as follows

\[
D'_\neq \phi := (\neg \delta) \phi \quad \text{and} \quad D'_\neq \alpha := (\neg \delta \circ \alpha \circ 1) \cup (1 \circ \alpha \circ \neg \delta).
\]

5. CHARACTERIZING PEIRCE FRAMES

As a prelude to the completeness result for two-sorted Peirce frames, we briefly present a characterization of two-sorted Peirce frames; we refer the reader to (De Rijke [22]) for proofs and details. We proceed in two steps.

5.1 A first approximation

In the first step we characterize a class of Peirce-like frames. We need a number of axioms governing the structure of Peirce frames. We first list the modal axioms handling the relational component of two-sorted frames plus the conditions they impose on such frames; they are simply the modal counterparts of the earlier relation algebraic axioms (R1)–(R8), and the corresponding conditions have been calculated by Lyndon [15] and Maddux [16]. We then
list the modal counterparts of the Peirce axioms (P1)–(P8), and calculate the corresponding conditions on frames. (Recall that a first-order condition $\gamma$ is said to correspond to a modal formula $\xi$ if for all frames $\mathfrak{F}$, $\mathfrak{F} \models \gamma$ iff $\mathfrak{F} \models \xi$; if $\gamma$ corresponds to $\xi$, we also say that $\xi$ defines or expresses $\gamma$.)

The first axiom states that $R$, the interpretation of $\&$, is a function; this is proved by standard arguments.

(MR0) $\&a \leftrightarrow \neg\neg-a$  \hspace{1cm} (CRE) $R$ is a function

So, in frames validating (MR0) we are justified in interpreting $\&$ using a unary function $f$, and evaluating formulas $\&\alpha$ as follows

$\mathcal{M}, x_r \models \&\alpha$ iff $\mathcal{M}, f(x_r) \models \alpha$.

A two-sorted arrow frame is simply a two-sorted frame $\mathfrak{F} = (W, W_r, I, f, C, F, \varphi)$ in which the binary relation $R$ used to interpret the operator $\&$ is a function from $W$ to $W_r$, denoted by $f$. A two-sorted arrow model is a two-sorted model based on a two-sorted arrow frame, where $\&$ is interpreted using the function $f$ as indicated above.

Here are the remaining axioms governing the behaviour of $\delta$, $\&$ and $\circ$, as well as the conditions expressed by these axioms.

(MR1) $a \rightarrow \&\&a$  \hspace{1cm} (CR1) $f(x_r) = x_r$

(MR2) $a \circ (b \circ c) \rightarrow (a \circ b) \circ c$  \hspace{1cm} (CR2) $\forall y_r z_r t_r v_r (C x_r y_r z_r \land C z_r u_r v_r \rightarrow \exists u_r (C x_r y_r z_r \land C w_r y_r u_r))$

(MR3) $(a \circ b) \circ c \rightarrow a \circ (b \circ c)$  \hspace{1cm} (CR3) $\forall y_r w_r u_r v_r (C x_r w_r v_r \land C w_r y_r u_r \rightarrow \exists z_r (C x_r y_r z_r \land C z_r u_r v_r))$

(MR4) $a \rightarrow \delta \circ a$, $a \rightarrow a \circ \delta$  \hspace{1cm} (CR4) $\exists y_r (I y_r \land C x_r y_r z_r)$, $\exists y_r (C x_r y_r z_r \land I y_r)$

(MR5) $\delta \circ a \rightarrow a$, $a \circ \delta \rightarrow a$  \hspace{1cm} (CR5) $\forall y_r z_r (C x_r y_r z_r \land I y_r \rightarrow x_r = z_r)$, $\forall y_r z_r (C x_r y_r z_r \land I x_r \rightarrow x_r = y_r)$

(MR6) $\& (a \circ b) \rightarrow (\& b \circ \& a)$  \hspace{1cm} (CR6) $\forall y_r z_r (C f(x_r) y_r z_r \rightarrow C x_r f(z_r) f(y_r))$

(MR7) $(\& b \circ \& a) \rightarrow (\& (a \circ b)$  \hspace{1cm} (CR7) $\forall y_r z_r (C x_r f(z_r) f(y_r) \rightarrow C f(x_r) y_r z_r)$

(MR8) $\& a \circ -(a \circ b) \land b \rightarrow 0$  \hspace{1cm} (CR8) $\forall y_r z_r (C x_r f(y_r) z_r \rightarrow C z_r y_r x_r)$

Next come the axioms governing the behaviour of the Peirce product and cylindrification.

(MP1) $(a \langle b \rangle p \rightarrow (a \circ b)p$  \hspace{1cm} (CP1) $\forall y_r y'_r z'_r z''_r (P x_r y_r z_r \land P z_r y'_r z''_r \rightarrow \exists y''_r (P x_r y''_r z'_r \land C y''_r y'_r y'_r))$

(MP2) $(a \circ b)p \rightarrow (a) \langle (b)p$  \hspace{1cm} (CP2) $\forall y_r y'_r z'_r z''_r (P x_r y_r z_r \land C y'_r y''_r z''_r \rightarrow \exists z'_r (P x_r y'_r z'_r \land P z'_r y'_r z'_r))$

(MP3) $(\langle b \rangle p \rightarrow p$  \hspace{1cm} (CP3) $\forall y_r z_r (P x_r y_r z_r \land I y_r \rightarrow x_r = z_r)$

(MP4) $p \rightarrow (\langle b \rangle p$  \hspace{1cm} (CP4) $\exists y_r (P x_r y_r z_r \land I y_r)$

(MP5) $(\& (a) \rightarrow (a)p \land p \rightarrow \bot$  \hspace{1cm} (CP5) $\forall y_r z_r (P x_r y_r z_r \rightarrow P z_r f(y_r) x_r)$

(MP6) $(\langle p \rangle \top \rightarrow p$  \hspace{1cm} (CP6) $\forall y_r z'_r z''_r (P x_r y_r z_r \land F y_r z'_r \rightarrow x_r = z''_r)$

(MP7) $p \rightarrow (\langle p \rangle \top$  \hspace{1cm} (CP7) $\exists y_r z_r (P x_r y_r z_r \land F y_r z_r)$

(MP8) $\langle (a) \top \rightarrow (a \circ 1$  \hspace{1cm} (CP8) $\forall y_r y'_r z'_r (P x_r y_r z_r \land P y_r y'_r z'_r \rightarrow \exists z'_r (C x_r y_r z'_r))$

(MP9) $(a \circ 1) \rightarrow \langle (a) \top$  \hspace{1cm} (CP9) $\forall y_r z_r (C x_r y_r z_r \rightarrow \exists z'_r (P x_r y'_r \land F y'_r z'_r)).$
5. Characterizing Peirce frames

It follows from the general results of [20] that the above axioms (MRI) and (MPI) correspond to the conditions (CMr1) and (CPi). All axioms listed here are so-called Sahlqvist formulas, and for such formulas there is an explicit algorithm computing the corresponding relational condition. Here we compute one such correspondence result 'by hand.' We consider axiom (MP7) and condition (CP7)s, i.e., \( p \to (\langle p \rangle T \land \exists y_r z_y (P x_s y_r z_y \land F y_r x_s)) \).

Assume first that \( x_s \) is a set element in some frame \( \mathfrak{F} \) in which (MP7) is valid. This means that (MP7) is true at \( x_s \) under the special valuation that assigns \( p \) to \( x_s \) (and only to \( x_s \)). It follows that \( x_s \models (\langle p \rangle T, \) that is: there are \( y_r, z_y \) with \( P x_s y_r z_y \) and \( y_r \models \langle p \rangle . \) By the latter conjunct, there must be a \( x'_s \) with \( F y_r x'_s \) and \( x'_s \models p \) — but as \( p \) is true at \( x_s \) only, we must have \( x_s = x'_s \):

\[
\begin{array}{c}
\text{y}_r \\
\text{x}_s \\
\text{z}_y
\end{array}
\]

For the converse, assume that we’re in a situation as depicted above, that is, \( P x_s y_r z_y \), \( F y_r x_s \) and \( x_s \models p \). We need that \( x_s \models (\langle p \rangle T) \). As \( F y_r x_s \) and \( x_s \models p \), it follows that \( y_r \models \langle p \rangle \). From this and \( P x_s y_r z_y \) we get \( x_r \models (\langle p \rangle T) \), as required.

Definition 5.1 A two-sorted arrow frame is Peirce like if it satisfies conditions (CR1)–(CR8), as well as (CP1)–(CP9). The class of Peirce like frames is denoted by TPLF.

Lemma 5.2 Let \( \mathfrak{F} \) be a two-sorted arrow frame. Then \( \mathfrak{F} \in \text{TPLF} \iff \mathfrak{F} \models (\text{MPI}) \).

5.2 Characterizing two sorted Peirce frames

We now arrive at the second stage in our characterization result: we narrow down the two-sorted Peirce like frames to two-sorted Peirce frames. Briefly, what we need, to show that a two-sorted Peirce like frame is a two-sorted Peirce frame, is the following

- With every relational element we can associate a unique set element as its first coordinate and a unique set element as its second coordinate.
- With every two set elements we can associate a unique relational element having these set elements as first and second coordinate.

This boils down to having the following conditions satisfied by our Peirce like frames:

(CP10) \( \forall x_r y_s y' \ (F x_r y_s \land F x_r y'_s \rightarrow y_s = y'_s) \)

(CP11) \( \forall x_r y_s y' \ (F f (x_r) y_s \land F f (x_r) y'_s \rightarrow y_s = y'_s) \)

(CP12) \( \forall x_r \exists y_s \ (F x_r y_s) \)

(CP13) \( \forall x_r \exists y_s \ (F f (x_r) y_s) \)

(CP14) \( \forall x_r \exists y_s \exists x_r \ (P x_s z_r y_s) \)

(CP15) \( \forall x_r y_s z_r \ z'_r \ (P x_s z_r y_s \land P x_s z'_r y_s \rightarrow z_r = z'_r) \).

Lemma 5.3 Let \( \mathfrak{F} \in \text{TPLF} \). Then \( \mathfrak{F} \models (\text{CP10}) \sim (\text{CP13}) \).

We leave it to the reader to check that Peirce-like frames validate each of (CP10)–(CP13), and that those conditions can be defined in the modal language \( \mathcal{M}_{\mathcal{C}_2} \). Observe that conditions (CP10)–(CP13) are expressed by the following four modal formulas, respectively:
The proof of this claim is left to the reader.

We will now give a representation result for full Peirce algebras. We like to think that our representation below is more elegant than the usual representations in relation algebra and arrow logic; the latter usually extract points from (a Cartesian product of) the diagonal to obtain a base set over which a full algebra can be built. In the case of Peirce algebras we already have our points available in the domain of set points; we will be able to simply map every relation point \( z_r \) in a Peirce frame onto a pair of set points \( x, y \) already present.

We need the following lemma.

**Lemma 5.4** Let \( \mathcal{F} \) be a two-sorted Peirce like frame. Then

1. \( \mathcal{F} \models \forall z_S y_S z_r (P_{x_S} z_r y_S \rightarrow F_{x_S} z_r F f(z_r) y_S) \), and
2. \( \mathcal{F} \models \forall z_S y_S z_r (F f(z_r) y_S \rightarrow P_{x_S} z_r y_S) \).

**Proof.** To prove (1) assume \( P_{x_S} z_r y_S \). By (CP12) \( F f(z_r) x_S' \), for some \( x_S' \). By (CP6) \( x_S = x_S' \), hence \( F f(z_r) x_S \). Likewise, by (CP13), (CP5) and (CP6) we have \( F f(z_r) y_S \). For (2), assume that \( F f(z_r) y_S \). By (CR4) there exists \( y_S' \) with \( C r y_S y_S' \). By (CP9) this implies there exist \( y_S' \) with \( P g_r y_S z_r y_S' \). (CP10) and (CP11) then yield \( x_S = y_S' \) and \( y_S = z_r \). Hence \( P_{x_S} z_r y_S \).

**Theorem 5.5** Let \( \mathcal{F} = (W_s, W_r, I, f, C, F, P) \) be a two-sorted Peirce like frame. If \( \mathcal{F} \models (CP14), (CP15) \), then \( \mathcal{F} \) is isomorphic to the two-sorted Peirce frame over \( W_s \).

**Proof.** If \( \mathcal{F} \) is a Peirce like frame satisfying (CP14) and (CP15), then, with every \( z_r \in W_r \) we can associate a unique \( x \) and \( y \) such that \( F f(z_r) x \) and \( F f(z_r) y \). Define a mapping \( g : W_r \rightarrow W_r \times W_s \) by \( g(z) = (x, z_1) \), where \( z_0, z_1 \) are the unique \( x \) and \( y \) with \( F f(z_r) x \) and \( F f(z_r) y \). We prove that \( g \) is an isomorphism.

- \( g \) is surjective. Let \( x_S, y_S \in W_s \). By (CP14) \( P_{x_S} z_r y_S \), for some \( z_r \). By Lemma 5.4 \( F f(z_r) y_S \) and \( F f(z_r) x_S \). Hence \( g(z) = (x_S, y_S) \).

- \( g \) is injective. Let \( z_r, z_r' \in W_r \), and assume \( g(z_r) = g(z_r') \). Then, for some \( x_S, y_S \), we have \( F f(z_r) y_S \) and \( F f(z_r') y_S \) and \( F f(z_r) x_S' \) and \( F f(z_r') y_S' \). By Lemma 5.4 this implies \( P_{x_S} z_r y_S \) and \( P_{x_r'} z_r y_S' \). Hence, by (CP15) \( z_r = z_r' \).

- \( g \) is a homomorphism. To establish this claim we need to consider 5 cases: \( I, f, C, P, F \).

**Case 1:** \( I \) such that \( g(z_r) = (x_S, y_S) \) for some \( x_S, y_S \). Choose \( x_S, y_S \) such that \( g(z_r) = (x_S, y_S) \). By definition \( F f(z_r) x_S \) and \( F f(z_r) y_S \) and so \( P_{x_S} z_r y_S \) by Lemma 5.4. By (CP3) this gives \( x_S = y_S \).

**Case 2:** \( f \) need to show that \( f(g(z_r)) = g(f(z_r)) \). If \( g(z_r) = (x_S, y_S) \), then \( P_{x_S} z_r y_S \), and, by (CP5), \( P_{x_S} z_r y_S \). Hence, \( g(f(z_r)) = (y_S, x_S) = f(g(z_r)) \).

**Case 3:** \( C \) need to show that \( C r x_S y_S \) implies that \( g(z_r) \) is the composition of \( g(y_r) \) and \( g(z_r) \). That is: if \( g(z_r) = (x, y) \), then \( x = y_0, y_1 = z_0, z_1 = x_1 \).
5. Characterizing Peirce frames

Observe that by (CP2) we have \( P_{x_0 y_0} z \), \( P_{z^*} z x_1 \), for some \( z' \). By Lemma 5.4, (CP5'), (CP10) and (CP11) this implies the three identities.

\( F \): here we need to show that \( F_{x_2} x_4 \) implies that if \( g(z) = (z_0, z_1) \) then \( z_0 = x_4 \). But this is immediate from the definition of \( g \) and (CP10).

\( P \): we need to show that \( P_{x_2} x_4 y_4 \) implies \( g(z) = (x_4, y_4) \); this is immediate by Lemma 5.4. \( g^{-1} \) is a homomorphism. Again, this requires us to consider 5 cases.

\( I \): we need to show that whenever \( g(x_4) = (x_4, x_5) \), then \( x_4 \in I \). If \( g(x_4) = (x_4, x_5) \), then \( P_{x_4} x_5 y_5 \). By (CP4) there is a \( y_5 \) such that \( P_{x_4} y_5 x_5 \) and \( I y_5 \). By (CP15) this implies \( y_5 = x_4 \); hence \( I x_4 \).

\( f \): this has already been proved above.

\( C \): assume \( g(x_4) \) is the composition of \( g(y_5) \) and \( g(x_4) \), that is, assume \( g(x_4) = (x_0, x_1), g(y_5) = (y_0, y_1), g(x_4) = (z_0, z_1) \); We need to show that \( C_{x_4} y_5 x_4 \). By definition \( x_0 = y_0, y_1 = z_0, z_1 = x_1 \); so \( P_{x_0} y_0 z_0 \) and \( P_{z_0} x_1 \). By (CM1) the latter implies that for some \( u_4 \), \( P_{x_0} u_4 x_1 \) and \( C_{u_4} y_5 u_4 \). By (CP15) \( u_4 = x_4 \), hence \( C_{x_4} y_5 x_4 \).

\( F \): assume \( g(x_4) = (x_4, y_5) \); we need to show that \( F_{x_4} y_5 x_4 \); but this is immediate from the definitions.

\( P \): assume that \( g(x_4) = (x_4, y_5) \); we have to show that \( P_{x_4} x_5 y_5 \). But \( g(x_4) = (x_4, y_5) \) implies \( F_{x_4} x_5 \) and \( F F_{x_4} x_5 \); now apply Lemma 5.4. \( \dashv \)

**Corollary 5.6** Let \( \mathcal{F} \) be a two-sorted arrow frame. Then

\[ \mathcal{F} \in \text{TPF} \iff \exists (\text{CR1} - (\text{CR8}), (\text{CP1}) - (\text{CP9}), (\text{CP14}), (\text{CP15}). \]

To pin down the Peirce frames we add two axioms, corresponding to (CP14) and (CP15); those axioms involve a difference operator. Recall from §4 that the operator \( E_s \) is short for \( E_s p \equiv p \lor D_s p \) (there exists a state where \( p \) holds), and that the operator \( O_s \) is short for \( O_s p \equiv (p \land \neg D_s p) \) (\( p \) is true only here).

We define the following two formulas:

\begin{align*}
(\text{MP14}) & \quad E_s p \rightarrow (1)p \\
(\text{MP15}) & \quad E_s O_s p \land (a \land b) p \rightarrow (a \land b) p.
\end{align*}

**Lemma 5.7** Let \( \mathcal{F} \) be a two-sorted Peirce like frame. Then \( \mathcal{F} \) satisfies (CP14) iff it validates (MP14); it satisfies (CP15) iff it validates (MP15).

**Proof.** We only prove that (CP15) is defined by (MP15). Assume \( \not\models (\text{CP15}) \). Then there are \( x_4, x'_4, x_5, y_5 \) such that \( P_{x_4} x'_4 y_5 \) and \( P_{x_4} x'_4 y_5 \), but \( x_4 \neq x'_4 \). Defining a valuation \( V \) such that \( V(p) = \{y_5\}, V(a) = \{x_4\}, V(b) = \{x'_4\} \) refutes (MP15) at \( x_4 \).

For the converse, if \( \not\models (\text{CP15}) \), then for some valuation \( V \) and \( x_4 \) in \( \mathcal{F} \) we have \( x_4 \models E_s O_s p \land (a \land b) p \) and \( x_4 \models (a \land b) p \). Hence, there exists a unique \( y_5 \) in \( \mathcal{F} \) with \( y_5 \models p \), and there exist \( x_4, x'_4 \) with \( P_{x_4} x'_4 y_5 \) and \( x_4 \models a, x'_4 \models b \). As \( x_4 \models (a \land b) p \), we must have \( x_4 \neq x'_4 \). So \( \not\models (\text{CP15}) \). \( \dashv \)

**Theorem 5.8** \( \text{TPF} = \{ \mathcal{F} \mid \mathcal{F} \models \wedge_{0 \leq i \leq 3} (\text{MRi}) \land \wedge_{0 \leq i \leq 9} (\text{MPi}) \land (\text{MP14}) \land (\text{MP15}) \} \).

**Proof.** This follows from 5.2, 5.6 and 5.7. \( \dashv \)
As pointed out in §4, the difference operator is definable on Peirce frames (cf. the operator $D'$). If we take versions of axioms (MP14), (MP15) in which $D'$ is replaced by $D''$, don't we get a characterization of Peirce frames in the original modal language $\mathcal{ML}_2$ from Theorem 5.8 after all? The answer is 'no.' And the reason is that the semantics of the difference operator as a primitive operator is based on the diversity relation $\neq$; for the defined difference operator this does not hold for all two-sorted frames for the language $\mathcal{ML}_2$.

6. Completeness

We will now use our characterization of Peirce frames to obtain a complete axiomatization of Peirce frames. We will use a strategy due to Yde Venema [26, 27] to prove completeness of derivation systems involving difference operators. For this strategy to be applicable our logic should satisfy the following conditions:

- It needs to have difference operators; these difference operators should satisfy certain axioms and rules.
- Converges, and more generally, conjugates for all modal operators (see below), and axioms expressing the appropriate relationships between conjugates.
- Inclusion axioms for all modal operators, stating that for any of the accessibility relations in our Peirce frames, a move along one of these relations must either lead to the same point or to a point that can be reached by using a difference operator.

To actually verify that the above conditions are satisfied we have to work our way through a number of cumbersome technicalities. Below we start by axiomatizing the Peirce like frames as a first approximation. We then add all the required operators and axioms, and apply Venema's strategy to arrive at our completeness result for Peirce frames. And finally we briefly discuss a slight simplification of the logic.

6.1 A first approximation

As a first step we axiomatize the logic of Peirce like frames in the language $\mathcal{ML}_2$.

Let $MLPL$ be the minimal normal modal axion system in $\mathcal{ML}_2(\delta, \circ, \otimes, \langle, \rangle)$ that has (MR0)–(MR8) and (MP1)–(MP9) as axioms. So, besides (MR0)–(MR8) and (MP1)–(MP9), $MLPL$ has the Boolean axioms for $\neg, \wedge, \bot, \top$; the Boolean axioms for $\neg, \cap, 0, 1$; and distribution axioms for the modal operators:

- $\otimes : \overline{\otimes}(a \rightarrow b) \rightarrow (\overline{\otimes}a \rightarrow \overline{\otimes}b)$, where $\overline{\otimes}\alpha \equiv \neg \neg \alpha$
- $\circ : (a \rightarrow b) \circ c \rightarrow ((a \circ c) \rightarrow (b \circ c))$, where $\circ \beta \equiv \neg \neg \alpha \circ \beta$
- $\otimes : (a \rightarrow c) \rightarrow ((a \circ c) \rightarrow (a \circ c))$
- $\langle \rangle : [a \rightarrow b]p \rightarrow ([a]p \rightarrow [b]p)$, where $[\alpha]\phi \equiv \neg(\neg \alpha)\neg \phi$
- $\langle \rangle : [\alpha](p \rightarrow q) \rightarrow ([\alpha]p \rightarrow [\alpha]q)$
- $\mathbb{I} : 1(p \rightarrow q) \rightarrow (\overline{\mathbb{I}}p \rightarrow \overline{\mathbb{I}}q)$, where $\overline{\mathbb{I}}\phi \equiv \neg \neg \phi$.

In addition, $MLPL$ has the derivation rules modus ponens (MP), substitution (SUB), and necessitation (NEC), for all modal operators. The latter covers the following:

- $\langle \text{NEC}_\otimes \rangle \alpha / \overline{\otimes}\alpha$
- $\langle \text{NEC}_\circ \rangle \alpha / [\alpha]\phi$
- $\langle \text{NEC}_\otimes \rangle \alpha / \alpha \circ \beta$
- $\langle \text{NEC}_\otimes \rangle \phi / \overline{\mathbb{I}}\phi$
- $\langle \text{NEC}_\circ \rangle \phi / [\alpha]\phi$
- $\langle \text{NEC}_\otimes \rangle \beta / \alpha \circ \beta $. 
6. Completeness

For $L$ a (two-sorted) modal logic we define an $L$-derivation to be a list of formulas from the language of $L$ such that every formula is either a substitution instance of an axiom of $L$, or obtained from earlier formulas in the list by means of a derivation rule of $L$. An $L$-theorem is any formula that can occur as the last item in a derivation. We write $\vdash_L \xi$ for $\xi$ is an $L$-theorem, and $\Sigma \vdash_L \xi$ for: there are $\sigma_1, \ldots, \sigma_n \in \Sigma$ such that $\vdash_L (\sigma_1 \land \ldots \land \sigma_n) \rightarrow \xi$ (if $\xi$ is a set formula), or $\vdash_L (\sigma_1 \cap \ldots \cap \sigma_n) \rightarrow \xi$ (if $\xi$ is a relation formula). (Compare Remark 4.3.)

**Theorem 6.1** MLPL is strongly sound and complete for TPLF.

**Proof.** To prove the theorem one may use the standard canonical model construction, or one may observe that all MLPL-axioms are Sahlqvist formulas, and derive immediately that MLPL is complete with respect to the two-sorted Peirce like frames satisfying the conditions (CRI) and (CPl) (see [23]).

### 6.2 A complete axiomatization of Peirce frames

To be able to apply Venema's strategy we need to add (to the logic MLPL and its language) difference operators with their axioms and rules, conjugates with their axioms and rules, and so-called inclusion axioms. We now consider each of these components. Fortunately, it will turn out that nearly all of the required additions can already be defined or derived within MLPL.

**Axioms for the difference operators.** Instead of adding primitive difference operators, we consider the derived ones $D'_\delta$ and $D'_\alpha$:

$$D'_\delta \phi := (-\delta) \phi \quad \text{and} \quad D'_\alpha \alpha := (-\delta \circ \alpha \circ 1) \cup (1 \circ \alpha \circ -\delta).$$

(Compare §4.) The difference operators are governed by the following axioms and rules:

- **(MD1)** $\overline{D}(k \rightarrow l) \rightarrow (\overline{D}k \rightarrow \overline{D}l)$, where $\overline{D} \equiv \neg D \neg$,
- **(MD2)** $D \overline{D}k \rightarrow k \lor Dk$
- **(MD3)** $k \rightarrow Dk$
- **(NEC\textsubscript{D})** $\xi \land \neg Dk \rightarrow \xi / \xi$, provided $k$ does not occur in $\xi$.
- **(IR\textsubscript{D})** $k \land \neg Dk \rightarrow \xi / \xi$, provided $k$ does not occur in $\xi$.

Axiom (MD1) is the usual distribution law for normal modal operators; (MD2) expresses that the diversity relation is pseudo-transitive (that is, it satisfies $\forall xyz (Rxz \land Ryz \rightarrow x = z \lor Rxz)$), and (MD3) expresses that it is asymmetric. To understand the irreflexivity rule (IR\textsubscript{D}) we first reason semantically: assume that $-\xi$ is satisfiable at some state $x$ and that $k$ is an atomic symbol that does not occur in $\xi$; as the diversity relation $\not\approx$ is irreflexive we find that by making $k$ true only at $x$ we can satisfy the conjunction $k \land \neg Dk \land -\xi$. So, if $-\xi$ is satisfiable, so is $-\xi \land k \land -Dk$ for any $k$ not occurring in $\xi$. Turning to validity: the rule 'if $\vdash k \land -Dk \rightarrow \xi$ then $\vdash \xi$ provided that $k$ does not occur in $\xi$' is sound.

We will show below that except for the irreflexivity rules each of the axioms (MD1)–(MD3) and the rule (NEC\textsubscript{D}) is derivable in MLPL for the defined difference operators.

**Axioms for conjugated operators.** Let $R$ be an $(n + 1)$-ary relation. A frame $\mathfrak{S} = (\ldots, R, \ldots)$ is called versatile for $R$ if there are relations $R_1, \ldots, R_n$ such that for all $x_1, \ldots, x_n$
one has \((x_0, \ldots, x_n) \in R\) iff \((x_1, \ldots, x_n, x_0) \in R_1\) iff \(\ldots\) iff \((x_n, x_0, \ldots, x_{n-1}) \in R_n\). Once we know that a frame is versatile for \(R\), it suffices to mention just a single \(R_s\) and suppress the other relations.

Let \(\#\) be an \(n\)-ary modal operator whose semantics is based on an \((n + 1)\)-ary relation \(R\); the conjugates of \(\#\) are \(n\) operators \(#_1, \ldots, #_n\) whose semantics are based on \((n + 1)\)-ary relations \(R_1, \ldots, R_n\), respectively, such that \(R, R_1, \ldots, R_n\) form a versatile system, and

\[ x \models #_i(\xi_1, \ldots, \xi_n) \quad \text{iff} \quad \exists y_1 \ldots y_n \ (Rxy_1 \ldots y_n \land \bigwedge_1 y_i \models \xi_i) . \]

Unary modal operators whose underlying relation is symmetric form their own conjugates; also, a frame is versatile for a binary relation \(B\) if it contains the converse \(B^{-1}\) of \(B\).

We will now define conjugated operators for all modal operators in \(\mathcal{ML}_2\). Note that the defined difference operators \(D'_s, D'_r\), and the converse operator \(\otimes\) are self-conjugated; so we don't need to add conjugates for them. For \(\mathbb{I}\) we define a conjugate \(\mathbb{F}\) by putting \(\mathbb{F}\alpha := (\alpha)^T\); so \(\mathbb{F}\) takes a relation formula and returns a set formula. For \(\langle \cdot \rangle\) we define two conjugate operators \(\langle \cdot \rangle_1\) and \(\langle \cdot \rangle_2\) by putting \(\langle \phi \rangle_1 \psi := \otimes (\mathbb{I} \phi) \cap \mathbb{I} \psi\), and \(\langle \phi \rangle_2 \alpha := \langle \otimes \alpha \rangle \phi\). For \(\circ\) we also add two operators, written \(\alpha_1\) and \(\alpha_2\), defined by \(\alpha \circ_1 \beta := \beta \circ \otimes \alpha\) and \(\alpha \circ_2 \beta := \otimes \beta \circ \alpha\).

All in all, we have have introduced the following abbreviations:

\[
\begin{align*}
\mathbb{F}\alpha & := (\alpha)^T \\
\langle \phi \rangle_1 \psi & := \otimes (\mathbb{I} \phi) \cap \mathbb{I} \psi \\
\langle \phi \rangle_2 \alpha & := \langle \otimes \alpha \rangle \phi \\
\alpha_1 \beta & := \beta \circ \otimes \alpha \\
\alpha_2 \beta & := \otimes \beta \circ \alpha.
\end{align*}
\]

To motivate the above, observe that \(y_r \models (p)_1 q\) means that there exist \(x_s, z_s\) with \(P_{x_s} y_r, x_s\) and \(z_s \models p, z_s \models q\); hence, \(y_r \models (\otimes p) \cap (\mathbb{I} q) (= (p)_1 q)\). Second, \(z_s \models (p)_2 a\) iff there exist \(x_s, y_r, z_s\) and \(x_s \models p, y_r \models a\). Hence \(z_s \models (\otimes a) p\ (= (p)_2 a)\). Similar remarks pertain to \(\alpha_1, \alpha_2\).

We force the appropriate modal operators to be each other's conjugates by imposing the axioms below; \(\mathbb{I} \phi\) abbreviates \(\neg \neg \phi\), and \(\mathbb{F}\alpha\) abbreviates \(\neg \circ \neg \alpha\).

\[
\begin{align*}
\text{(MP16)} & \quad a \to \mathbb{I} \mathbb{F} a \\
\text{(MP17)} & \quad p \to \mathbb{F} \mathbb{I} p \\
\text{(MP18)} & \quad p \land \neg (\neg q)_1 p \land q \to \bot \\
\text{(MP19)} & \quad a \cap \neg (\neg (p)_2 a)_1 p \to 0 \\
\text{(MP20)} & \quad p \land \neg (\neg (a)_p)_2 a \to \bot \\
\text{(MP21)} & \quad a \cap \neg (b \circ_1 a) \circ b \to 0 \\
\text{(MP22)} & \quad a \cap \neg (b \circ_2 a) \circ_1 b \to 0 \\
\text{(MP23)} & \quad a \cap \neg (b \circ a) \circ_2 b \to 0.
\end{align*}
\]

The first of the above two axioms are well-known from temporal logic; they simply express that \(\mathbb{I}\) and \(\mathbb{F}\) are interpreted using converse relations; likewise, axioms (MP18)–(MP23) express that \(\langle \cdot \rangle_1, \langle \cdot \rangle_2\) and \(\circ, \circ_1, \circ_2\) form conjugate triples, cf. [26].

**Inclusion axioms.** One special feature of the difference operators is that if you move along one of the accessibility relations using the modal operators \(\circ, \mathbb{I}, \otimes,\) or \(\langle \cdot \rangle\), you either get back to the starting point or to a point that you must be able to reach using one of the difference operators. This feature is implemented by the following so-called inclusion axioms:
6. Completeness

(INC1) \((a)\) \(p \rightarrow E'_s p\)
(INC2) \(\otimes a \rightarrow E'_s a\)
(INC3) \(a \circ b \rightarrow E'_s a \land E'_s b\)
(INC4) \(\langle D'_s \langle \|q\rangle \rangle \rightarrow E'_s q\)
(INC5) \(\langle D'_s \langle a \rangle \rangle \rightarrow E'_s a\).

The logic of Peirce frames. We are ready now to define the modal logic of two-sorted Peirce frames and prove its completeness.

Definition 6.2 We define one more axiom system: MLP. Its language is \(\mathcal{ML}_2\). Its axioms are those of MLPL (§6.1), and its rules of inference are those of MLPL plus the following two irreflexivity rules:

\((IR_p)\) \(p \land \neg D'_s p \rightarrow \phi / \phi\), where \(p\) does not occur in \(\phi\)
\((IR_r)\) \(a \land \neg D'_r a \rightarrow \alpha / \alpha\), where \(a\) does not occur in \(\alpha\).

To prove MLP complete using Venema's strategy, we need to show that it derives the axioms and rules for the difference operators and conjugate operators, as well as the inclusion axioms. Showing this is in fact the heart of the completeness proof.

Theorem 6.3 Let \(\Delta \cup \{\xi\}\) be a set of \(\mathcal{ML}_2\)-formulas. Then \(\Delta \vdash \xi\) in MLP iff \(\Delta \models_{\text{TPF}} \xi\).

Proof. Proving soundness is left to the reader. As to completeness, by Lemmas A.1 and A.2 MLP satisfies all the requirements needed for an application of Venema's strategy as described at the start of this section: both the axioms for the difference operators and the conjugated operators and the inclusion axioms are derivable in derivable in MLP, and the necessitation rules for the difference operators and the conjugated operators are all derived rules of MLP. So by Venema [27, Theorem 7.7] MLP is strongly complete for the class of frames validating axioms (MR1)-(MR8) and (MP1)-(MP15). Therefore by our characterization result Theorem 5.8, MLP is strongly complete for Peirce frames. That is, \(\Delta \models_{\text{TPF}} \xi\) implies \(\Delta \vdash \xi\).

To conclude this section we briefly discuss the number of irreflexivity rules that we need. From our observations in §3 we know that we need at least some non-standard means to get a complete axiomatization for Peirce frames. To rephrase this somewhat inaccurately, we need at least one irreflexivity rule. Further, by Theorem 6.3 we know that we need at most two. However, we can get by with just one, namely the irreflexivity rule for \(D'_s\). The one for \(D'_s\) can be replaced by the derived rule

\[\langle a \land \neg D'_s (a \land \delta) \rangle \land \neg D'_s (a \land \neg D'_s (a \land \delta)) \rightarrow \phi / \phi, \text{ provided } a \text{ does not occur in } \phi.\]  

As with the irreflexivity rule for \(D'_s\), the intuition is that if \(\phi\) is consistent then it is consistent to have \(\phi\) together with a 'unique name'. With the rule (6.2) a unique name for a set element (namely \(\langle a \land \neg D'_s (a \land \delta) \rangle \land \neg D'_s (a \land \neg D'_s (a \land \delta)) \rangle\) is borrowed from a unique name for a unique name for a diagonal element — and being relation elements such elements will get unique names by the irreflexivity rule for \(D'_s\).

It can be shown that (6.2) is a derived rule in MLP minus the irreflexivity rule for \(D'_s\), but the proof is too cumbersome (and uninformative) to be included here.
7. EXPRESSIVE POWER

Consider the triangle ‘algebraic logic – modal logic – first-order logic’ depicted in the introduction again. In sections 4–6 we concentrated on the ‘algebraic logic – modal logic’ side of the triangle to arrive at a complete axiomatization. In the present section we will put together some results from the modal literature that bear on the ‘modal logic – first-order logic’ side. Concretely, we will characterize the fragment of first-order logic that corresponds to the modal language $\mathcal{ML}_2$ using an appropriate notion of bisimulation; along the way we will obtain a definability result.

When interpreted on Peirce models (that is, on models based on Peirce frames), $\mathcal{ML}_2$-formulas become equivalent to first-order formulas of the following kind. Let $\tau$ be the (first-order) vocabulary $\{P_1, P_2, \ldots, A_1, A_2, \ldots\}$, where the $P_i$’s are unary relation symbols corresponding to the atomic set variables $p_i$ in our language, and the $A_i$’s are binary relation symbols corresponding to the atomic relation variables. Let $\mathcal{L}(\tau)$ be the set of all first-order formulas over $\tau$ (with identity).

We now define a translation $ST$ taking $\mathcal{ML}_2$-formulas to formulas in $\mathcal{L}(\tau)$. Fix three distinct individual variables $x_1, x_2, x_3$, and let $i, j, k$ denote distinct objects in $\{1, 2, 3\}$.

\[
\begin{align*}
ST_i(\top) &= (x_i = x_i) \\
ST_i(p) &= P(x_i) \\
ST_i(\neg \phi) &= \neg ST_i(\phi) \\
ST_i(\phi \land \psi) &= ST_i(\phi) \land ST_i(\psi) \\
ST_i((\alpha)\phi) &= \exists x_j (ST_{ij}(\alpha) \land ST_j(\phi)) \\
ST_{ij}(1) &= (x_i = x_j) \land (x_j = x_j) \\
ST_{ij}(\delta) &= (x_i = x_j) \\
ST_{ij}(\alpha) &= A(x_i, x_j) \\
ST_{ij}(\neg \alpha) &= \neg ST_{ij}(\alpha) \\
ST_{ij}(\alpha \land \beta) &= ST_{ij}(\alpha) \land ST_{ij}(\beta) \\
ST_{ij}(\alpha \theta \beta) &= ST_{ij}(\alpha) \\
ST_{ij}(\alpha \circ \beta) &= \exists x_k (ST_{ik}(\alpha) \land ST_{kj}(\beta)) \\
ST_{ij}(\phi) &= ST_i(\phi) \land (x_j = x_j).
\end{align*}
\]

Table 2: The first-order translation for $\mathcal{ML}_2$.

Then, for $\phi$ a set formula in $\mathcal{ML}_2$, we define the standard translation $ST(\phi)$ of $\phi$ by $ST(\phi) := ST_{12}(\phi)$; and $\alpha$ is a relation formula in $\mathcal{ML}_2$, we define its standard translation by $ST(\alpha) := ST_{12}(\alpha)$.

Peirce models may be viewed as models for $\mathcal{L}(\tau)$: to interpret the predicate symbols in $\mathcal{L}(\tau)$ we simply use the the values that the valuation assigns to the corresponding modal symbols.

**Proposition 7.1** Let $\phi$ be a set formula in $\mathcal{ML}_2$, $\alpha$ a relation formula in $\mathcal{ML}_2$, and $\mathfrak{M}$ a Peirce model. For any $x$ in $\mathfrak{M}$, $\mathfrak{M}, x \models \phi$ iff $\mathfrak{M} \models ST(\phi)[x]$. For any $x, y$ in $\mathfrak{M}$, $\mathfrak{M}, (x, y) \models \alpha$ iff $\mathfrak{M} \models ST(\alpha)[xy]$.

So, on models every $\mathcal{ML}_2$-formula is equivalent to a first-order formula, or more precisely, to a first-order formula containing at most three variables. A natural question at this point
7. Expressive power

is to ask for special semantic features that isolate the fragment of \( \mathcal{L}(\tau) \) that corresponds to \( \mathcal{ML}_2 \). Similar questions have been raised and answered before in the literature on modal logic; we refer the reader to \([2, 3, 18, 21]\) for examples. Bisimulations have turned out to be an important tool in answering such questions — and in our case too we will use an appropriate notion of bisimulation.

**Definition 7.2** A bisimulation for \( \mathcal{ML}_2 \) between \( M_1 \) and \( M_2 \) is a non-empty relation \( Z \subseteq (W_1 \times W_2) \cup (W_1^2 \times W_2^2) \) such that

1. \( \bar{x}\bar{y} \) implies \( \text{lh}(\bar{x}) = \text{lh}(\bar{y}) \), where \( \text{lh}(\bar{x}) \) is the length of \( \bar{x} \),
2. if \( Z(x_1x_2)(y_1y_2) \) then \( Zx_1y_1, Zx_2y_2 \), and \( Z(x_2x_1)(y_2y_1) \),
3. if \( Zx_1y_1 \) then \( x_1 \) and \( y_1 \) agree on all set variables \( \rho \),
4. if \( Z(x_1x_2)(y_1y_2) \) then \( (x_1, x_2) \) and \( (y_1, y_2) \) agree on all relation variables \( \alpha \) and on \( \delta \),
5. if \( Zx_1y_1 \) and \( x_2 \in M_1 \), then there exists \( y_2 \) in \( M_2 \) such that \( Z(x_1x_2)(y_1y_2) \), and similarly in the opposite direction,
6. if \( Z(x_1x_2)(y_1y_2) \) and \( x_3 \in M_1 \), then there exists \( y_3 \) in \( M_2 \) such that \( Z(x_1x_3)(y_1y_3) \) and \( Z(x_2x_3)(y_2y_3) \), and similarly in the opposite direction.

(As an aside, it is clear from the standard translation that \( \mathcal{ML}_2 \) contains the equivalent of the full 2-variable fragment of \( \mathcal{L}(\tau) \). Hence, as the latter is characterized by its invariance under 2-partial isomorphisms (see \([3, \text{Chapter 15}]\)), any relation between models that is to preserve truth of \( \mathcal{ML}_2 \)-formulas should at least act like a family of 2-partial isomorphisms. This is indeed the case.)

**Proposition 7.3** Bisimilar states satisfy the same \( \mathcal{ML}_2 \)-formulas. More precisely, let \( M_1, M_2 \) be Peirce models, and let \( Z \) be a bisimulation for \( \mathcal{ML}_2 \) between \( M_1 \) and \( M_2 \).

For any \( x_1 \) in \( M_1 \), \( y_1 \) in \( M_2 \), and for any set formula \( \phi \), if we have \( Zx_1y_1 \), then \( M_1, x_1 \models \phi \) iff \( M_2, y_1 \models \phi \).

Likewise, for any \( x_1, x_2 \) in \( M_1 \), \( y_1, y_2 \) in \( M_2 \), and for any relation formula \( \phi \), if we have \( Z(x_1x_2)(y_1y_2) \), then \( M_1, (x_1, x_2) \models \alpha \) iff \( M_2, (y_1, y_2) \models \alpha \).

In fact, the above preservation result Proposition 7.3 is characteristic for \( \mathcal{ML}_2 \)-formulas; below we will briefly sketch a proof of this claim. We first observe that the converse of Proposition 7.3 does not hold: \( \mathcal{ML}_2 \)-equivalent models need not be bisimilar (see \([9, 12, 21]\) for a counterexample). Following Goldblatt \([9]\) and Hollenberg \([12]\), we call a class \( K \) of \( \mathcal{ML}_2 \)-models a Hennessy-Milner class if every pair of models in \( K \) is \( \mathcal{ML}_2 \)-equivalent if it is bisimilar. As an example, the class of finite \( \mathcal{ML}_2 \)-models is a Hennessy-Milner class, as is the class of \( \omega \)-saturated models (in the sense of standard first-order model theory); we refer to \([9, 12, 21]\) for further details.

Call a first-order formula \( \lambda \in \mathcal{L}(\tau) \) invariant for bisimulations if for all \( \mathcal{L}(\tau) \)-models \( M, M' \) and all tuples \( \bar{x}, \bar{x}' \) in \( M \) and \( M' \), respectively, and all binary relations \( Z \) between \( M \) and \( M' \), we have that if \( Z \) is bisimulation linking \( \bar{x} \) and \( \bar{x}' \), then \( \models M = \lambda[\bar{x}] \) iff \( M' = \lambda[\bar{x}'] \).
Theorem 7.4 Let $\lambda(\vec{x}) \in L(\tau)$ be a first-order formula in one or two free variables. The $\lambda(\vec{x})$ is invariant for bisimulations iff it is equivalent to (the translation of) an $ML_2$-formula.

Proof. The direction from right to left is Proposition 7.3. Proving the converse requires more work. We will sketch the proof. Assume $\lambda(\vec{x}) = \lambda(x)$ has just a single free variable, and assume $\lambda(x)$ is invariant for bisimulations. Consider the set of modal consequences of $\lambda$ in a single free variable:

$$\text{MOD-CON}(\lambda) = \{ ST(\phi) \mid \lambda \models ST(\phi)(x), \quad \phi \text{ is a set formula in } ML_2 \}.$$ 

By compactness it suffices to show that $\text{MOD-CON}(\lambda) \models \lambda$. For then there exists a finite $\Gamma \subseteq \text{MOD-CON}(\lambda)$ such that $\Gamma \models \lambda$ (and conversely) and $\bigwedge \Gamma$ is an $ML_2$-formula.

Assume $\mathcal{M} \models \text{MOD-CON}(\lambda)[w]$. We have to show that $\mathcal{M} \models \lambda[w]$. Our first observation is that by a simple compactness argument the set $X := \{ \lambda \} \cup \{ ST(\psi) \mid \mathcal{M}, w \models \psi \}$ is consistent. Let $\mathcal{N}$ be a model with $\mathcal{N} \models X[v]$, for some $v$. Note that $w$ in $\mathcal{M}$ and $v$ in $\mathcal{N}$ satisfy the same $ML_2$-formulas.

If $\mathcal{M}$ and $\mathcal{N}$ both lived in a Hennessy-Milner class, then the the fact that $w$ and $v$ satisfy the same $ML_2$-formulas would imply that there exists a bisimulation between $\mathcal{M}$ and $\mathcal{N}$ that links $w$ and $v$, and from this we would be able to infer $\mathcal{M} \models \lambda[w]$, which would complete the proof. We can get away with slightly less: it’s enough to make a detour through a Hennessy-Milner class, as follows. By general model-theoretic considerations from first-order logic, both $\mathcal{M}$ and $\mathcal{N}$ have $\omega$-saturated elementary extensions $\mathcal{M}^*$ and $\mathcal{N}^*$; it follows that $w$ in $\mathcal{M}^*$ and $v$ in $\mathcal{N}^*$ satisfy the same $ML_2$-formulas, and that $\mathcal{N}^* \models \lambda[w]$. The class of $\omega$-saturated models is a Hennessy-Milner class, hence there exists a bisimulation between $\mathcal{M}^*$ and $\mathcal{N}^*$ that links $w$ and $v$. By invariance under bisimulations we get $\mathcal{M}^* \models \lambda[w]$. As $\mathcal{M}^*$ is an elementary extension of $\mathcal{M}$ we infer that $\mathcal{M} \models \lambda[w]$ — and we are done. \hfill \top

Corollary 7.5 Let $K$ be a class of $ML_2$-models that is defined by a set of first-order formulas. Then $K$ is modally definable iff it is closed under bisimulations.

Corollary 7.6 A class of $ML_2$-models is modally definable iff it is closed under bisimulations and ultraproducts, and its complement is closed under ultrapowers.

Further definability results along these lines may be obtained using general techniques from modal model theory; see [21] for details.

8. Concluding remarks

In this paper we studied Peirce algebras via modal logic. Known techniques from modal completeness theory supplied us with a finite axiomatization of Peirce frames, and thereby of the equational theory of full Peirce algebras, and general results from modal model theory helped us to analyze the expressive power of Peirce algebras.

A lot remains to be done. Some questions were already mentioned in the main body of the paper. To conclude, here are further questions.

In §2 we briefly mentioned a connection between a system of Arrow Logic and Peirce algebras. There is a whole hierarchy of calculi in between this Arrow Logic and $MLP$, the logic of Peirce frames, just like there is a hierarchy of subsystems of relation algebra. About the former hierarchy one can ask the same kind of questions as for the latter. For example, where
does undecidability strike? Is there an arrow version of Peirce algebras which is sufficiently expressive for applications (say, in terminological logic), but still decidable? Recent work by Marx [17] presents partial answers, but more work needs to be done.

Another point in connection with the use of Peirce algebras in terminological logic is this. In terminological reasoning one often needs to be able to count the number of objects related to a given object; this is done using so-called number restrictions [7]. The modal logic of such counting expressions is analyzed by Van der Hoek and De Rijke [11]. One direction for further work is to combine the results of the latter with the results of the present paper.

Finally, a more general point. We have used unorthodox derivation rules like the irreflexivity rule to arrive at our completeness result. To which extent do such rules capture our operators? We know from [18] that the irreflexivity rule goes a long way towards determining the difference operator. But what about the other operators, like o, ⌣, :? Which aspects of their behaviour are determined by our unorthodox derivation rules?

Acknowledgments
I would like to thank Johan van Benthem, Chris Brink and Yde Venema for valuable comments and advice. This research was supported by the Netherlands Organization for Scientific Research (NWO), project NFO 102/62-356 ‘Structural and Semantic Parallels in Natural Languages and Programming Languages’.

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A. Appendix


A. appendix

We still need two lemmas to round off the completeness theorem for MLP, one saying that the axioms for the difference operators and the conjugated operators, as well as the inclusion axioms are derivable in MLP, and another lemma showing that the necessitation rules for the difference operators and the conjugated operators are derived rules of MLP.

Lemma A.1 MLP proves axioms (MD1)–(MD3) for the difference operators $D'_x$ and $D'_y$. It also proves the distribution axioms for the defined operators $(\cdot)_1$, $(\cdot)_2$, $\circ_1$, and $\circ_2$, as well as (MP14), (MP15), (MP16)–(MP23) and (INC1)–(INC5).

Proof. We use the completeness of MLPL established in Theorem 6.1 to argue semantically that MLP, and hence MLP, proves the axioms mentioned in the lemma.

Showing that MLP proves the distribution axioms is left to the reader. (MD3), (MD4) are easy consequences of (CP1) and (CP3); (MD1)–(MD3) are proved by Venema [26, Proposition 3.3.38]; (INC1) is easy and (INC2), (INC3) are dealt with by Venema [26, Proposition 3.3.38]; (INC5) is an easy consequence of (CP1) and (CP8); (MP14), (MP15)–(MP20) are easy, and (M21)–(M23) are, again, in Venema [26, Proposition 3.3.38]. This only leaves (MD3), (INC4) and (MP15) to prove.

(MD3) $p \rightarrow \neg(\delta \neg(\neg(\delta))p$. Assume that $x \models p$, $(\neg(\delta \neg(\neg(\delta)p$. We will derive a contradiction. As $x \models (\neg(\delta \neg(\neg(\delta)p$ there are $y_r$, $z_s$ with $Px_y y_r z_s$, $y_r \not\in I$ and $z_s \models \neg(\neg(\delta)p$. By (CP5) this implies $Pz_s f(y_r) z_s$. Now, if $f(y_r) \not\in I$ then $z_s \models (\neg(\delta)p$, and we have arrived at the desired contradiction. So it suffices to show $f(y_r) \not\in I$. Assume $f(y_r) \in I$, then, as $Px_y y_r z_s$ and $Pz_s f(y_r) z_s$ there exists $y'_r$ with $C y'_r y_r f(y_r)$, by (CP1). Hence, by (CR5) and
\( f(y_r) \in I \) \( C_{y_r} y_r f(y_r) \). By (CR1) and (CR7) this implies \( C_f(y_r) y_f j(y_r) \), and by (CR5) this in turns yields \( f(y_r) = y_r \). Therefore \( y_r \in I \) \( \notin q \) a contradiction.

\( (INC) \) \( \langle (\delta \circ \uparrow q \circ 1) \cup (1 \circ \uparrow q \circ -\delta) \rangle \top \to q \lor (-\delta)q \). Assume \( x_s \) satisfies the antecedent of the axiom, say \( x_s = (\delta \circ \uparrow q \circ 1) \top \). Then there are \( y_r, z_s \) with \( P x_s y_r z_s \) and \( y_r \models (\delta \circ \uparrow q \circ 1) \). This means that there are \( y_r', y_r'', z_r', z_r'' \) such that \( C_{y_r} y_r y_r'', C_{y_r} y_r z_r'' \) and \( y_r' \models (\delta \circ \uparrow q \circ 1) \). The latter implies that there is an \( x_s' \) with \( P x_s' x_s'' \) and \( x_s'' \models q \).

It suffices to show that \( P x_s y_r z_s \), for then \( x_s \models (\delta)q \). Now, to see that \( P x_s y_r x_s'' \), observe that

\[
P x_s y_r z_s \land C_{y_r} y_r y_r'' \Rightarrow P x_s y_r x_s'' \land P z_r y_r z_s,
\]

for some \( z_r' \) by (CP2). Furthermore, \( P z_r y_r z_s \) and \( C_{y_r} y_r z_r'' \) imply that for some \( z_r'', P z_r y_r z_r'' \), by (CP2). Next, \( P z_r y_r z_r'' \) implies \( P x_s y_r x_s'' \) by Proposition 5.4. On the other hand, we already have that \( F x_s y_r z_s \), so by (CP10) it follows that \( x_s = x_s' \). But, then, by (A.3) we must have \( P x_s y_r x_s'', \) as required.

The case that \( x_s = (\circ \uparrow q \circ -\delta) \top \) is proved entirely analogously.

\( (MP) \) \( (p \land \neg (-\delta)p) \lor (-\delta)(p \land \neg (-\delta)p) \land (a \land b)p \to (a \land b)p \). Assume first that \( x_s \models (p \land \neg (\delta)p) \land (a \land b)p \). Then, for some \( y_r, y_r', z_s, z_s' \) we have \( P x_s y_r z_s, P x_s y_r x_s'' \) and \( y_r \models a, y_r' \models b, \) and \( z_s, z_s' \models p \).

Observe that \( x_s \models \neg (\delta)p \) implies \( y_r, y_r' \in I \). Hence, by (CP3), \( x_s = z_s = z_s' \). Furthermore, \( P x_s y_r z_s \) implies \( P z_s f(y_r) x_s \), together with \( P x_s y_r z_s' \) and (CP1) this yields a \( y_r'' \) such that \( C_{y_r} f(y_r) y_r'' \), and so by (CR8) such that \( C_{y_r} y_r y_r'' \). By (CR5) we find \( y_r' = y_r'' \) and \( y_r'' = f(y_r) \). So we have \( C f(y_r) f(y_r) y_r f(y_r) \), and by (CR8) \( C f(y_r) y_r f(y_r) \). As \( y_r \in I \) implies \( f(y_r) \in I \), (CR5) now gives \( y_r = f(y_r) \). All in all we find \( y_r = f(y_r) = y_r'' = y_r' \). Hence \( x_s \models (a \land b)p \).

Assume next that \( x_s \models (\neg (\delta)p \land (a \land b)p \land (a \land b)p \). Then there are \( y_r, y_r', y_r'' \) and \( z_s, z_s', z_s'' \) with \( P x_s y_r z_s, P x_s y_r z_s', P x_s y_r z_s'' \), and \( y_r \notin I, y_r' \models a, y_r'' \models b, \) and \( z_s, z_s', z_s'' \models p \), and \( z_s \models \neg (\delta)p \).
By (CP1) there is a $y''_r$ with $Pz_sy''_r x_s$ and $Cy''_r f(y_r)y'_r$. If $y''_r \notin I$, then $z'_s \neq p$ — a contradiction. Hence $y''_r \in I$. Likewise we find a $y'''_r \in I$ with $Cy'''_r f(y_r)y''_r$. By (CR8) and (CR5) we have $y'_r = y_r$ and $y''_r = y_r$. Hence $y_r \models a \cap b$, and $x_s \models (a \cap b)p$. \qed

**Lemma A.2** The necessitation rules for the defined operators $\mathcal{J}$, $\langle \cdot \rangle_1$, $\langle \cdot \rangle_2$, $\diamond_1$, $\diamond_2$ and $D'$, $D'_s$ are derived rules in MLP.

**Proof.** By way of example we show that the two necessitation rules for $\langle \cdot \rangle_2$ are derived rules. Assume $\vdash_{MLP} \alpha$; we need $\vdash_{MLP} [\phi]_2 \alpha$. Now, by (NEC$_\mathcal{J}$) and (MR0), $\vdash_{MLP} \alpha$ implies $\vdash_{MLP} \otimes \alpha$. Hence, by (NEC$_\langle \cdot \rangle$) we have $\vdash_{MLP} [\otimes \alpha] \phi$, and by definition of $\langle \cdot \rangle_2$ this means $\vdash_{MLP} [\phi]_2 \alpha$. Next, assume $\vdash_{MLP} \phi$; we need $\vdash_{MLP} [\phi]_2 \alpha$, but this is immediate from (NEC$_\mathcal{J}$). \qed