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Bisimulations for Temporal Logic

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Abstract

We define bisimulations for temporal logic with Since and Until. We compare our notion to existing notions of bisimulations, and we use it to develop the basic model theory of temporal logic with Since and Until, which includes preservation and definability results. We conclude with a brief discussion of the wider applicability of our ideas.

1 Introduction

The simplest structures used to model dynamic phenomena are (labeled) transition systems: structures equipped with a collection of states and one or more transition relations that indicate how one state can evolve into another. Numerous languages have been proposed as suitable description tools for talking about transition systems. *Process algebraic* languages take an external view on transition systems in that each process algebraic term denotes an entire transition system. *Modal* and *temporal* languages, on the other hand, offer an internal perspective on transition systems, as they describe (local) properties of states and transitions between them.

This paper deals with the model theory of one particular 'internal' description language for transition systems: the temporal language with Since and Until. This language, and languages closely related to it, have been proposed by a number of authors as suitable for describing dynamic phenomena. For example, Van Benthem [2] suggests that we use Since and Until to describe operations of theory change. Also, information change often involves an 'economy principle' saying that one should change as little information as possible when accommodating new data; languages with Since and Until (or Since and Until-like operators) are the obvious candidates if one wants to express this idea of minimal change, and, indeed, in most of the more powerful dynamic languages one can define them; see for example [4, 9, 19].

In a properly developed theory of dynamics the relation between the models of dynamic phenomena on the one hand, and the description language used to specify such models is a central issue. In this paper we analyze the model theory of the temporal language with Since and Until; the main tool in our analysis is a special kind of bisimulations.

The relevance of bisimulations to dynamics lies in the answer one can give to the following question: when do two transition systems represent the same process? Obviously, this depends on the features of transition systems that one finds important. Typically, a minimal requirement is that states to be identified have the same choice of actions enabled. But in the presence of Since and Until we want more, as we'll see below; among other things, if an action is enabled in a state s, then we should not only find the same action enabled in any state t that we want to identify with s, but we should also ensure that the 'interval' or 'period' leading from s to the result of the action can be matched by a similar interval starting from t.

In addition there are also more technical reasons to work with bisimulations in trying to understand the model theory of Since and Until. Recent work in the model theory of modal languages is characterized by a pervasive use of bisimulations. Van Benthem [2] first observed the close resemblance of bisimulations to partial isomorphism. This observation has inspired a systematic investigation of the model theory of basic poly-modal logic along the lines of first-order model theory in De Rijke [21], whose results take the following 'heuristic equation' as their starting point:

$$\frac{\text{partial isomorphims}}{\text{first-order logic}} = \frac{\text{bisimulations}}{\text{modal logic}}.$$

Andréka, van Benthem and Németi [1] further explore the links between modal logic and first-order logic using bisimulations as a central tool, and the investigations of Van Benthem, Van Eijck and Stebletsova [4], Van Benthem and Bergstra [3], and De Rijke [20] also revolve around the use of bisimulations in the model theory of modal logic.

Most of the results in the papers cited above concern only basic modal diamonds $\langle\alpha\rangle$ and boxes $[\alpha]$ with their familiar truth definitions, or simple variations thereof. The model theory of modal and temporal languages with more complex operators isn't as well developed. In particular, in the case of the temporal language with Since and Until, there is no proper notion of bisimulation that allows for the development of its model theory in analogy with basic poly-modal logic; this has been observed by a number of authors (see [3, 4, 21]). In this paper we address this issue by introducing a notion of bisimulation that 'works' for the temporal language with Since and Until. That is, we define a notion of bisimulation that can serve as a central tool in the model theory of temporal logic by allowing us to prove basic preservation and definability results.

The structure of the paper is as follows. In Section 2 we recall some basic concepts; in Section 3 we introduce a notion of bisimulations for Since and Until, and compare it to related equivalence relations on models. Section 4 considers the question when temporal equivalence implies bisimilarity, and Section 5 then uses bisimulations to to establish basic model-theoretic results on preservation and definability for the temporal language with Since and Until. We conclude with some questions and suggestions for future work.

2 Definitions

This section introduces the various concepts needed. First, *SU-formulas* are built up using propositional variables p, q, ..., the constants $\top$ and $\bot$, boolean connectives $\neg$, $\wedge$, and the binary temporal operators S (Since) and U (Until). We use $\mathcal{L}_{SU}$ to denote this language. We use the usual abbreviations: $F\phi \equiv U(\phi, \top)$, $G\phi \equiv \neg F \neg \phi$, $P\phi \equiv S(\phi, \top)$, $H\phi \equiv \neg P \neg \phi$. The *mirror image* of a formula is obtained by simultaneously replacing S by U and U by S.

A *flow of time*, *temporal order* or *frame* is a pair $F = (W, <)$, where W is a non-empty set of *time points* or *states*, and $<$ is a binary relation on W. A *valuation* is a function assigning a subset of W to every proposition letter. A *model* is a pair (F, V) where F is a frame and V a valuation.

The *satisfaction relation* is defined in the familiar way for the atomic and boolean cases, while for the temporal connectives we put

$$\begin{array}{lll} M, t \models S(\phi, \psi) & \text{iff} & \text{there exists } v < t \text{ such that } M, v \models \phi, \text{ and} \\ & & \text{for all } u \text{ with } v < u < t\text{: } M, u \models \psi \\ M, t \models U(\phi, \psi) & \text{iff} & \text{there exists } v > t \text{ such that } M, v \models \phi, \text{ and} \\ & & \text{for all } u \text{ with } v > u > t\text{: } M, u \models \psi. \end{array}$$

To talk about the points involved in interpreting temporal formulas, the notion of an interval proves useful. Let $M = (W, <, V)$ be a model. An *interval* in M is simply a pair of points $w, v \in W$. An interval wv is called a *pseudo-interval* if there is no $u \in W$ such that $w < u$ and $u < v$. If wv is an interval, and ϕ a temporal formula, then *truth* of ϕ in wv by putting

$$wv \models \phi \text{ iff for all } u \text{ with } w < u < v \text{ we have } u \models \phi.$$

Using our notion of intervals we can rewrite the truth condition for S as $w \models S(\phi, \psi)$ iff there exists $v < w$ with $v \models \phi$ and $wv \models \psi$.

The *temporal theory* of a point w is the set $tp(w) = \{\phi \in \mathcal{L}_{SU} \mid w \models \phi\}$, and the *temporal theory* an interval wv is the set $tp(wv) = \{\phi \in \mathcal{L}_{SU} \mid wv \models \phi\}$. If we want to emphasize the model M in which w (or wv) lives, we write $tp_M(w)$ (or $tp_M(wv)$). Observe that if wv is a pseudo-interval, then its temporal theory is simply the set of all temporal formulas. Two points

w, v are *temporally equivalent* if $tp(w) = tp(v)$ (notation $w \equiv v$); temporal equivalence for intervals is defined analogously.

Let $\mathcal{L}_1$ be the first-order language with unary predicate symbols corresponding to the proposition letters in $\mathcal{L}_{SU}$, and with one binary relation symbol $<$. $\mathcal{L}_1$ is called the *correspondence language* for $\mathcal{L}_{SU}$. $\mathcal{L}_1(x)$ denotes the set of all $\mathcal{L}_1$-formulas having one free variable x.

Models can be viewed as $\mathcal{L}_1$-structures in the usual first-order sense. The *standard translation* takes temporal formulas ϕ into equivalent formulas $ST(\phi)$ in the correspondence language. It maps proposition letters p onto unary predicate symbols Px, it commutes with the booleans, and the temporal case is

$$ST(S(\phi,\psi)) = \exists y\,(y < x \wedge ST(\phi)(y) \wedge \forall z\,(y < z < x \rightarrow ST(\psi)(z))).$$

For all models M and points t we have $M, t \models \phi$ iff $M \models ST(\phi)[t]$, where the latter denotes first-order satisfaction of $ST(\phi)$ under the assignment of t to the free variable of $ST(\phi)$.

3 Bisimulations for S and U

In this section we introduce our notion of bisimulation for Since and Until, and compare it to related equivalence relations on models; our findings are summarized in a diagram at the end of the section (Figure 6).

To define bisimulations that work for temporal logic, we will use relations that link points to points and intervals to intervals.

Definition 3.1 (Bisimulations) Let $M_1 = (W_1, <_1, V_1)$ and $M_2 = (W_2, <_2, V_2)$ be two models. A non-empty relation $Z \subseteq (W_1 \times W_2) \cup (W_1^2 \times W_2^2)$ is a relation of *bisimulation* if $Z \cap (W_1 \times W_2) \neq \emptyset$, and in addition the following hold

1. If $x_1 Z x_2$ then x_1 and x_2 satisfy the same proposition letters.
2. If $x_1 Z x_2$ and $y_1 <_1 x_1$, then there exists y_2 in M_2 with $y_2 <_2 x_2$, $y_1 Z y_2$ and $y_1 x_1 Z y_2 x_2$.
3. If $y_1 x_1 Z y_2 x_2$ and $y_1 <_1 z_1 <_1 x_1$, then there exists z_2 with $y_2 <_2 z_2 <_2 x_2$ and $z_1 Z z_2$.
4. Clauses 2 and 3 with $>_1$ (and $>_2$) instead of $<_1$ (and $<_2$).
5. Clauses 2, 3 and 4 going from M_2 to M_1.

If there is a relation of bisimulation between x_1 and x_2, then we say that x_1 and x_2 are *bisimilar* (notation $x_1 \leftrightarrows x_2$), and similarly for intervals $y_1 x_1$ and $y_2 x_2$. If necessary, the models in which x_1 and x_2 live will also be included in the notation: $M_1, x_1 \leftrightarrows M_2, x_2$.

In the semantics of dynamic formalisms both states and transitions play an important role; the semantics of Since and Until may seem to suggest that the transitions only have a secondary role to play in determining the truth value of a formula involving Since and Until. Our notion of bisimulation, however, clearly shows that both properties of states and of intervals are important.

It is easily verified that arbitrary unions of bisimulation relations are again bisimulations, and that $\leftrightarroweq$ is the maximal bisimulation and an equivalence relation.

In Section 5 we show that a first-order formula in the correspondence language $\mathcal{L}_{SU}$ is equivalent to a temporal formula with Since and Until iff it is invariant for the notion of bisimulation defined in Definition 3.1. In the remainder of the present section we compare our notion of bisimulation to closely related equivalence relations on models. Such comparisons can take place at two levels: one can compare particular instances of bisimulation relations, but at a more abstract level one can also compare the equivalence classes of models modulo the various notions of bisimilarity.

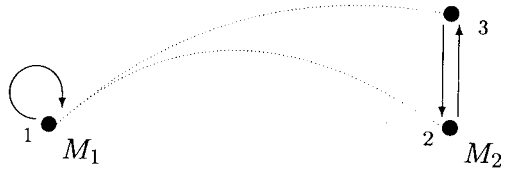
Our goal in comparing these equivalence relations is to locate our notion in the wider landscape of such relations, and show that our notion of bisimulation is the weakest one that allows for a direct development of the model theory of Since and Until without a detour through richer languages.

Modal Bisimulations

We start with bisimulations for standard modal languages, often called *strong* bisimulations in the computational literature (see [13]). These are defined by clause 1 of Definition 3.1 together with clause 2 with the last conjunct ('and $y_1x_1Zy_2x_2$') left out. Strong bisimulations are much weaker than our bisimulations: they don't take the 'past' of nodes into account. An obvious way of taking the past into account is by extending the language so as to include the familiar forward looking modality F and backward looking modality P. The corresponding notion of bisimulation is defined as follows. Let M_1, M_2 be two models; a non-empty relation $Z \subseteq W_1 \times W_2$ is a relation of F, P-*bisimulation* if it satisfies condition 1 of Definition 3.1 and a trimmed down version of its condition 2 in which references to intervals have been deleted:

2′. If x_1Zx_2 and $y_1 <_1 x_1$, then there exists y_2 in M_2 with $y_2 <_2 x_2$ and y_1Zy_2,

and a similar condition with $>_1$ instead of $<_1$, and going from M_2 to M_1. We write $x_1 \leftrightarroweq_{F,P} x_2$ to denote that there exists a F, P-bisimulation between x_1 and x_2. Clearly, $x_1 \leftrightarroweq x_2$ implies $x_1 \leftrightarroweq_{F,P} x_2$, but the converse need not hold, as is witnessed by the following example.



Here we have $M_1 \leftrightarroweq_{F,P} M_2$ via the relation indicated with dotted lines; but $M_1 \not\leftrightarroweq M_2$, because any candidate bisimulation Z should link 1 to both 2 and 3; so it would follow that $11Z23$, and by the definition of bisimulations, there would be state z between 2 and 3 — a contradiction.

All in all, then, we have the following.

Proposition 3.2 *1. Bisimilarity implies F, P-bisimilarity.*

2. F, P-bisimilarity does not imply bisimilarity.

$\mathcal{U}$-bisimulations

Next we consider so-called $\mathcal{U}$-bisimulations. These were defined by Van Benthem, Van Eijck and Stebletsova [4, Definition 4.2] as candidate bisimulations for temporal logic. A relation $Z \subseteq W_1 \times W_2$ is a *$\mathcal{U}$-bisimulation* if it satisfies clause 1 of Definition 3.1, clause $2'$ above, and

$3'$. if $x_1 Z x_2$, $x_1 < y_1$, $x_2 < y_2$, $y_1 Z y_2$, and $x_1 < z_1 < y_1$, then there exists a z_2 in W_2 such that $x_2 < z_2 < y_2$ and $z_1 Z z_2$,

as well as similar conditions with $>$ instead of $<$, and going from M_2 to M_1. We use $x_1 \leftrightarroweq_{\mathcal{U}} x_2$ to denote that there exists a $\mathcal{U}$-bisimulation between x_1 and x_2.

It is easily verified that $M_1, w \leftrightarroweq_{\mathcal{U}} M_2, v$ implies $M_1, w \leftrightarroweq M_2, v$: any $\mathcal{U}$-bisimulation can be extended to a bisimulation in our sense. Let Z be a $\mathcal{U}$-bisimulation, and define Z' by $y_1 x_1 Z y_2 x_2$ iff $y_1 < x_1$, $y_2 < x_2$, $y_1 Z y_2$ and $x_1 Z x_2$ all hold. Put $Y := Z \cup Z'$. Then Y is a bisimulation in our sense. By way of example, let us check clause 3 of Definition 3.1. Assume $y_1 x_1 Y y_2 x_2$ and $y_1 < z_1 x_1$; we need to find a z_2 in between y_2 and x_2 with $z_1 Y z_2$. By definition we have $y_1 Z y_2$, $y_1 < x_1$, $y_1 Z y_2$, and $x_1 Z x_2$. So, as Z was a $\mathcal{U}$-bisimulation there exists z_2 with $y_2 < z_2 < x_2$ and $z_1 Z z_2$, and we are done.

Figure 1 below depicts a bisimulation in our sense (indicated with dotted lines), that is not a $\mathcal{U}$-bisimulation. Here, $M_1 = (\mathbb{N}, <, V)$, where V is arbitrary, and $<$ is the usual less-than relation; $M_2 = (\mathbb{N}, <, V)$, where V and $<$ are as in M_1.

Define a relation $Z \subseteq \mathbb{N}^2 \cup (\mathbb{N}^2 \times \mathbb{N}^2)$ by putting

$$Z := \{(n, n') \mid n \in \mathbb{N}\} \cup \{(n, (n+1)') \mid n \in \mathbb{N}\} \cup \{(n\,m, (n+1)'\,(m+1)') \mid n < m \in \mathbb{N}\}.$$

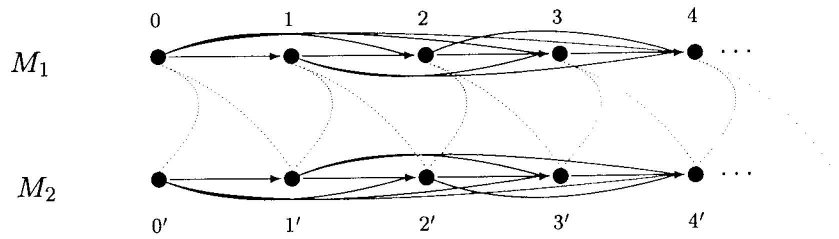


Figure 1: A bisimulation which is not a $\mathcal{U}$-bisimulation

We leave it to the reader to check that $Z : M_1, 0 \leftrightarroweq M_2, 0'$. Now, to see that $Z : M_1, 0 \not\leftrightarroweq_{\mathcal{U}} M_2, 0'$, observe that because of $1 < 3$, $1 < 2 < 3$, $1Z2'$, and $2Z3'$, for Z to be a $\mathcal{U}$-bisimulation we should be able find a z with $2' < z < 3'$ and $2Zz$ — and there is no such point.

Proposition 3.3 *1. $\mathcal{B}$-bisimilarity implies bisimilarity.*

2. Bisimilarity does not imply $\mathcal{U}$-bisimilarity. (Cf. Proposition 3.6 below.)

$\mathcal{B}$-bisimulations

Van Benthem, Van Eijck and Stebletsova [4] also consider an alternative notion, called $\mathcal{B}$-bisimulation, which relates points to points and pairs of points to pairs of points, much like our notion of bisimulation; the notion of $\mathcal{B}$-bisimulation is used to analyze a two-dimensional counterpart of the language of temporal logic with S and U. To be precise, a relation $Z \subseteq (W_1 \times W_2) \cup (W_1^2 \times W_2^2)$ is a *$\mathcal{B}$-bisimulation* if it satisfies clause 1 of Definition 3.1 and

2″. if $x_1 Z x_2$ and $x_1 < y_1$, then there exists y_2 with $x_2 < y_2$ and $x_1 y_1 Z x_2 y_2$

3″. if $x_1 y_1 Z x_2 y_2$, then $x_1 Z x_2$ and $y_1 Z y_2$

4″. if $x_1 y_1 Z x_2 y_2$ and $x_1 < z_1 < y_1$, then there exists z_2 with $x_2 < z_2 < y_2$ and both $x_1 z_1 Z x_2 z_2$ and $z_1 y_1 Z z_2 y_2$,

and similar conditions with $>$ instead of $<$, and going from M_2 to M_1. We use $x_1 \leftrightarroweq_{\mathcal{B}} x_2$ to denote that there exists a $\mathcal{B}$-bisimulation between x_1 and x_2. Van Benthem, Van Eijck and Stebletsova [4, Proposition 4.8] show that $x_1 \leftrightarroweq_{\mathcal{U}} x_2$ implies $x_1 \leftrightarroweq_{\mathcal{B}} x_2$: any $\mathcal{U}$-bisimulation can be extended to a $\mathcal{B}$-bisimulation. What about the relation between $\leftrightarroweq$ and $\leftrightarroweq_{\mathcal{B}}$? It is clear that any $\mathcal{B}$-bisimulation is a bisimulation in our sense. To see that the converse doesn't hold, look at Figure 1 again, but redefine the relations in the models to arrive at the picture in Figure 2. That is, define $M_1 = (\mathbb{N}, R_1, V)$, where

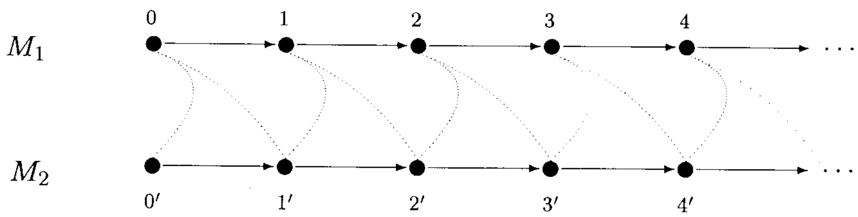


Figure 2: A bisimulation which is not a $\mathcal{B}$-bisimulation

V is arbitrary, and R_1nm iff $m = n+1$; and $M_2 = (\mathbb{N}, R_2, V)$, where V and R_2 are as in M_1.

Define a relation $Z \subseteq \mathbb{N}^2 \cup (\mathbb{N}^2 \times \mathbb{N}^2)$ by putting

$$\begin{aligned} Z \;:=\; & \{(n, n') \mid n \in \mathbb{N}\} \cup \{(n, (n+1)') \mid n \in \mathbb{N}\} \cup \\ & \{(n\,n+1, m'\,(m+1)') \mid n, m \in \mathbb{N}\}. \end{aligned}$$

We leave it to the reader to check that $Z : M_1, 0 \leftrightarrows M_2, 0'$. Now, to see that $Z : M_1, 0 \not\leftrightarrows_{\mathcal{B}} M_2, 0'$, observe that if $Z : (0\,1) \leftrightarrows_{\mathcal{B}} (5'\,6')$ were to hold, we would also have $Z : 0 \leftrightarrows_{\mathcal{B}} 5'$, which is not the case.

The above observations can be strengthened: there are models that are bisimilar in our sense, but not $\mathcal{B}$-bisimilar (and hence, not $\mathcal{U}$-bisimilar either). Here is an example. For the purposes of this example we will define a new, unary modal operator **3-step**, and prove that it is preserved under $\mathcal{B}$-bisimulations; we will then display two models M_1 and M_2 that are bisimilar in our sense, but that don't agree on a formula involving the **3-step**-operator; hence, M_1 and M_2 cannot be $\mathcal{B}$-bisimilar.

We need some preliminary definitions. First of all, let $M = (W, <, V)$ be a model, s, $t \in W$, and $s < t$. Then $(s\,t)$ is called a *locally linear chain* if the following condition is satisfied:

if $s < u_1 < t$ and $s < u_2 < t$ then $<$ is transitive on $\{s, t, u_1, u_2\}$ and $s < u_1 < u_2 < t$ or $s < u_2 < u_1 < t$.

Next, we call M a *locally linear model* if for every s, $t \in W$ we have that $s < t$ implies that $(s\,t)$ is locally linear chain. The *length* of an interval $(s\,t)$ is the largest n such that there exist $u_0, \ldots, u_n$ with $s = u_0 < \cdots < u_n = t$ (if such n exists). Call a structure *discrete* if every interval in the structure has finite length.

Claim 3.4 *Let M_1, M_2 be two discrete, locally linear models. Assume that s, $t \in W_1$, x, $y \in W_2$, and both $s < t$ and $x < y$. If $(s\,t)$ and $(x\,y)$ are $\mathcal{B}$-bisimilar, then $(s\,t)$ and $(x\,y)$ have the same length.*

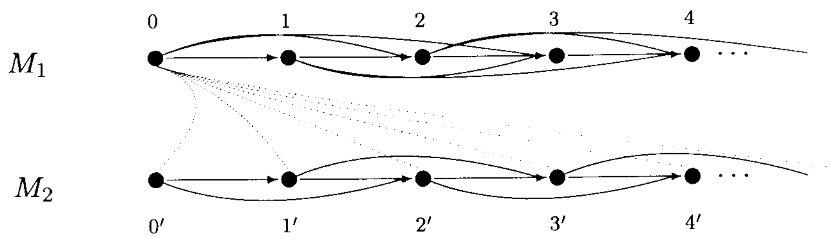


Figure 3: Bisimilar models but not $\mathcal{U}$-bisimilar

Proof. Use induction on the length of $(s\,t)$. $\dashv$

Let us define our new operator **3-step** now:

$s \models$ **3-step** ϕ iff there is a locally linear chain $s < u_1 < u_2 < b$ with $b \models \phi$.

Claim 3.5 *Formulas involving the* **3-step** *-operator are preserved under $\mathcal{B}$-bisimulations between discrete, locally linear models.*

Proof. Let M_1 and M_2 be bisimilar models, say $Z : s \leftrightarrow x$, where $s \in M_1$ and $x \in M_2$. Suppose $s \models$ **3-step** ϕ; we need $x \models$ **3-step** ϕ. Then there is a locally finite chain $s < u_1 < u_2 < t$ such that $t \models \phi$. Since $s < t$ and sZx there exists y such that $x < y$ and $(s\,t)Z(x\,y)$ and tZy. So $y \models \phi$. From Claim 3.4 we get that $(s\,t)$ and $(x\,y)$ have the same length. It follows that there is a local chain $x < z_1 < z_2 < y$, and because $y \models \phi$, it follows that $x \models$ **3-step** ϕ, as required. $\dashv$

Next, consider the models M_1 and M_2 in Figure 3. That is, both have $\mathbb{N}$ as their domain, and in M_1 the relation $<_1$ is given by $\{(n, n+i) \mid i \in \{1,2,3\}\}$. In M_2 the relation $<_2$ is given by $\{(n, n+1) \mid i \in \{1,2\}\}$. Choose valuations such that all proposition letters are true in all states.

Then, the models M_1 and M_2 are bisimilar in our sense: use the relation Z that links all points in M_1 to all points in M_2, and it should link minimal intervals (i.e., of length 0) in M_1 to minimal intervals in M_2, and all non-minimal intervals in M_1 should be linked to all non-minimal intervals in M_2. We leave it to the reader to check that this does indeed establish a bisimulation between M_1 and M_2.

Finally, we show that there cannot be a $\mathcal{B}$-bisimulation between M_1 and M_2 that links 0 to 0′. Observe that $M_1, 0 \models$ **3-step** p (for all p), but in M_2 there are no locally linear chains of length 2, hence $M_2, 0' \not\models$ **3-step** p. By Claim 3.5 it follows that M_1 and M_2 are not $\mathcal{B}$-bisimilar.

Proposition 3.6 *1. $\mathcal{B}$-bisimilarity implies bisimilarity*

2. Bisimilarity does not imply $\mathcal{B}$-bisimilarity. (Hence it does not imply $\mathcal{U}$-bisimilarity either.)

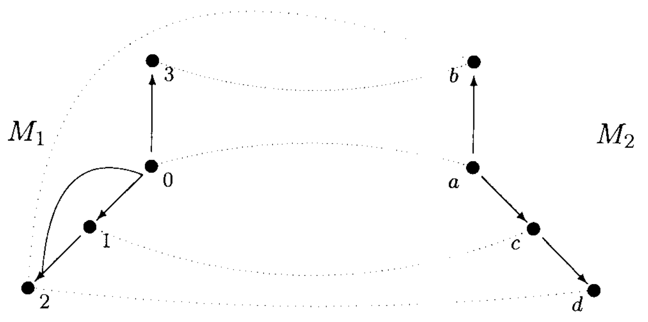


Figure 4: S-similar but not bisimilar

S-bisimulations

Sturm [22] defines a notion of bisimulation, called S-simulation, for the forward looking fragment of our temporal language as follows. Let M_1, M_2 be two models; a non-empty relation $Z \subseteq W_1 \times W_2$ is a relation of *S-simulation* if it satisfies condition 1 of Definition 3.1 as well as

- If $x_1 Z x_2$ and $x_1 < y_1$, then there exists y_2 in M_2 with $y_1 Z y_2$ and $x_2 < y_2$ such that for every z_2 in M_2 with $x_2 < z_2 < y_2$ there exists z_1 in M_1 with $x_1 < z_1 < y_1$ and $z_1 Z z_2$.
- A similar clause going from M_2 to M_1.

Observe that S-simulations only 'look forward'; they don't take the converse $>$ of $<$ into account. Sturm [22, Lemma 2.11.16] shows that all forward looking temporal formulas (that is: formulas without occurrences of Since) are preserved under S-similarity.

Clearly, bisimilarity in our sense implies S-similarity. For the converse implication, consider the two models in Figure 4. That is: $M_1 = (\{0,1,2,3\}, \{(0,1), (0,2), (1,2), (0,3)\}, V_1)$ and $M_2 = (\{a,b,c,d\}, \{(a,b), (a,c), (a,d)\}, V_2)$ where V_1 and V_2 are such that all proposition letters are true at all states of both models. Furthermore, $Z = \{(0,a), (1,c), (2,b), (2,d), (3,b)\}$. Then $Z : M_1, 0 \leftrightarroweq_S M_2, a$. On the other hand, there can't be a bisimulation in our sense between M_1 and M_2 that links 0 and a; observe, for example, that $0 \models FFPFF\top$ while $a \not\models FFPFF\top$, hence by Lemma 3.10 $M_1, 0 \not\leftrightarroweq M_2, a$. To be fair, though, we should compare S-similarity to *forward looking* bisimulations only — after all, S-simulations only 'look forward.' So, define a *forward bisimulation* to be a relation Z between models that satisfies clauses 1, 2, and 3 of Definition 3.1, going from the one model to the other, and vice versa. We will now show that there is no forward bisimulation linking the states 0 in M_1 and a in M_2 in Figure 4; this will establish that S-similarity is weaker than (forward) bisimilarity.

To see that 0 and 1 can't be forward bisimilar, argue as follows. Define a new binary modal operator **after** (ϕ, ψ) by

$$w \models \textbf{after}\,(\phi,\psi) \text{ iff } \exists vu\,(w < v < u \wedge w < u \wedge v \models \phi \wedge u \models \psi).$$

(This operator is related to a number of operators in the literature, including dynamic composition, Lambek and arrow logic product, and Venema's *chop*-operator [23].)

We leave it as an exercise to show that formulas with this new operator are preserved by forward bisimilarity as defined above. Observe, finally, that $M_1, 0 \models$ **after** $(\top, \top)$ but $M_2, a \not\models$ **after** $(\top, \top)$. Hence 0 and a can't be forward bisimilar.

Proposition 3.7 *1. (Forward looking) bisimilarity implies S-similarity.*

2. S-similarity does not imply (forward) bisimilarity.

To conclude our discussion of S-similarity we want to emphasize the following. S-similarity preserves truth of (forward looking) temporal formulas, and by the above counterexample it is weaker than our notion of bisimulation (at least when we only consider the forward looking part). This may seem to be a reason to prefer S-similarity over our notion of bisimilarity, especially since our bisimulations involve both points and intervals, while temporal formulas are evaluated at points only. However, as we will show below, it is precisely this special two-sorted character of our notion of bisimulation that allows us to develop the model theory of Since and Until in a direct way (without detours through richer languages); S-simulations don't have this feature. And thus, S-simulations seem to be too weak.

3-Back-and-forth Equivalence

The following notion of an equivalence relation on models is taken from Van Benthem [2]. First, a *partial isomorphism* from M_1 to M_2 is a partial map $\theta : W_1 \to W_2$ such that

- for all proposition letters p and all states w, $w \in V_1(p)$ iff $f(w_1) \in V_2(p)$,
- for all states w_1, $v_1 \in W_1$ and all quantifier-free formulas $\alpha(x, y)$ in $<$ and $=$ we have $M_1 \models \alpha[w_1 v_1]$ iff $M_2 \models \alpha[f(w_1)f(v_1)]$.

Next, a *κ-back-and-forth system* ($\kappa \leq \omega$) from M_1 to M_2 is a non-empty set C of partial isomorphisms from M_1 to M_2 such that

1. if $\theta \in C$ then $|dom(\theta)| \leq \kappa$
2. if $\theta \in C$ then any restriction of θ to a subset of its domain is also in C

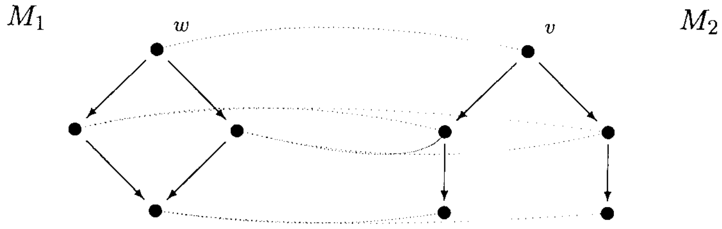


Figure 5: Bisimilar but not 3-back-and-forth-equivalent.

3. if $\theta \in C$, $w \in W_1 \setminus dom(\theta)$ and $|dom(\theta)| < \kappa$, then there exists θ^+ in C with $\{w\} \cup dom(\theta) \subseteq dom(\theta^+)$

4. if $\theta \in C$, $v \in W_2 \setminus rng(\theta)$ and $|dom(\theta)| < \kappa$, then there exists θ^+ in C with $\{v\} \cup rng(\theta) \subseteq rng(\theta^+)$.

Let $\overline{w} \in M_1$ and $\overline{v} \in M_2$ be tuples of equal length. The structures $(M_1, \overline{w})$ and $(M_2, \overline{v})$ are *κ-back-and-forth equivalent* if there exists a κ-back-and-forth system C from M_1 to M_2 containing a map θ such that $\theta(\overline{w}) = \overline{v}$; notation $C : M_1 \simeq_\kappa M_2$.

Van Benthem [2] shows that a first-order formula (in $<$, $=$) can be written with at most 3 variables iff it is invariant under 3-back-and-forth equivalence. The relevance of this result for temporal logic is that temporal formulas with Since and Until can be translated into the 3-variable fragment of $\mathcal{L}_1$, the first-order correspondence language.

Clearly, $M_1, w \simeq_3 M_2, v$ implies $M_1, w \sim M_2, v$ for all $\sim \in \{\underline{\leftrightarrow}_{\mathcal{U}}, \underline{\leftrightarrow}_{\mathcal{B}}, \underline{\leftrightarrow}, \underline{\leftrightarrow}_S, \underline{\leftrightarrow}_{F,P}\}$, but none of the converse implications holds, as is witnessed by the example in Figure 5.

We leave it to the reader to check that $M_1 \underline{\leftrightarrow}_{\mathcal{U}} M_2, v$ via the dotted lines (and from this the other bisimilarities follow). However, the single 'end point' in M_1 satisfies the 3-variable statement

$$\exists y \exists z\, (y \neq z \wedge y < x \wedge z < x)$$

which is not satisfied by any node in M_2, so M_1 and M_2 can not be 3-back-and-forth equivalent.

Proposition 3.8 *1. 3-Back-and-forth equivalence implies bisimilarity.*

2. Bisimilarity does not imply 3-back-and-forth equivalence.

Temporal Equivalence

Finally, we compare temporal equivalence to bisimilarity.

Proposition 3.9 *If ϕ cannot distinguish between bisimilar points, then it cannot distinguish between bisimilar intervals.*

Proof. Assume $w_1 v_1 \models \phi$ and $w_1 v_1 \leftrightarroweq w_2 v_2$. We have to show that $w_2 v_2 \models \phi$. So choose u_2 such that $w_2 < u_2 < v_2$. We need to show that $u_2 \models \phi$. As $w_1 v_1 \leftrightarroweq w_2 v_2$, there exists u_1 such that $w_1 < u_1 < v_1$ and $u_1 \leftrightarroweq u_2$. Then $u_1 \models \phi$, so by the assumption on ϕ we have $u_2 \models \phi$. ⊣

Lemma 3.10 *If $M_1 = (W_1, <_1, V_1)$ and $M_2 = (W_2, <_2, V_2)$ are two models, and $x_1 \in W_1$, $x_2 \in W_2$, are such that $x_1 \leftrightarroweq x_2$, then $x_1 \equiv x_2$. In other words: bisimilarity implies temporal equivalence.*

Proof. We argue by induction on the structure of formulas. The atomic and boolean cases are easy. So let us consider the temporal case. Assume $w_1 \models S(\phi, \psi)$ and $w_1 \leftrightarroweq w_2$. We need to show that $w_2 \models S(\phi, \psi)$. By definition there exists a v_1 such that (i) $v_1 < w_1$, (ii) $v_1 \models \phi$, and (iii) $v_1 w_1 \models \psi$. From (i) and $w_1 \leftrightarroweq w_2$ we obtain a v_2 such that (iv) $v_2 < w_2$, (v) $v_1 \leftrightarroweq v_2$, and (vi) $v_1 w_1 \leftrightarroweq v_2 w_2$. By the inductive hypothesis, (v) and (ii) we get $v_2 \models \phi$. From the inductive hypothesis, Proposition 3.9, (iii) and (vi) it follows that $v_2 w_2 \models \psi$. By (iv) this implies $w_2 \models S(\phi, \psi)$, as required. ⊣

The converse of the implication proved in Lemma 3.10 ('Does temporal equivalence imlpy bisimilarity?') will be examined in Section 4 below.

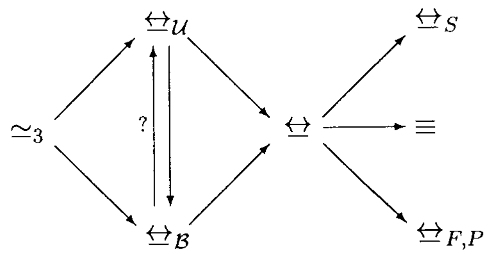


Figure 6: The findings of this section.

Summarizing the findings of this section, we arrive at the diagram of inclusions depicted in Figure 6, where an arrow $\sim \;\rightarrow\; \approx$ denotes that $\sim$-bisimilarity implies $\approx$-bisimilarity. The upward arrow marked with a question mark represents an open problem due to Van Benthem, Van Eijck and Stebletsova [4, Open Problem 4.7].

4 Hennessy-Milner classes

In this section we consider the converse of Lemma 3.10: when does temporal equivalence imply bisimilarity? Using a standard example from the literature on modal logic, it is easily seen that this is not the case in general.

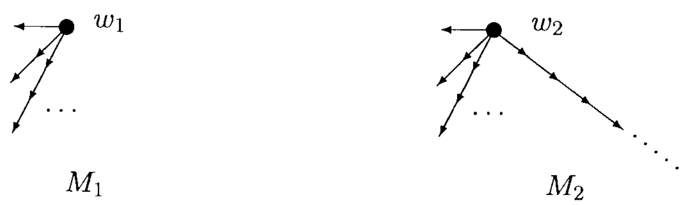


Figure 7: Equivalent but not bisimilar.

The two models in Figure 7 satisfy the same temporal formulas in their root nodes, but there is no bisimulation linking the two root nodes.

To get a handle on situations where temporal equivalence *does* imply bisimilarity, we need the following definition.

Definition 4.1 (Hennessy-Milner classes) A class K of models is called a *Hennessy-Milner class* if for M_1, $M_2 \in \mathsf{K}$, and all $w_1 \in M_1$ and $w_2 \in M_2$, $w \mathrel{\underline{\leftrightarrow}} v$ iff $w \equiv v$. That is, if temporal equivalence is a bisimulation between M_1 and M_2.

We say that a model M has the *Hennessy-Milner property* whenever the class $\{M\}$ has the Hennessy-Milner property.

For the standard modal language with $\Diamond$ and $\Box$ the above notion is due to Goldblatt [12] and Hollenberg [16]. The standard example of a modal Hennessy-Milner class in which modal equivalence and modal bisimilarity coincide, is the class of all image-finite models — models for which the set of $<$-successors is finite for any point in the model.

It turns out that a natural way to determine whether a class of models is a Hennessy-Milner class involves the concept of temporal saturation. Let $\Delta \subseteq_{\mathsf{fin}} \Phi$ denote that Δ is a finite subset of Φ.

Definition 4.2 Let $M = (W, <, V)$ be a model. M is called *t-saturated* if it satisfies the following conditions:

> **If** $\forall \Delta \subseteq_{\mathsf{fin}} \Phi \forall \Gamma \subseteq_{\mathsf{fin}} \Psi \exists v \in W\ (v < w \text{ and } v \models \bigwedge \Delta \text{ and } vw \models \bigwedge \Gamma)$
> **then** $\exists v \in W\ (v < w \text{ and } v \models \bigwedge \Phi \text{ and } vw \models \bigwedge \Psi)$; and
>
> **If** $\forall \Gamma \subseteq_{\mathsf{fin}} \Psi \exists u \in W\ (v < u < w \text{ and } u \models \bigwedge \Gamma)$
> **then** $\exists u \in W\ (v < u < w \text{ and } u \models \bigwedge \Psi)$.

(And similarly, with $>$ instead of $<$.) We use $\mathsf{T\text{-}SAT}$ to denote the class of all t-saturated models.

The notion of m-saturation considered in the literature on modal logic arises if one only takes the first condition for $>$ in the definition of t-saturation, with $\Gamma = \emptyset$ (see [10, 12, 16]).

Theorem 4.3 T-SAT *is a Hennessy-Milner class.*

Proof. Assume that M_1, M_2 are in T-SAT, and that $w_1 \in M_1$, $w_2 \in M_2$ are such that $w_1 \equiv w_2$. We will show that $w_1 \leftrightarrows w_2$ by showing that $\equiv$ is a bisimulation.

The first clause in the definition of a bisimulation is trivially satisfied. For the second one, assume $w_1 \equiv w_2$ and $v_1 < w_1$. We need to find a v_2 such that $v_2 < w_2$, $v_1 \equiv v_2$ and $v_1w_1 \equiv v_2w_2$. Consider $\Delta \subseteq_{\text{fin}} tp_{M_1}(v_1)$ and $\Gamma \subseteq_{\text{fin}} tp(v_1w_1)$. Then $w_1 \models S(\bigwedge\Delta, \bigwedge\Gamma)$, and so, as $w_1 \equiv w_2$, we have $w_2 \models S(\bigwedge\Delta, \bigwedge\Gamma)$. Thus, there exists v in M_2 such that $v < w_2$, $v \models \bigwedge\Delta$, and $vw_2 \models \bigwedge\Gamma$. By t-saturation there must be a $v_2 < w_2$ such that $v_2 \models \bigwedge tp_{M_1}(v_1)$ and $v_2w_2 \models tp_{M_1}(v_1w_1)$. But then $v_1 \equiv v_2$ and $v_1w_1 \equiv v_2w_2$, as required.

For the third clause in the definition of a bisimulation, assume that $v_1 < u_1 < w_1$, and $v_1w_1 \equiv v_2w_2$. We need to find a u_2 such that $v_2 < u_2 < w_2$ and $u_1 \equiv u_2$. Consider $\Gamma \subseteq_{\text{fin}} tp_{M_1}(u_1)$. Then $v_1w_1 \not\models \neg\bigwedge\Gamma$, and so $v_2w_2 \not\models \neg\bigwedge\Gamma$. This implies that there exists a u in M_2 such that $v_2 < u < w_2$ and $u \models \bigwedge\Gamma$. Applying the second clause in the definition of t-saturation, we find a u_2 in M_2 such that $v_2 < u_2 < w_2$ and $u_1 \equiv u_2$, and we are done. ⊣

Proposition 4.4 *Let M be a finite model, then it is t-saturated.*

Proof. Consider set of formulas Φ and Ψ such that for all $\Delta \subseteq_{\text{fin}} \Phi$ and $\Gamma \subseteq_{\text{fin}} \Psi$ there exists a v such that

$$v < w \text{ and } v \models \bigwedge\Delta \text{ and } vw \models \bigwedge\Gamma. \tag{1}$$

Suppose that there is no v such that (1) holds for all of Φ and Ψ. Then, for every $v < w$, there find either a $\phi_v \in \Phi$ with $v \not\models \phi_v$ or a $\psi_v \in \Psi$ with $vw \not\models \psi_v$. As M is finite we can collect these formulas together in finite sets $\Delta \subseteq_{\text{fin}} \Phi$, $\Gamma \subseteq_{\text{fin}} \Psi$ (where $\Delta \cup \Gamma \neq \emptyset$) for which (1) does not hold — a contradiction!

By similar arguments one can show that M satisfies the remaining conditions in Definition 4.2. ⊣

We need the following form of saturation from first-order logic. Recall first that M_1 is an *elementary extension* of M_2 if $W_1 \supseteq W_2$ and for all $\mathcal{L}_1$-formulas $\alpha(x_1, \ldots, x_n)$ and all tuples $w_1, \ldots, w_n$ of M_2,

$$M_1 \models \alpha(x_1, \ldots, x_n)[w_1, \ldots, w_n] \text{ iff } M_2 \models \alpha(x_1, \ldots, x_n)[w_1, \ldots, w_n].$$

We write $M_2 \preceq M_1$ in this case.

Let κ be a cardinal number. A model M is *κ-saturated* in the sense of first-order logic if whenever Φ is a set of formulas in a $\mathcal{L}_1'(x)$-formulas, where

$\mathcal{L}'_1$ extends $\mathcal{L}_1$ by the addition of fewer then κ many individual constants, and Φ is finitely satisfiable in an $\mathcal{L}'_1$-expansion of M, then Φ itself is satisfiable in this expansion.

To show that M is t-saturated it suffices to show that M is 3-saturated. Below we will need the stronger assumption of ω-saturation.

Proposition 4.5 *Let M be an ω-saturated model, then it is t-saturated.*

One can construe ω-saturated models as ultrapowers over a special kind of ultrafilters. We assume that the reader is familiar with the definition of ultraproducts and ultrapowers of models (consult Hodges [14] if necessary). An ultrafilter is called *ω-incomplete* if it is not closed under countable intersections. As a result, if U is an ω-incomplete ultrafilter and M is a model, then the ultrapower $\prod_U M$ is an ω-saturated elementary extension of M.

Theorem 4.6 *Let M_1, M_2 be two models, and let w_1, w_2 be elements of M_1, M_2, respectively. If $w_1 \equiv w_2$ then M_1 and M_2 have bisimilar ultrapowers.*

Proof. The proof is similar to the proof of [21, Theorem 5.7]. We confine ourselves to a sketch of the proof. Let I be an infinite index set; by Chang and Keisler [7, Proposition 4.3.5] there is an ω-incomplete ultrafilter U over I. By our previous remarks the ultrapowers $\prod_U(M_1, w_1) =: (M'_1, w'_1)$ and $\prod_U(M_2, w_2) =: (M'_2, w'_2)$ are ω-saturated.

Observe that $tp_{M'_1}(w'_1) = tp_{M'_2}(w'_2) = tp_{M_1}(w_1)$. Hence, $M'_1, w'_1 \equiv M'_2, w'_2$; as M'_1, w'_1 and M'_2, w'_2 are ω-saturated, it follows that $M'_1, w'_1 \leftrightarrow M_2, w'_2$, as required. $\dashv$

Thus, temporal equivalence implies that there exist bisimilar ultrapowers. Hennessy-Milner classes can be characterized in terms of a stronger connection between temporal equivalence and bisimilar ultrapowers. We need two lemmas to arrive at this characterization.

Lemma 4.7 *Let I be an index set, and U an ultrafilter over I. Then*

1. *If $M_i, w_i \leftrightarrow N_i, v_i$ holds for all $i \in I$, then $\prod_U(M_i, w_i) \leftrightarrow \prod_U(N_i, v_i)$.*
2. *If $M, w \leftrightarrow N, v$, then $\prod_U(M, w) \leftrightarrow \prod_U(N, v)$.*

Proof. We only prove the first item. For each $i \in I$, let Z_i be a bisimulation linking M_i and N_i: $Z_i : M_i, w_i \leftrightarrow N_i, v_i$. Define a relation Z between points of $\prod_U(M_i, w_i)$ and $\prod_U(N_i, v_i)$, and pairs of points of $\prod_U(M_i, w_i)$ and $\prod_U(N_i, v_i)$ in the obvious way by putting

$$\begin{aligned} x_1 Z x_2 &\text{ iff } \{i \in I \mid x_1(i) Z_i x_2(i)\} \in U, \text{ and} \\ x_1 y_1 Z x_2 y_2 &\text{ iff } \{i \in I \mid x_1(i) y_1(i) Z_i x_2(i) y_2(i)\} \in U \end{aligned}$$

We claim that this is a bisimulation. First of all, it is clearly non-empty (take $x_1 : i \mapsto w_i$, and $x_2 : i \mapsto v_i$; then $x_1 Z x_2$). Next, if x in $\prod_U(M_i, w_i)$ has $x \models p$ and xZy, then, by the definition of ultraproducts $\{i \in I \mid x(i) \in V_i(p)\} \in U$. As xZy this implies

$$X := \{i \in I \mid x(i) \in V_i(p) \text{ and } x(i) Z_i y(i)\} \in U.$$

As each Z_i is a bisimulation it follows that $X \subseteq \{i \in I \mid y(i) \in V_i(p)\}$, hence the latter set is in U, from which we get $y \models p$, as required.

The remaining clauses may be proved by similar arguments. ⊣

Lemma 4.8 *Let* K *be a Hennessy-Milner class, and* M_1, $M_2 \in K$. *Let* w_1, w_2 *be elements of* M_1, M_2, *respectively, such that* $w_1 \equiv w_2$. *Then* $\prod_U(M_1, w_1) \leftrightarrows \prod_U(M_2, w_2)$ *for all index sets* I *and ultrafilters* U *over* I.

Proof. By Lemma 4.7 we have both $M_1, w_1 \leftrightarrows \prod_U(M_1, w_1)$ and $M_2, w_2 \leftrightarrows \prod_U(M_2, w_2)$. As M_1 and M_2 live in a Hennessy-Milner class, $M_1, w_1 \equiv M_2, w_2$ implies $M_1, w_1 \leftrightarrows M_2, w_2$, hence $\prod_U(M_1, w_1) \leftrightarrows \prod_U(M_2, w_2)$. ⊣

Corollary 4.9 *Let* K *be a class of models. Then* K *is a Hennessy-Milner class iff the following are equivalent for all models* M_1, $M_2 \in \mathsf{K}$:

1. $M_1, w_1 \equiv M_2, w_2$
2. *all ultrapowers of* M_1, w_1 *and* M_2, w_2 *are bisimilar.*

For the standard modal language with $\Diamond$ and $\Box$, Marco Hollenberg has characterized the *maximal* Hennessy-Milner classes in terms of submodels of canonical models. No such characterization has been obtained for Hennessy-Milner classes for the temporal language with Since and Until; in fact, it is not always clear whether canonical models for Since and Until form a Hennessy-Milner class. For example, the lack of a uniform definition of an accessibility relation in the completeness proofs for logics with Since and Until due to Burgess [6] and Xu [24] makes it hard to determine whether their canonical models form a Hennessy-Milner class. On the other hand, it is easy to see that the canonical models defined by Gabbay and Hodkinson [11] using the so-called irreflexivity rule do form a Hennessy-Milner class.

5 Applications to temporal model theory

In this section we apply the tools developed in Sections 3 and 4 to arrive at model-theoretic results for temporal logic on preservation and definability. We give quick proofs of definability, separation, and interpolation theorems, as well as a preservation theorem characterizing the first-order translations of temporal formulas.

To smoothen the presentation of our results, we will be working with so-called *pointed models*; these are structures of the form (M, w), where w lives in the domain of M; w is called the *distinguished point* of (M, w). We will assume that a bisimulation between two pointed models links their distinguished points.

We will also be using the following operations on classes of models: **Pr**, **Po**, **B**. Here $\mathbf{Pr}(\mathsf{K})$ is the class of ultraproducts of models in K; $\mathbf{Po}(\mathsf{K})$ is the class of ultrapowers of models in K; and $\mathbf{B}(\mathsf{K})$ is the class of all models that are bisimilar to a model in K.

Lemma 5.1 *Let K be a class of pointed models.*

1. K *is closed under bisimulations and ultraproducts iff* $\mathsf{K} = \mathbf{BPr}(\mathsf{K})$,
2. K *is closed under bisimulations and ultrapowers iff* $\mathsf{K} = \mathbf{BPo}(\mathsf{K})$.

Proof. We only prove the first item, and to prove the first item it suffices to show that $\mathbf{PrB}(\mathsf{K}) \subseteq \mathbf{BPr}(\mathsf{K})$. So, assume $(M, w) \in \mathbf{PrB}(\mathsf{K})$. Then there are an index set I, models (M_i, w_i) and (N_i, v_i) $(i \in I)$ such that $(N_i, v_i) \in \mathsf{K}$, $(M_i, w_i) \leftrightarrows (N_i, v_i)$, and $(M, w) = \prod_U (M_i, w_i)$, for some ultrafilter U over I. Trivially, $\prod_U (N_i, v_i) \in \mathbf{Pr}(\mathsf{K})$. By Lemma 4.7, $(M, w) = \prod_U (M_i, w_i) \leftrightarrows \prod_U (N_i, v_i)$. Hence, $(M, w) \in \mathbf{BPr}(\mathsf{K})$, as required. $\dashv$

We will say that a class K of pointed models is *S, U-definable*, or simply *definable*, by means of a set of temporal formulas if there exists a set of temporal formulas T such that $\mathsf{K} = \{(M, w) \mid (M, w) \models T\}$. A class of pointed models K is *definable by means of a single formula* if it is definable by means of a singleton set.

Let K be a class of pointed models; we use $\overline{\mathsf{K}}$ to denote the class of pointed models that are not in K.

Theorem 5.2 *Let K be a class of pointed models. Then*

1. K *is definable by means of a set of temporal formulas iff* $\mathsf{K} = \mathbf{BPr}(\mathsf{K})$ *and* $\overline{\mathsf{K}} = \mathbf{BPo}(\overline{\mathsf{K}})$,
2. K *is definable by means of a single temporal formula iff* $\mathsf{K} = \mathbf{BPr}(\mathsf{K})$ *and* $\overline{\mathsf{K}} = \mathbf{BPr}(\overline{\mathsf{K}})$.

Proof. 1. The *only if* direction is easy. For the converse, we can 'bisimulate' familiar arguments from first-order model theory. Assume K is closed under ultraproducts and bisimulations, while $\overline{\mathsf{K}}$ is closed under ultrapowers. Let $T = \bigcap \{tp_{(M,w)}(w) \mid (M, w) \in \mathsf{K}\}$.

We will show that T defines K. First, $\mathsf{K} \models T$. Second, assume that $(M, w) \models T$; we need to show $(M, w) \in \mathsf{K}$. Consider $tp_{(M,w)}(w)$, and define $I = \{\sigma \subseteq tp_{(M,w)}(w) \mid |\sigma| < \omega\}$. For each $i = \{\sigma_1, \ldots, \sigma_n\} \in I$ there is a model (M_i, w_i) of i in K. By standard model-theoretic arguments there exists an ultraproduct $\prod_U (M_i, w_i)$ which is a model of $tp_{(M,w)}(w)$; hence $\prod_U (M_i, w_i) \equiv (M, w)$. As $\mathbf{Pr}(\mathsf{K}) \subseteq \mathsf{K}$, $\prod_U (M_i, w_i) \in \mathsf{K}$. By Theorem 4.6 there is an ultrafilter U' such that $\prod_{U'}(\prod_U (M_i, w_i)) \leftrightarroweq \prod_{U'}(M, w)$. Hence, the latter is in K, and, by the closure condition on $\overline{\mathsf{K}}$, this implies $(M, w) \in \mathsf{K}$, as required.

2. Again, the *only if* direction is easy. Assume K, $\overline{\mathsf{K}}$ satisfy the stated conditions. Then both are closed under ultrapowers, hence, by item 1, there are sets of temporal formulas T_1, T_2 defining K and $\overline{\mathsf{K}}$, respectively. Obviously, $T_1 \cup T_2 \models \bot$, so by compactness for some $\phi_1, \ldots, \phi_n \in T_1$, $\psi_1, \ldots, \psi_m \in T_2$, we have $\bigwedge_i \phi_i \models \bigvee_j \neg\psi_j$. Then K is defined by $\bigwedge_i \phi_i$. ⊣

Corollary 5.3 (Separation) *Let* K, L *be classes of pointed models such that* $\mathsf{K} \cap \mathsf{L} = \emptyset$.

1. *If* K *is closed under bisimulations and ultraproducts, and* L *is closed under bisimulations and ultrapowers, then there exists a class of models* M *that is definable by means of a set of temporal formulas and such that* $\mathsf{K} \subseteq \mathsf{M}$ *and* $\mathsf{L} \cap \mathsf{M} = \emptyset$.

2. *If both* K *and* L *are closed under bisimulations and ultraproducts, then there exists a class of models* M *that is definable by means of a single temporal formula and such that* $\mathsf{K} \subseteq \mathsf{M}$ *and* $\mathsf{L} \cap \mathsf{M} = \emptyset$.

Proof. We only prove the first item. Let K' be the class of all pointed models (M, w) such that for some $(N, v) \in \mathsf{K}$, $(M, v) \equiv (N, w)$. Define L' similarly. Then $\mathsf{K} \subseteq \mathsf{K}'$, $\mathsf{L} \subseteq \mathsf{L}'$ and K' and L' are both closed under $\equiv$. Moreover, $\mathsf{K}' \cap \mathsf{L}' = \emptyset$. For suppose $(M, w) \in \mathsf{K}' \cap \mathsf{L}'$; then there exist $(N, v) \in \mathsf{K}$, $(N', v') \in \mathsf{L}$ such that $(N, v) \equiv (M, w) \equiv (N', v')$. By Theorem 4.6 (N, v) and (N', v') have bisimilar ultrapowers $\prod_U (N, v)$ and $\prod_U (N', v')$. As K, L are closed under **B** and **Po**, this implies $\prod_U (N, v) \in \mathsf{K} \cap \mathsf{L}$ — a contradiction.

To complete the proof, let $T = \bigcap\{tp_{(M,w)}(w) \mid (M, w) \in \mathsf{K}'\}$. Then T defines K'. As $\mathsf{K} \subseteq \mathsf{K}'$ and $\mathsf{K}' \cap \mathsf{L} = \emptyset$, we are done. ⊣

Corollary 5.4 (Interpolation) *If* K, L *are both definable by single temporal formulas* ϕ *and* ψ, *respectively, and* $\mathsf{K} \cap \mathsf{L} = \emptyset$, *then there is a class* M *that is definable by a temporal formula in the common language of* ϕ *and* ψ *with* $\mathsf{K} \subseteq \mathsf{M}$ *and* $\mathsf{M} \cap \mathsf{L} = \emptyset$.

To obtain a characterization of the first-order formulas that are equivalent to a temporal formula, we use the following notion. A first-order formula $\alpha(x)$ in $\mathcal{L}_1(x)$ is *invariant for bisimulations* iff for any two pointed models (M, w) and (N, v), any two states $w' \in M$ and $v' \in N$, and any bisimulation Z such that $w'Zv'$, we have that $M \models \alpha[w']$ iff $N \models \alpha[v']$.

Corollary 5.5 *Let $\alpha(x)$ be an $\mathcal{L}_1(x)$-formula. Then the following are equivalent.*

1. *$\alpha(x)$ is invariant for bisimulations*
2. *$\alpha(x)$ is equivalent to the standard translation of a temporal formula.*

Proof. The implication from 2 to 1 is Lemma 3.10. For the converse implication, let $\alpha(x)$ be invariant for bisimulations. Let K be the class of (pointed) models of $\alpha(x)$. Then K and $\overline{\mathsf{K}}$ (being defined by $\neg\alpha(x)$) are closed under ultraproducts. As $\alpha(x)$ is invariant for bisimulations, both K and $\overline{\mathsf{K}}$ must also be closed under bisimulations. Hence, by Theorem 5.2, K must be definable by a single temporal formula ϕ. This means that $\alpha(x)$ is (equivalent to) the standard translation of ϕ. ⊣

6 Concluding Remarks

In this paper we have introduced a notion of bisimulation for temporal logic with Since and Until that allows one to develop the basic model theory for temporal logic. We established a preservation result that characterizes the first-order formulas that correspond to temporal formulas with Since and Until, thereby answering Open Problem 4.4 from Van Benthem, Van Eijck and Stebletsova [4]. In addition we proved definability and interpolation results.

A lot remains to be done. First of all, we believe that our notion of bisimulation may be a useful tool in obtaining further results in the model theory of Since and Until. In particular, Hans Kamp's famous result of the expressive completeness of Since and Until over dedekind-complete linear order is an important one, for which multiple proofs should be available. One of the most recent proofs, due to Ian Hodkinson [15] uses games that seem to be quite close to our notion of bisimulation; it therefore seems feasible to try and prove Kamp's theorem using our bisimulations.

Next, we think that our general methodology of involving more complex patterns of states in the definition of bisimulation for Since and Until also indicates the way to go when attempting to define suitable bisimulations for other complex modal operators whose truth definition involves both universal and existential quantification. In particular, our ideas seem applicable to the *minimality operator* min whose semantics is given by

$$w \models \min(\phi) \text{ iff } \exists y\,(w < y \wedge y \models \phi \wedge \forall z\,(w < z < y \rightarrow z \not\models \phi)).$$

Obviously the min-operator is definable using Since and Until, and as a result we have that states that are bisimilar in our sense agree on formulas involving the min-operator — but what about a notion of bisimulation that exactly characterizes the fragment involving min in the sense of Corollary 5.5? Further examples along these lines could include the temporal operators found in Manna and Pnueli [18]. But more exotic modal operators might also be analyzed using our strategy. Take, for example, the binary *interpretability* operator $\rhd$ whose truth definition is based on a binary relation R and a ternary relation S as follows:

$$w \models \phi \rhd \psi \text{ iff } \exists y\,(Rxy \wedge y \models \phi \wedge \forall z\,(Sxyz \rightarrow z \models \psi)).$$

See Berarducci [5] for further details on this operator.

In our comparisons in this paper we focused on equivalence relations between models that were defined by fairly simple first-order conditions. De Nicola and Vaandrager [8] study so-called *branching bisimulations* whose definition involves non first-order definable concepts like 'finitely many silent steps'; they show that on certain transition systems branching bisimulations and several temporal logics induce the same equivalence relations. The exact connection hasn't been determined, though, and to obtain a precise description of the connections one needs other tools than the ones we have used in this paper as these are essentially first-order.

Finally, in this paper we have given the first notion of bisimulation that allowed for an exact characterization theorem in the sense of Corollary 5.5 of modal operators whose truth definition is not of the simple $\exists \cdots \exists \alpha$ or $\forall \cdots \forall \alpha$ format (for α quantifier-free). Do our ideas of introducing bisimulations that link states to states and sequences to sequences generalize to the extent that we can handle any first-order definable modal operator, no matter how complex its truth definition is? Recent work by Andréka, van Benthem and Németi [1] and by Hollenberg [17] is relevant here.

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