A Note on Graded Modal Logic

Maarten de Rijke

Dept. of Computer Science, University of Warwick
Coventry CV4 7AL, England
mdr@dcs.warwick.ac.uk

Abstract

We introduce a notion of bisimulation for graded modal logic. Using these bisimulations the model theory of graded modal logic can be developed in a uniform manner. We illustrate this by establishing the finite model property, and proving invariance and definability results.

1 Introduction

The language of graded modal logic (GML) has modal operators $\Diamond_i$ (for $i \in \mathbb{N}$) that can count the number of successors of a given state: a state w in a model (W, R, V) satisfies $\Diamond_i\phi$ iff there exist at least n R-related states that satisfy ϕ. Originally introduced in the early 1970s [9, 10], the language has enjoyed an increased interest during the past few years, especially because of its considerable expressive power. Formal logical and algebraic results on axiomatizability, decidability, and expressive completeness over bounded trees have been reported in a number of papers [2, 3, 5, 7, 8, 12, 20], and the language has shown up in various guises in knowledge representation, generalized quantifier theory, algebraic logic, and fuzzy reasoning [6, 13, 14, 17, 18].

This note is concerned with graded modal logic as a description language for reasoning about models. It is part of a larger enterprise to study the model theory — and in particular, the expressive power — of restricted description languages such as modal and temporal languages, terminological logics and feature logics (cf. [1, 15, 16, 19]). Bisimulations have proved to be a very powerful tool in this area, but so far a version of bisimulation that is appropriate for graded modal logic has not been proposed. As a consequence, the model theory of graded modal logic is not as well developed as the model theory of, say, standard modal or temporal logic. In this note we propose a notion of bisimulation, called g-bisimulation that 'fits' GML exactly in the sense that a first-order formula is invariant under g-bisimulations iff it is equivalent to a graded modal formula (cf. Theorem 4.3 below).

The remainder of this note is organized as follows. The next section introduces the main notions needed. In Section 3 g-bisimulations are defined. In Section 4 we first give a quick and intuitive proof for the finite model property of GML using g-bisimulations, and then prove the above invariance theorem, as well as two results on definability. Section 5 contains some concluding comments.

2 Basic Definitions

Graded modal formulas are built up using propositional variables p, q, ..., the constants $\top$ and $\bot$, boolean connectives $\neg$, $\wedge$, and the unary temporal operators $\Diamond_i$ and $\Box_i$. We use $\mathcal{L}_{GML}$ to denote this language.

A *model* is a triple $M = (W, R, V)$, where W is a non-empty set of states, R is a binary relation on W, and V is a valuation, that is: a function assigning a subset of W to every proposition letter.

The *satisfaction relation* is defined in the familiar way for the atomic and boolean cases, while for the modal operators we put

$$M, w \models \Diamond_i \phi \text{ iff}$$
$$\exists v_1 \dots v_i \left(\bigwedge_{1 \leq j \neq k \leq i} (v_j \neq v_k) \wedge \bigwedge_{1 \leq j \leq i} Rwv_j \wedge \bigwedge_{1 \leq j \leq i} M, v_j \models \phi \right)$$

and $M, w \models \Box_i \phi$ iff $M, w \models \neg \Diamond_i \neg \phi$.

The *graded modal type* of a state is simply the set of all graded modal formulas satisfied by the state: $tp(w) = \{\phi \mid w \models \phi\}$; if necessary we record the model M in which w lives as a subscript: $tp_M(w)$. Two states w, v are *graded modally equivalent* if $tp(w) = tp(v)$ (notation: $w \equiv_g v$). If X is a set of states, we write $X \models \phi$ to denote that for all $x \in X$, $x \models \phi$.

Let $\mathcal{L}_1$ be the first-order language with unary predicate symbols corresponding to the proposition letters in $\mathcal{L}_{GML}$, and with one binary relation symbol R.

Models can be viewed as $\mathcal{L}_1$-structures in the usual first-order sense. The *standard translation* takes graded modal formulas ϕ into equivalent formulas $ST_x(\phi)$ in $\mathcal{L}_1$. It maps proposition letters p onto unary predicate symbols Px, it commutes with the booleans, and the modal cases are given by

$$ST_x(\Diamond_i \phi) \;=\; \exists y_1 \dots y_i \left(\bigwedge_{1 \leq j \neq k \leq i} (y_j \neq y_k) \wedge \bigwedge_{1 \leq j \leq i} (Rxy_j \wedge ST_{y_j}(\phi)) \right),$$

and similarly for the box operators $\Box_i$. For all models M and states w we have $M, w \models \phi$ iff $M \models ST_x(\phi)[w]$, where the latter denotes first-order satisfaction of $ST_x(\phi)$ under the assignment of w to the free variable of $ST_x(\phi)$.

3 G-bisimulations

In this section we introduce the main notion of this note: g-bisimulations. In [19] bisimulations are advocated as the central tool in the model theory of modal logic; see [15, 16] for case studies implementing this strategy for Since, Until logic, and for negation-free modal logics. In Section 4 below we will use g-bisimulations to establish the finite model property, and to prove invariance and definability results for graded modal logic, thus showing that g-bisimulations can play a similar central role in the model theory of graded modal logic.

By way of introduction we first consider bisimulations.

Definition 3.1 Let $M_1 = (W_1, R_1, V_1)$, $M_2 = (W_2, R_2, V_2)$ be two models. A *bisimulation* between M_1 and M_2 is a relation $Z \subseteq (W_1 \times W_2)$ of relations satisfying the following requirements:

1. Z is non-empty;
2. if xZy, then $x \models p$ iff $y \models p$, for all proposition letters p;
3. if xZy and R_1xx', then there exists $y' \in W_2$ with R_2yy' and $x'Zy'$;
4. if xZy and R_2yy', then there exists $x' \in W_1$ with R_1xx' and $x'Zy'$.

We write $Z : M_1, x \leftrightarroweq M_2, y$ to denote that Z is a bisimulation with xZy.

Graded modal formulas are not preserved under bisimulations. To see this, consider the following two models M_1 and M_2, where $M_1 = (\{0,1,2\}, \{(0,1),\ (0,2)\},\ V_1)$, $M_2 = (\{3,4\},\ \{(3,4)\},\ V_2)$, and V_1 and V_2 verify all proposition letters true in all states; see Figure 1). The relation indicated by the dotted line in Figure 1 is a bisimulation between M_1 and M_2. But $0 \not\equiv_g 3$, as $0 \models \Diamond_2\top$, while $3 \not\models \Diamond_2\top$.

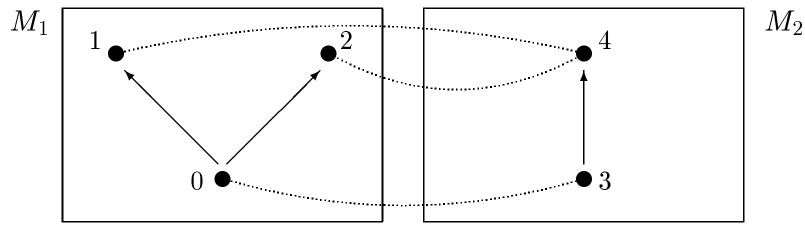

Figure 1: Bisimilar but not equivalent.

To define a truth-preserving notion of bisimulation for graded modal logic, we need the following definitions. If X is a set, we write $\mathcal{P}^{<\omega}(X)$ to denote the collection of all finite subsets of X, and $|X|$ to denote its cardinality. Also, we write $R^\bullet xX$ to denote that for all $x' \in X$, Rxx' holds.

Definition 3.2 Let $M_1 = (W_1, R_1, V_1)$, $M_2 = (W_2, R_2, V_2)$ be two models. A *g-bisimulation* between M_1 and M_2 is a tuple $Z = (Z_1, Z_2, \ldots)$ of relations satisfying the following requirements:

1. Z_1 is non-empty;
2. for all i, $Z_i \subseteq \mathcal{P}^{<\omega}(W_1) \times \mathcal{P}^{<\omega}(W_2)$;
3. if XZ_iY, then $|X| = |Y| = i$;
4. if $\{x\}Z_1\{y\}$, then $x \models p$ iff $y \models p$, for all proposition letters p;
5. if $\{x\}Z_1\{y\}$ and $R_1^\bullet xX$, where $|X| = i \geq 1$, then there exists $Y \in \mathcal{P}^{<\omega}(W_2)$ with $R_2^\bullet yY$ and XZ_iY;
6. if $\{x\}Z_1\{y\}$ and $R_2^\bullet yY$, where $|Y| = i \geq 1$, then there exists $X \in \mathcal{P}^{<\omega}(W_1)$ with $R_1^\bullet xX$ and XZ_iY;
7. if XZ_iY, then
   (a) for all $x \in X$ there exists $y \in Y$ with $\{x\}Z_1\{y\}$, and
   (b) for all $y \in Y$ there exists $x \in X$ with $\{x\}Z_1\{y\}$.

We write $Z : M_1, x \leftrightarroweq_g M_2, y$ to denote that Z is a g-bisimulation with $\{x\}Z_1\{y\}$.

To grasp the intuition behind Definition 3.2, reconsider the definition of a (normal) bisimulation. There, bisimilar states satisfy the same (standard) modal formulas in $\Diamond$, $\Box$ because they satisfy the same proposition letters (Definition 3.1, item 2), and because the relevant relational patterns present in the one model are mirrored in the other model (Definition 3.1, items 3 and 4). To guarantee that g-bisimilar states satisfy the same graded modal formulas, one requires, firstly, that they satisfy the same proposition letters (Definition 3.2, item 4). Next, to preserve formulas of the form $\Diamond\phi$, sets of successors of size i present in the one model should be mirrored in the other (Definition 3.2, items 5 and 6). If two such sets 'mirror' each other, and all the states in the one set agree on a formula, then all the states in the other should do so as well (Definition 3.2, items 7(a), (b)).

Proposition 3.3 *Let M_1, M_2 be two models, and let Z be a bisimulation between M_1 and M_2 with $Z : w_1 \leftrightarroweq_g w_2$. Then, $w_1 \equiv_g w_2$.*

Proof. The proof is by induction on formulas. The atomic and boolean cases are trivial. For the modal case, assume that $w_1 \models \Diamond_i\phi$. Then there exists $X_1 \in \mathcal{P}^{<\omega}(W_1)$ with $R_1^\bullet w_1 X_1$, $|X_1| = i$ and $X_1 \models \phi$. By Definition 3.2, items 5 and 3, there exists $X_2 \in \mathcal{P}^{<\omega}(W_2)$ with $X_1 Z_i X_2$, $R_2^\bullet w_2 X_2$, and $|X_2| = i$. We're done once we've shown that $X_2 \models \phi$, for then $w_2 \models \Diamond_i\phi$.

To this end, pick any $v_2 \in X_2$. By Definition 3.2, item 7(b), there exists $v_1 \in X_1$ with $\{v_1\}Z_1\{v_2\}$. As $X_1 \models \phi$, we get $v_1 \models \phi$, and by the inductive hypothesis this implies $v_2 \models \phi$. $\dashv$

As a corollary, the models M_1 and M_2 depicted in Figure 1 are not g-bisimilar.

By restricting the definition of g-bisimulation to just a *finite* tuple $(Z_1, \ldots, Z_k)$ we arrive at the notion of *g_k-bisimulation*; we write $M_1, w \leftrightarroweq_{g_k} M_2, v$ to denote that there is a g_k-bisimulation between w and v. This notion of bisimulation is appropriate for the fragment $\mathcal{L}_{GML}$ in which all modal operators $\Diamond_i$ and $\Box_i$ have subscripts $i \leq k$. In particular, for $k = 1$ we get a notion that is equivalent to the standard notion of bisimulation defined in Definition 3.1.

Another restriction, which does not limit the length of the tuple $(Z_1, \ldots)$, but rather the number of times the clauses in Definition 3.2 can be applied starting from a given pair of points.

Definition 3.4 Let $M_1 = (W_1, R_1, V_1)$, $M_2 = (W_2, R_2, V_2)$ be two models, and let m be a natural number. A *g-bisimulation up to m* between M_1 and M_2 is a sequence of tuples of relations $Z^0 = (Z^0_1, Z^0_2, \ldots)$, $Z^1 = (Z^1_1, Z^1_2, \ldots)$, $\ldots$, $Z^m = (Z^m_1, Z^m_2, \ldots)$ satisfying the following requirements:

1. Z^0_1 is non-empty;
2. $Z^m_i \subseteq \cdots \subseteq Z^0_i \subseteq \mathcal{P}^{<\omega}(W_1) \times \mathcal{P}^{<\omega}(W_2)$;
3. if XZ^j_iY, then $|X| = |Y| = i$ $(j \leq m)$;
4. if $\{x\}Z^0_1\{y\}$, then $x \models p$ iff $y \models p$, for all proposition letters p;
5. if $\{x\}Z^{j+1}_1\{y\}$, where $j + 1 \leq m$, and $R^\bullet_1 xX$, where $|X| = i \geq 1$, then there exists $Y \in \mathcal{P}^{<\omega}(W_2)$ with $R^\bullet_2 yY$ and XZ^j_iY;
6. similar to item 5;
7. like item 7 of Definition 3.2, but with Z^j_i and Z^j_1 instead of Z_i and Z_1 $(j \leq m)$.

The notion of a *g_k-bisimulation up to m* is defined similarly.

We write $M_1, x \leftrightarroweq^m_g M_2, y$ to denote that there is a g-bisimulation up to m between M_1 and N_2, say $Z^0, \ldots, Z^m$, such that $\{x\}Z^0_1\{y\}$. The notation $\leftrightarroweq^m_{g_k}$ has the obvious meaning.

Let $M = (W, R, V)$ be a model, and assume $w \in W$. For each $n \in \mathbb{N}$ we define the *n-hull* $H_n(w)$ around w in M as follows. The 0-hull H_0 is simply $\{w\}$; the $(n+1)$-hull is the set $H_{n+1} := \{u \mid \exists v \in H_n\ (Rvu)\}$.

We write M_w to denote the submodel of M that is generated by w. That is, M_w is the submodel of M whose domain is $\bigcup_n H_n(w)$. Clearly, for any model M and state w in M, $M \leftrightarrow_g M_w$.

If M is generated by w, we define the *restriction of M to depth m*, notation: $M \restriction m$, to be the submodel of M whose domain is the set $\bigcup_{0 \leq j \leq m} H_j(w)$.

Proposition 3.5 *Let M be generated by a w. Then $M, w \leftrightarrow_g^m (M \restriction m), w$.*

The *degree* of a graded modal formula is simply the largest number of nested modal operators occurring in it. The *index* of a formula is the highest natural number i such that the modal operator $\Diamond_i$ occurs in the formula.

Proposition 3.6 *Let M_1, M_2 be two models, and let w, v be states in M_1 and M_2, respectively. If $M_1, w \leftrightarrow_g^m M_2, v$, then w and v verify the same graded modal formulas of degree at most m.*

4 Results

In this section we first give a new and intuitive proof of the finite model property for graded modal logic using g-bisimulations. We then use g-bisimulations to prove the main results of this note: invariance and definability.

§4.1. Finite Model Property. The finite model property for graded modal logic was first established in [11]; see also [3, 12]. The proof presented below is attractive because it clearly brings out the two obvious reasons why $\mathcal{L}_{GML}$ has the finite model property; to determine the truth of falsehood of a graded modal formula only R-paths $wR \cdots Rv$ of finite length are needed, and every state on such a path only needs finitely many successors.

Let's get to work. Fix a satisfiable formula ϕ with degree m and index k. Let M and w be such that $M, w \models \phi$. We will construct a finite submodel of M that is still a model for ϕ. First, we may assume that $M = M_w$. Consider $M \restriction m$; it only has finite R-paths, and $(M \restriction w), w \models \phi$. Now $(M \restriction m)$ need not be finite, as it may be infinitely branching.

Consider the sublanguage $\mathcal{L}_{GML}(\phi)$ in which all formulas are built up using only proposition letters that occur in ϕ. It is easily verified that there are only finitely many non-equivalent formulas in $\mathcal{L}_{GML}(\phi)$ with degree at most m and index at most k.

Our final model $(M \restriction m)^{\leq k}$ is defined as follows. Its domain is the union of certain subsets H_0', ..., H_m' of the domain of $(M \restriction m)$. Here $H_0' = \{w\}$, and to define H_{j+1}' $(j + 1 \leq m)$ do the following:

set $H'_{j+1} = \emptyset$
for all $x \in H'_j$
for each of the finitely many non-equivalent $\mathcal{L}_{GML}(\phi)$-formulas ψ
select as many as possible (but at most k) R-successors y of x
with $y \models \psi$
add these states to H'_{j+1}
end.

The relation and valuation of $(M \restriction m)^{\leq k}$ are simply the restrictions to the domain of $(M \restriction m)^{\leq k}$. Clearly, $(M \restriction m)^{\leq k}$ is finite, and $(M \restriction m)^{\leq k} \leftrightarrows^m_{g_k} (M \restriction m)$.

Putting things together, we arrive at the following result:

Theorem 4.1 *$\mathcal{L}_{GML}$ has the finite model property.*

§4.2. Invariance. We need the following notion. A model M is ω-*saturated model* (in the sense of first-order logic) if whenever Δ is a set of formulas in $\mathcal{L}'_1$, where $\mathcal{L}'_1$ extends $\mathcal{L}_1$ by the addition of fewer than ω new individual constants, and Δ is finitely satisfiable in an $\mathcal{L}'_1$-expansion of M, then Δ is satisfiable in this expansion.

Lemma 4.2 *Let M_1, M_2 be two ω-saturated models, and let $w_1 \in W_1$ and $w_2 \in W_2$. Then $tp(w_1) = tp(w_2)$ iff $w_1 \leftrightarrows_g w_2$.*

Proof. The right-to-left implication is Proposition 3.3. For the left to right implication, assume that $w_1 \equiv_g w_2$, and define a series of relations $Z = (Z_1, \ldots)$ between the finite subsets of W_1 and W_2 by putting (for $i \geq 1$):

$$\begin{aligned} X_1 Z_i X_2 \quad \text{iff} \quad & |X_1| = |X_2| = i \text{ and} \\ & \forall x_1 \in X_1 \exists x_2 \in X_2\, tp(x_1) = tp(x_2) \text{ and} \\ & \forall x_2 \in X_2 \exists x_1 \in X_1\, tp(x_1) = tp(x_2). \end{aligned}$$

Let us check that this defines a graded bisimulation between w_1 and w_2. First, as $tp(w_1) = tp(w_2)$, Z_1 is non-empty. Conditions 2, 3, and 4 from Definition 3.2 are trivially fulfilled.

As to condition 5, assume $w_1 Z_1 w_2$ and $R_1^{\bullet} w_1 X_1$, where $|X_1| = i$. We need to find a finite set $X_2 \subseteq W_2$ with $R_2^{\bullet} w_2 X_2$ and $X_1 Z_i X_2$. Assume that $X_1 = \{v_{11}, \ldots, v_{1i}\}$. Consider the types generated by the states in X_1; clearly, some of them may coincide. Let $tp_1, \ldots, tp_n$ $(n \leq i)$ be a minimal collection of types such that every tp_j coincides with one of $tp(v_{11}), \ldots, tp(v_{1i})$, and such that for every v_{1j} there exists a tp_k with $tp_k = tp(v_{1j})$. Next, we need to record, for each type $tp_1, \ldots, tp_n$, by how many states in X_1 it is generated; for $j = 1, \ldots, n$ let

$$m_j = |\{v_{1k} \mid tp_j = tp(v_{1k}),\ 1 \leq k \leq i\}|.$$

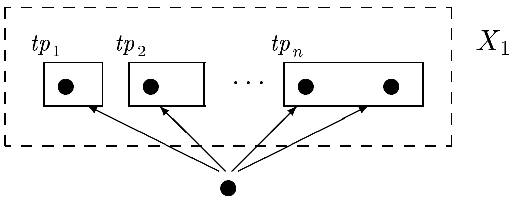

Figure 2: The successor types of w_1.

Then $\sum_{j=1}^{n} = i$; see Figure 2.

Consider the following collection of formulas:

$$(1)\quad \bigcup_{1\le j\le n}\Big(\{Rxy^j_k \mid 1\le k\le m_j\} \ \cup\ \{(y^j_k \neq y^j_l) \mid 1\le k\neq l\le m_j\} \ \cup\ \{ST_{y^j_k}(\phi) \mid \phi\in tp_j,\ 1\le k\le m_j\}\Big).$$

We want to satisfy the set of formulas (1) at w_2 in M_2. If we succeed in doing so, then, for each type tp_j we have forced the existence of m_j successors of w_2 satisfying tp_j. Putting these successors together gives us a set X_2 of size i, as required. (To see this, observe first that for each tp_j we will have m_j states at which it is realized; and, second, that no state can realize two different types, as types are maximal.) Moreover, it is obvious that for each state x_1 in X_1, there will be a state $x_2 \in X_2$ with $\{x_1\}Z_1\{x_2\}$, and conversely. Thus $X_1 Z_i X_2$, and we have established condition 5.

Let us see why all of (1) is satisfiable at w_2. Since M_2 is ω-saturated, it suffices to show that (1) is *finitely* satisfiable at w_2. Assume for the sake of contradiction that this is not the case. Then there exist finite sets $\Phi_1 \subseteq tp_1$, $\ldots$, $\Phi_n \subseteq tp_n$ such that

$$(2)\qquad M_2 \models \neg \bigwedge_{1\le j\le n} \exists y^j_1 \ldots y^j_{m_j}\left(\bigwedge_{1\le k\le m_j} Rxy^j_k \wedge \bigwedge_{1\le k\neq l\le m_j}(y^j_k \neq y^j_l) \wedge \bigwedge_{1\le k\le m_j} ST_{y^j_k}(\Phi_j)\right)[w_2]$$

is not satisfiable. Now (2) is equivalent to

$$M_2, w_2 \models \neg\left(\Diamond_{m_1}\left(\bigwedge \Phi_1\right) \wedge \ldots \wedge \Diamond_{m_n}\left(\bigwedge \Phi_n\right)\right).$$

But as

$$M_1, w_1 \models \Diamond_{m_1}\left(\bigwedge \Phi_1\right) \wedge \ldots \wedge \Diamond_{m_n}\left(\bigwedge \Phi_n\right),$$

this contradicts $\{w_1\}Z_1\{w_2\}$. Hence, (1) is finitely satisfiable in w_2, as required.

Finally, condition 6 is proved analogously to condition 5, and condition 7 is immediate from the definition of Z. $\dashv$

An $\mathcal{L}_1$-formula $\alpha(x)$ is *invariant under g-bisimulations* if for all models M_1 and M_2, all states w_1 in M_1 and w_2 in M_2, and g-bisimulations $Z = (Z_1, \ldots)$ between M_1 and M_2, $\{w_1\}Z_1\{w_2\}$ implies that $M_1 \models \alpha[w_1]$ iff $M_2 \models \alpha[w_2]$.

Theorem 4.3 (Invariance) *Assume that $\mathcal{L}_1$ is countable. An $\mathcal{L}_1$-formula is (equivalent to the translation of) a graded modal formula iff it is invariant under g-bisimulations.*

Proof. The right-to-left implication is simply Proposition 3.3. For the other direction, assume that $\alpha(x)$ is preserved under directed simulations. By a simple compactness argument it suffices to show that

(3) $\quad \textit{GML-Cons}(\alpha) := \{ST_x(\phi) \mid \alpha \models ST_x(\phi) \text{ and } \phi \in \mathcal{L}_{GML}\} \models \alpha.$

To prove (3), assume that $M \models \textit{GML-Cons}(\alpha)[w]$; we have to show that $M \models \alpha[w]$.

The proof of the following claim is left to the reader:

Claim 1. *The set $\{\alpha(x)\} \cup \{ST_x(\phi) \mid \phi \in tp(w)\}$ is satisfiable.*

Using Claim 1, we find a model N and state v with $N \models \alpha[v]$ and $N, v \models tp(w)$. The following is immediate:

Claim 2. $tp_N(v) = tp_M(w)$.

Now, to conclude the proof we want to 'lift' α from N, v to M, w. To do so, take two ω-saturated elementary extensions N^+, v and M^+, w of N, v and M, w, respectively (cf. [4, Theorem 6.1]. Then $tp_{M^+}(w) = tp_{N^+}(v)$, and so by Lemma 4.2 we get that $M^+, w \mathrel{\underline{\leftrightarrow}}_g N^+, v$. A walk around the following diagram completes the proof:

$$\begin{array}{ccc}
tp_M(w) & = & tp_N(v) \\
\Big| & & \Big| \\
M, w & & N, v \\
{\scriptstyle\preceq}\Big| & & \Big|{\scriptstyle\preceq} \\
M^+, w & \mathrel{\underline{\leftrightarrow}}_g & N^+, v.
\end{array}$$

That is, $N \models \alpha[v]$ implies $N^+ \models \alpha[v]$ by elementary extension. As $N^+, v \mathrel{\underline{\leftrightarrow}}_g M^+, w$ it follows that $M^+ \models \alpha[w]$, and hence $M \models \alpha[w]$, as required. $\dashv$

§4.3. **Definability.** To simplify the presentation, we will work with *pointed models*; these are structures of the form (M, w), where w is a state in M, called the *distinguished point* of (M, w). We will assume that g-bisimulations between two pointed models link the singletons containing their distinguished points.

Let K be a class of pointed models. Then K is *definable* by a set of graded modal formulas if there exists a set of formulas Δ such that $\mathsf{K} = \{(M, w) \mid (M, w) \models \Delta\}$; K is *definable by a single formula* if it is definable by means of a singleton set; $\overline{\mathsf{K}}$ denotes the class of pointed models outside K.

K is *closed under ultraproducts* (*ultrapowers*) if every ultraproduct (ultrapower) of models in K is itself in K; K is *closed under g-bisimulations* if every model g-bisimilar to a model in K is in K.

Theorem 4.4 (Definability 1) *Assume that $\mathcal{L}_{GML}$ is countable, and let K be a class of pointed models. Then*

1. *K is definable by a set of graded modal formulas iff K is closed under g-bisimulations and ultraproducts, while $\overline{\mathsf{K}}$ is closed under ultrapowers.*
2. *K is definable by a single graded modal formula iff K is closed under g-bisimulations and ultraproducts, while $\overline{\mathsf{K}}$ is closed under ultraproducts.*

Proof. 1. The *only if* direction is easy. For the converse, we can 'bisimulate' familiar arguments from first-order model theory. Assume K is closed under ultraproducts and g-bisimulations, while $\overline{\mathsf{K}}$ is closed under ultrapowers. Let $\Delta = \bigcap\{tp_{(M,w)}(w) \mid (M, w) \in \mathsf{K}\}$.

We will show that Δ defines K. First, $\mathsf{K} \models \Delta$. Second, assume that $(M, w) \models \Delta$; we need to show $(M, w) \in \mathsf{K}$. Consider $tp_{(M,w)}(w)$, and define $I = \{\Sigma \subseteq tp_{(M,w)}(w) \mid |\Sigma| < \omega\}$. For each $i = \{\sigma_1, \ldots, \sigma_n\} \in I$ there is a model (N_i, v_i) of i in K. By standard model-theoretic arguments there exists an ultraproduct $(N, v) = \prod_U (N_i, v_i)$ such that $tp_{(N,v)}(v) = tp_{(M,w)}(w)$. As K is closed under ultraproducts $(N, v) \in \mathsf{K}$.

Now, let U' be a countably incomplete ultrafilter, and consider the ultrapowers

$$(N^*, v^*) := \prod_{U'} (N, v) \quad \text{and} \quad (M^*, w^*) := \prod_{U'} (M, w).$$

Both (N^*, v^*) and (M^*, w^*) are ω-saturated (cf. [4, Theorem 6.1]), and $tp(w^*) = tp(v^*)$. Hence, by Lemma 4.2, $(N^*, v^*) \leftrightarroweq_g (M^*, w^*)$. By closure under ultraproducts $(N^*, v^*) \in \mathsf{K}$, and by closure under g-bisimulations $(M^*, w^*) \in \mathsf{K}$. Since $\overline{\mathsf{K}}$ is closed under ultrapowers, we get $(M, w) \in \mathsf{K}$, as required.

2. Again, the *only if* direction is easy. Assume K, $\overline{\mathsf{K}}$ satisfy the stated conditions. Then both are closed under ultrapowers, hence, by item 1, there are sets of graded modal formulas Δ_1, Δ_2 defining K and $\overline{\mathsf{K}}$, respectively.

Obviously, $\Delta_1 \cup \Delta_2 \models \bot$, so by compactness for some $\phi_1, \ldots, \phi_n \in \Delta_1$, $\psi_1, \ldots, \psi_m \in \Delta_2$, we have $\bigwedge_i \phi_i \models \bigvee_j \neg\psi_j$. Then K is defined by $\bigwedge_i \phi_i$. $\dashv$

To conclude this section we present an alternative and more manageable characterization of the properties definable in graded modal logic.

Theorem 4.5 (Definability 2) *Assume that $\mathcal{L}_{GML}$ contains only finitely many proposition letters, and let K be a class of pointed models. Then K is definable by a single graded modal formula iff, for some k, $m \in \mathbb{N}$, K is closed under g_k-bisimulations up to m.*

Proof. Clearly, if K is negation-free definable by a single formula of degree m and index k, then it is closed under g_k-bisimulations up to m. To prove the converse, let $(M_1, w_1) \in \mathsf{K}$, and define $\phi^{k,m}_{M_1,w_1}$ to be the conjunction of all formulas in $tp_M(w)$ of index at most k and degree at most m — as we are working in a finite language, we can assume that there are only finitely many non-equivalent graded modal formulas of index at k and degree at most m, hence we may assume $\phi^{k,m}_{M_1,w_1}$ to be a (finitary) formula in $\mathcal{L}_{GML}$.

Using the finite character of the language again, we find that there are only finitely many non-equivalent formulas $\phi^{k,m}_{M,w}$ for $(M, w) \in \mathsf{K}$. Let $\Phi^{k,m}$ be their disjunction. Then $\Phi^{k,m}$ defines K. For, assume that $(M_1, w_1) \models \Phi^{k,m}$; we need to show that $(M_1, w_1) \in \mathsf{K}$. First, from $(M_1, w_1) \models \Phi^{k,m}$ it follows that for some $(M_2, w_2) \in \mathsf{K}$, (M_1, w_1) agrees with (M_2, w_2) on all graded modal formulas of index at most k and degree at most m. Second, the latter fact implies that $M_1, w_1 \leftrightarrows^m_{g_k} M_2, w_2$. To see this, define tuples of relations $Z^0 = (Z^0_1, \ldots, Z^0_k), \ldots, Z^m = (Z^m_1, \ldots, Z^m_k)$ by

- $\{x\}Z^j_1\{y\}$, for $j = 0, \ldots, m$, iff x and y satisfy the same graded modal formulas of index at most k and degree at most j; and
- XZ^j_iY, for $i = 2, \ldots, k$ and $j = 0, \ldots, m-1$, iff $|X| = |Y| = i$ and $\forall x \in X \exists y \in Y\ \{x\}Z^j_1\{y\}$ and $\forall y \in Y \exists x \in X\ \{x\}Z^j_1\{y\}$.

Then $Z^0, \ldots, Z^m$ is a g_k-bisimulation up to m that links M_1, w_1 to M_2, w_2. As $(M_2, w_2) \in \mathsf{K}$ and K is closed under g_k-bisimulations up to m, this implies $(M_1, w_1) \in \mathsf{K}$, and we are done. $\dashv$

5 Conclusion

In this note g-bisimulations were introduced as a tool for exploring the model theory of graded modal logic. Their usefulness was demonstrated by their use in obtaining both known results (the finite model property) and new ones (invariance and definability).

Now that a working notion of bisimulation is available for graded modal logic, it may be used to obtain further results on the model (and frame) theory of graded modal logic. Obvious questions to be answered next include the following: Can g-bisimulations be used to prove a Goldblatt-Thomason style result about the classes of frames definable in $\mathcal{L}_{GML}$? What is the appropriate kind of Ehrenfeucht-Fraïssé style games needed to prove analogs of the results in this note for the class of *finite* models? Fragments of $\mathcal{L}_{GML}$ have been used in terminological reasoning [6]; can these fragments be characterized by adapting the notion of g-bisimulation?

Acknowledgment. This work was partially supported by the Research and Teaching Innovation Fund at the University of Warwick.

References

[1] J. van Benthem. *Exploring Logical Dynamics.* Studies in Logic, Language and Information, CSLI Publications, Stanford, 1996.

[2] C. Cerrato. General Canonical Models for Graded Normal Logics (Graded Modalities IV). *Studia Logica*, 49:241–252, 1990.

[3] C. Cerrato. Decidability by Filtrations for Graded Normal Logics (Graded Modalities V). *Studia Logica*, 53:61–74, 1994.

[4] C.C. Chang and H.J. Keisler. *Model Theory.* North-Holland, Amsterdam, 1973.

[5] F. De Caro. Graded Modalities II. *Studia Logica*, 47:1–10, 1988.

[6] F.M. Donini, M. Lenzerini, D. Nardi, and A. Schaerf. Reasoning in Description Logics. In G. Brewka, editor, *Principles of Knowledge Representation*. Studies in Logic, Language and Information. CSLI Publications, Stanford, 1996.

[7] M. Fattorosi-Barnaba and F. De Caro. Graded Modalities I. *Studia Logica*, 44:197–221, 1985.

[8] M. Fattorosi-Barnaba and C. Cerrato. Graded Modalities III. *Studia Logica*, 47:99–110, 1988.

[9] K. Fine. In So Many Possible Worlds. *Notre Dame Journal of Formal Logic*, 13:516–520, 1972.

[10] L.F. Goble. Grades of Modality. *Logique et Analyse*, 13:323–334, 1970.

[11] W. van der Hoek. *Modalities for Reasoning about Knowledge and Quantities.* Phd thesis, Free University of Amsterdam, 1992.

[12] W. van der Hoek. On the Semantics of Graded Modalities. *Journal of Applied Non-Classical Logic*, 2:81–123, 1992.

[13] W. van der Hoek and M. de Rijke. Generalized Quantifiers and Modal Logic. *Journal of Logic, Language and Information*, 2:19–57, 1993.

[14] W. van der Hoek and M. de Rijke. Counting Objects. *Journal of Logic and Computation*, 5:325–345, 1995.

[15] N. Kurtonina and M. de Rijke. Bisimulations for Temporal Logic. *Journal of Logic, Language and Information*, to appear

[16] N. Kurtonina and M. de Rijke. Simulating without Negation. *Journal of Logic and Computation*, to appear.

[17] M. Marx. *Algebraic Relativization and Arrow Logic.* PhD thesis, ILLC, University of Amsterdam , 1995.

[18] A. Nakamura. On a Logic Based on Graded Modalities. *IECE Trans. Inf. & Syst.*, E76-D(5):527–532, 1992.

[19] M. de Rijke. Modal Model Theory. *Annals of Pure and Applied Logic*, to appear.

[20] H. Schlingloff. Expressive Completeness of Temporal Logic of Trees. *Journal of Applied Non-Classical Logics*, 2:157–180, 1992.