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Abstract

In certain areas of logic dealing with information there is a need to make statements not only about cognitive states, but also about transitions between such states. In this paper a modal logic designed with this purpose in mind is analyzed. On top of an abstract informational ordering on states it has instructions to move forward or backward along this ordering, to states where a certain assertion holds or fails, while also allowing the combination of such instructions by means of the standard operations from relational algebra. In addition, the logic has devices for expressing whether or not in a given state a certain instruction can be carried out, and whether or not that state can be arrived at by carrying out a certain instruction.

This paper deals mainly with technical aspects of this dynamic modal language. It gives a precise description of the expressive power of this language. It also contains results on decidability for the language with 'arbitrary' structures and for the special case with a restricted class of admissible structures. In addition a complete axiomatization is given of the validities over arbitrary structures as well as of the validities over this restricted class of structures. The paper concludes with a remark about modal algebras appropriate for the language studied here, plus some speculations and suggestions for further work.

Although the paper contains some minor examples showing how the logic can be used to capture situations of dynamic interest, far more detailed applications are given in a companion to this paper.

Key words: dynamics of information, modal logic, axiomatic completeness, expressiveness, bisimulations, decidability.

1 Introduction

Over the last 10 or 15 years logicians have paid more and more attention to dynamic aspects of reasoning. Motivated by examples taken from such diverse disciplines as the

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semantics of natural language, linguistic analysis of discourse, the philosophy of science, a multitude of logical systems have been proposed, each of them equipped with the predicate "dynamic". In Van Benthem [3] a general perspective on dynamic matters is put forward, and a somewhat informal description is given of a dynamic modal language designed for reasoning about the processes of expanding and contracting with a given piece of information. This language is not meant as just another device for reasoning about information and its dynamics, but rather as a more general framework in which other proposals can be described and compared.

Quite a number of such descriptions and comparisons have been given in Van Benthem [2, 3]. They include an analysis within our logic of various dynamic styles of inference, and it is also shown there how a number of dynamic connectives that have been proposed in the literature can be formulated in our logic. Furthermore, in a companion to the present paper I show, among other things, how several sets of postulates governing contractions that have been proposed in the literature, fare in our language (cf. De Rijke [27], or §3 below for a short sketch).

The dynamic modal language of this paper derives its generality from the following observation. What most dynamic proposals have in common is a notion of *states* and a notion of *transitions* between those states; of course, what these states and transitions are may differ from one particular proposal to another. Now, our modal language has the right means to deal with these two notions. To be precise, it is a *two sorted* language in that it not only has the usual Boolean part, i.e. propositions with the usual connectives to talk about states, but also a relational part containing procedures that may be combined using the relation algebra operations $\cap$, ;, $-$, and $\check{}$ to talk about transitions. In addition these two realms are connected via *modes* and *projections* as depicted in Figure 1.

$$\begin{array}{c} \text{propositions} \\ (\mathsf{BA}) \end{array} \underset{\text{projections}}{\overset{\text{modes}}{\rightleftarrows}} \begin{array}{c} \text{procedures} \\ (\mathsf{RA}) \end{array}$$

Figure 1: Propositions and procedures.

The choice of the projections and modes may depend on the particular application one has in mind. Here, I will choose a very basic, and rather natural set of modes and projections, one that is suggested in [3]. Let me motivate this choice somewhat. The underlying idea is that we have some abstract informational structure about whose static aspects we reason using the Boolean component of the language, while the procedural part is to be used to reason about its dynamic aspects. The minimal requirements such structures are usually supposed to satisfy in the literature are those of a pre-order. Of course, pre-orders have a long tradition as informational structures, viz. their use as models for intuitionistic logic. The elements of our informational structures may intuitively be thought of as the cognitive states an agent passes through searching for knowledge.

Following Katsuno and Mendelzon [19] I use the term *update* to refer to any change to an agent's cognitive state.[1] As an agent's perception of the world as it is encoded by his current cognitive state changes, his cognitive state will be updated. Several kinds of updates may be distinguished. If we simply acquire additional knowledge about the world, and the new knowledge does not conflict with the current beliefs, we can *expand* with the additional information. A different change occurs when a sentence previously believed becomes questionable and has to be given up; such an operation is called *contraction*. The basic updates an agent is able to perform in my set up are expansion with a formula φ (moving along the informational ordering to state where φ holds) and contraction with a formula (moving backwards along the ordering to a state where φ no longer holds). In addition there are *tests* to see whether a given formula holds. Using these three 'basic procedures' (expansions, contractions and tests), additional procedures may be defined using the standard operations from relational algebra. Some examples of compound procedures will be given in §3 below.

Going from procedures to propositions I will consider projections that return, given a procedure α as input, its *domain*, *range* and *fix points*. Given our interest in dynamic matters they are a natural choice, expressing, for instance, whether or not in a given state a certain transition is at all possible.

So much for an introduction. The main purpose of this paper is to study the above modal language and its logical properties in precise and formal detail. After some initial definitions in §2, I give some quick examples of the use of the language in §3. Then, in §4, the expressive power of our modal language is studied; a precise syntactic description is given of its first-order counterparts, as well as a semantic characterization by means of an appropriate kind of bisimulations. In §5 I establish (un-)decidability results for satisfiability in this language, both when interpreted on arbitrary structures and when interpreted on a restricted class of structures only. In §6 complete axiomatizations are given for validity on arbitrary structures, and, again, for validity over this special frame class. Some quick remarks about the kind of modal algebras appropriate for the language studied here are made in §7, and, finally, §8 contains some concluding remarks, speculations, and suggestions for further work.

[1] A technical distinction between update and revision is sometimes made in the literature (cf. Katsuno and Mendelzon [18]); here I will ignore that distinction.

2 Some definitions

Although I will use a slightly different version of the language, here's the version that appears in [3]:

Atomic formulas:	$p \in \Phi$,
Formulas:	$\varphi \in Form(\Phi)$,
Procedures:	$\alpha \in Proc(\Phi)$.

$$\varphi ::= p \mid \bot \mid \top \mid \varphi_1 \to \varphi_2 \mid \mathsf{do}(\alpha) \mid \mathsf{ra}(\alpha) \mid \mathsf{fix}(\alpha),$$
$$\alpha ::= \mathsf{exp}(\varphi) \mid \mathsf{con}(\varphi) \mid \mu\text{-}\mathsf{exp}(\varphi) \mid \mu\text{-}\mathsf{con}(\varphi) \mid \alpha_1 \cap \alpha_2 \mid \alpha_1 ; \alpha_2 \mid -\alpha \mid \alpha^{\smile} \mid \varphi?.$$

I will refer to elements of $Form(\Phi) \cup Proc(\Phi)$ as *expressions*.

The intended interpretation of the above connectives and mappings is the following. A formula $\mathsf{do}(\alpha)$ ($\mathsf{ra}(\alpha)$) is true at a state x iff x is in the domain (range) of α, and $\mathsf{fix}(\alpha)$ is true at x if x is a fixed point of α. The interpretation of $\mathsf{exp}(\varphi)$ (read: expand with φ) in a model $\mathfrak{M}$ is the set of all moves along the "informational ordering" in $\mathfrak{M}$ that take you to a state where φ holds; the interpretation of $\mathsf{con}(\varphi)$ (read: contract with φ) consists of all moves *backwards* along the ordering to states where φ *fails*. The modes μ-exp and μ-con are *minimal* versions of exp and con, respectively: the interpretation of μ-$\mathsf{exp}(\varphi)$ consists of all moves (x, y) along the ordering such that y satisfies φ, while there is no point in between x and y that also satisfies φ; the interpretation of μ-con is defined likewise. As usual, $\varphi?$ is the "test-for-φ" relation, while the intended interpretation of the operators left unexplained should be clear.

The models for this language are structures of the form $\mathfrak{M} = (W, \sqsubseteq, [\![\cdot]\!], V)$, where $\sqsubseteq \subseteq W^2$ is transitive and reflexive (the informational ordering), $[\![\cdot]\!] : Proc(\Phi) \to 2^{W \times W}$, and $V : \Phi \to 2^W$. The interpretation of the projections in our modal language is the following:

$$\begin{array}{lll} \mathfrak{M}, x \models \mathsf{do}(\alpha) & \text{iff} & \exists y\, ((x, y) \in [\![\alpha]\!]), \\ \mathfrak{M}, x \models \mathsf{ra}(\alpha) & \text{iff} & \exists y\, ((y, x) \in [\![\alpha]\!]), \\ \mathfrak{M}, x \models \mathsf{fix}(\alpha) & \text{iff} & (x, x) \in [\![\alpha]\!]. \end{array}$$

A model $\mathfrak{M}$ is called *standard* if the relational part of our language is interpreted as follows in $\mathfrak{M}$:

$$
\begin{aligned}
[\![\mathsf{exp}(\varphi)]\!] &= \lambda xy.\,(x \sqsubseteq y \wedge \mathfrak{M}, y \models \varphi),\\
[\![\mathsf{con}(\varphi)]\!] &= \lambda xy.\,(x \sqsupseteq y \wedge \mathfrak{M}, y \not\models \varphi),\\
[\![\mu\text{-}\mathsf{exp}(\varphi)]\!] &= \lambda xy.\,(x \sqsubseteq y \wedge \mathfrak{M}, y \models \varphi \wedge \neg\exists z\,(x \sqsubseteq z \sqsubset y \wedge \mathfrak{M}, z \models \varphi)),\\
[\![\mu\text{-}\mathsf{con}(\varphi)]\!] &= \lambda xy.\,(x \sqsupseteq y \wedge \mathfrak{M}, y \not\models \varphi \wedge \neg\exists z\,(x \sqsupseteq z \sqsupset y \wedge \mathfrak{M}, z \not\models \varphi)),\\
[\![\alpha \cap \beta]\!] &= [\![\alpha]\!] \cap [\![\beta]\!],\\
[\![\alpha ; \beta]\!] &= [\![\alpha]\!] ; [\![\beta]\!],\\
[\![-\alpha]\!] &= -[\![\alpha]\!],\\
[\![\alpha^{\smallsmile}]\!] &= \{\, (x,y) : (y,x) \in [\![\alpha]\!] \,\},\\
[\![\varphi?]\!] &= \{\, (x,x) : \mathfrak{M}, x \models \varphi \,\}.
\end{aligned}
$$

Note that the minimal procedures μ-$\mathsf{exp}(\cdot)$ and μ-$\mathsf{con}(\cdot)$ are definable using $\mathsf{exp}(\cdot)$ (or $\mathsf{con}(\cdot)$) and $\cap, -$ and ; as follows:

$$(x,y) \in [\![\mu\text{-}\mathsf{exp}(\varphi)]\!] \text{ iff } (x,y) \in [\![\mathsf{exp}(\varphi) \cap -(\mathsf{exp}(\varphi); (\mathsf{exp}(\top) \cap -(\top?)))]\!],$$

and similarly for μ-$\mathsf{con}(\varphi)$. Consequently, I will leave out the 'minimal' versions of $\mathsf{exp}(\cdot)$ and $\mathsf{con}(\cdot)$ from the official definition of the language.

Of course, ra and fix are definable using the other operators, however, for conceptual and notational convenience they will be part of the official definition of the language.

As a second change to the original definition I will add a modal operator to the language. But before I do this, note that if α is some procedure, then the modal operator $\langle\alpha\rangle$ whose semantics is based on α, may be defined as $\langle\alpha\rangle\varphi := \mathsf{do}(\alpha; \varphi?)$. The additional operator I will add below will be defined in this way. Thus, this addition is only a cosmetic one that will facilitate direct reference at the Boolean level to a procedure that is available at the relational level anyway. Note that we can define the *diagonal* relation $\delta = \{\, (x,y) : x = y \,\}$ as $\delta := \top?$. Hence the inequality relation is definable as $-\delta$. The operator I want to add to the language is the so-called $\mathbf{D}$ operator, whose semantics is given by $\lambda P.\, \exists y\,(x \neq y \wedge P(y))$ (cf. De Rijke [25]). Using the $\mathbf{D}$ operator some other useful operators may be defined as well: $\mathbf{A}\varphi := \varphi \wedge \neg\mathbf{D}\neg\varphi$ (φ holds at *all* points), and $\mathbf{E}\varphi := \varphi \vee \mathbf{D}\varphi$ (there *exists* a point at which φ holds). The main reason for adding the $\mathbf{D}$ operator is that it will greatly simplify my completeness proofs in §6.

Here, finally, is the full language; I will refer to this language as the *DML*-language, and in more official parts of this paper also as *DML*(Φ), where Φ is the set of proposition letters.

$$
\begin{aligned}
\varphi &::= p \mid \bot \mid \top \mid \varphi_1 \to \varphi_2 \mid \mathsf{do}(\alpha) \mid \mathsf{ra}(\alpha) \mid \mathsf{fix}(\alpha) \mid \mathbf{D}\varphi,\\
\alpha &::= \mathsf{exp}(\varphi) \mid \mathsf{con}(\varphi) \mid \alpha_1 \cap \alpha_2 \mid \alpha_1 ; \alpha_2 \mid -\alpha \mid \alpha^{\smallsmile} \mid \varphi?.
\end{aligned}
$$

(Here, $p \in \Phi$, $\varphi \in \mathit{Form}(\Phi)$ and $\alpha \in \mathit{Proc}(\Phi)$, as before.)

There are obvious connections between *DML* and *propositional dynamic logic* (*PDL*, cf. Harel [16]). For a start, as pointed out above, the 'old diamonds' $\langle\alpha\rangle$ from *PDL* can be simulated in *DML* by putting $\langle\alpha\rangle\varphi := \mathsf{do}(\alpha;\varphi?)$. And conversely, the expansion and contraction operators are definable in a particular mutation of *PDL* where taking converses of program relations is allowed and a name for the informational ordering is available: $[\![\mathsf{exp}(\varphi)]\!] = [\![\sqsubseteq;\varphi?]\!]$ and $[\![\mathsf{con}(\varphi)]\!] = [\![\sqsubseteq^{\smallsmile};\neg\varphi?]\!]$. The domain operator $\mathsf{do}(\alpha)$ can be simulated in standard *PDL* by $\diamond\top$. An obvious difference between *DML* and *PDL* is that (at least in it's more traditional mutations) *PDL* only has the *regular* program operations $\cup, ;$ and *, while *DML* has the full relational repertoire $\cup, -, ^{\smallsmile}$ and ;, but not the Kleene star. Another difference is not a technical difference, but one in emphasis; whereas in *PDL* the Boolean part of the language clearly is the primary component of the language, in *DML* some effort is made to give the relational part the status of a first-class citizen as well by shifting the notation towards one that more clearly reflects the aspects of relations which we usually consider to be important.

A related formalism whose relational apparatus is more alike that of *DML* is the *Boolean modal logic* (*BML*) studied by Gargov and Passy [12]. This system has atomic relations $\rho_1, \rho_2, \ldots$, a constant for the cartesian product $W \times W$ of the underlying domain W, and relation-forming operators $\cap, \cup$ and $-$. Relations are referred to within the *BML*-language by means of the *PDL*-like diamonds $\langle\alpha\rangle$. Since *BML* does *not* allow either ; or $^{\smallsmile}$ as operators on relations, it is a strict subsystem of $DML(\Phi)$ with multiple base relations $\{\rho_1, \rho_2, \ldots\}$.

3 Some examples

It is high time for an example or two. Here's a simple-minded one. Suppose you're sitting in a room, waiting for the start of a talk by a famous logician who is known for his lively presentation, and who has done a lot of work on non-monotonic logic. So, after some time the lights are dimmed and logician comes in (l). You can see that he's carrying a birds cage with a bird in it, although you can not see what kind of bird it is. Having read the relevant literature you conclude that the bird must be a penguin (p) called Tweety (t). However, the first thing the speaker says, while holding up the cage and pointing at the bird in it, is: "This bird is not called Tweety". In that case, you think, it's probably not a penguin either. The speaker continues: "I want to do a little experiment with you. I want you to think of a name for this bird; any name will do, as long as it's not Tweety". Being a cooperative member of the audience you think of a name other than Tweety, say Bob (b) ... Some of the changes brought about in your initial informational state during this story may schematically be represented as

$$\mathsf{exp}(l); (\mathsf{exp}(p) \cap \mathsf{exp}(t)); \mathsf{con}(t); \mathsf{con}(p); (\mathsf{exp}(b) \cap -\mathsf{exp}(t)),$$

where ; is the usual relational composition.

Here's another, more serious example having to do with Theory Change or Belief Revision. Assuming the so-called Levi identity (cf. Gärdenfors [11]), *revisions*, that is, operations to somehow resolve conflicts that arise when new knowledge is acquired that is inconsistent with the old beliefs, such revisions are usually explained as "contract with $\neg\varphi$, and then expand with φ, while changing *as little as possible* from the old theory." In [27] this idea is implemented by defining a revision operator $[*\varphi]\psi$ ("ψ belongs to every theory that results from revising by φ") as

$$[*\varphi]\psi := \mu\text{-}\mathsf{con}(\neg\varphi); \mu\text{-}\mathsf{exp}(\varphi).$$

It can be shown that for many sets of postulates that have been proposed for theory change, this revision operator satisfies nearly all of the individual postulates in such a set.

Further formalisms to which *DML* has been linked include conditionals and other systems that somehow involve a notion of change. But, whereas the applications to Theory Change and conditionals do not require the states in *DML*-models to have any particular structure, others do. For example, one version of Frank Veltman's update semantics [28] may be seen as a formalism for reasoning about models of the modal system $S5$ and certain transitions between such models. By imposing the structure of $S5$-models on the individual states in a *DML*-model, the latter becomes a class of $S5$-models in which the *DML*-apparatus can be used to reason about *global* transitions between $S5$-models, while the language of $S5$ can be used to reason about the *local* structure of the states. When used in this way *DML* becomes a super-system of one particular brand of Veltman's update semantics. Other brands of update semantics can be interpreted as being formalisms for reasoning about certain bi-modal, or even poly-modal models; following the strategy sketched above these formalisms too can be can be interpreted in a version of *DML* with appropriately structured states. Further applications given in [27] in which the states need to be equipped with some kind of structure include discourse representation and minimization.

Many of the dynamic operators that have been proposed in the literature can be defined in *DML*. The underlying reason for this is that most dynamic proposals have some kind of two-dimensional structures in common as their underlying models, and that the *DML*-language is strong enough to define all the standard operations on binary relations, and many more besides. For instance, the *residuals* of Vaughan Pratt's action logic [22] can be defined in the *DML*-language:

$$\begin{array}{rclcl}
\alpha \Rightarrow \beta & = & \{\,(x,y) : \forall z\,((z,x) \in [\![\alpha]\!] \rightarrow (z,y) \in [\![\beta]\!])\,\} & = & -(\alpha^{\vee}; -\beta), \\
\alpha \Leftarrow \beta & = & \{\,(x,y) : \forall z\,((y,z) \in [\![\alpha]\!] \rightarrow (x,z) \in [\![\beta]\!])\,\} & = & -(-\beta; \alpha^{\vee}).
\end{array}$$

As pointed out in [3] the *test negation* proposed in Groenendijk and Stokhof [13] becomes

$$\sim\alpha \quad = \quad \{\, (x,x) : \neg\exists y\,((x,y) \in [\![\alpha]\!]\,\} \quad = \quad \delta \cap -(\alpha; \top?).$$

A logical system can be dubbed dynamic for a number of reasons: because it has dynamic connectives of some sort (as in the above examples), or because it has a dynamic notion of inference. Quite often these too can be simulated in the *DML*-language. Here are some examples taken from [3]. The standard notion of inference $\models_1$ ("every state that models all of the premises, should also model the conclusion") may be represented as

$$\varphi_1 \wedge \ldots \wedge \varphi_n \models_1 \psi \quad \text{iff} \quad \mathsf{fix}(\varphi_1?) \cap \ldots \cap \mathsf{fix}(\varphi_n?) \to \mathsf{fix}(\psi?).$$

A more dynamic notion $\models_2$ taken from [13], which may be paraphrased as "process all premises consecutively, then you should be able to reach a state where the conclusion holds", has the following transcription in the *DML*-language:

$$\varphi_1 \wedge \ldots \wedge \varphi_n \models_2 \psi \quad \text{iff} \quad \mathsf{ra}\Big(\mathsf{exp}(\varphi_1); \ldots; \mathsf{exp}(\varphi_n)\Big) \to \mathsf{do}(\mathsf{exp}(\psi)).$$

A third notion of inference, $\models_3$, found for example in Van Eijck and de Vries [7] which reads "whenever it is possible to consecutively expand with all premises, then it should be possible to expand with the conclusion", can be given the following representation:

$$\varphi_1 \wedge \ldots \wedge \varphi_n \models_3 \psi \quad \text{iff} \quad \mathsf{do}\Big(\mathsf{exp}(\varphi_1); \ldots; \mathsf{exp}(\varphi_n)\Big) \to \mathsf{do}(\mathsf{exp}(\psi)).$$

4 The connection with classical logic

When interpreted on models ordinary modal formulas are equivalent to a special kind of first order formulas. To be precise, these first order translations form a restricted 2-variable fragment of the full first order language, one that can easily be described syntactically, and for which a semantic characterization can be given in terms of so-called p-relations or bisimulations (cf. Van Benthem [1] for details). Likewise, the first order transcriptions of modal formalisms used to reason about relation algebras live in a 3-variable fragment of the full first order language; they too can be given precise syntactic and semantic descriptions (cf. De Rijke [26]).

Of course, the above two are special cases of a much more general phenomenon, namely the relation between patterns or important features of structures and bisimulations that precisely preserve these patterns on the one hand, and (extended) modal formulas whose validity is invariant under such bisimulations on the other hand (again, cf. [26]). In the present case of the *DML*-language it is also possible to give a precise syntactic description of its first order transcriptions (this will be done in §4.1), and the notion of bisimulation can be adapted to obtain a semantic characterization of these first order transcriptions (in §4.2).

4.1 Translation into first order logic

The usual translation $(\cdot)^*$ taking modal formulas to first order ones (over a vocabulary $\{R, P_1, P_2, \ldots\}$) can be extended to the full *DML*-language without too much trouble (cf. [1] for the standard modal case). However, whereas standard modal formulas translate into formulas having one free variable in a two-variable fragment, expressions in the *DML*-language translate into formulas of a three-variable fragment that may contain up to two free variables.

My approach will be a bit more general than the one suggested by the truth definition given in §2; instead of $\sqsubseteq$ I will use an abstract binary relation symbol R to translate the modal operators and the 'dynamic' constructs.

Definition 4.1 Let τ be the (first order) vocabulary $\{R, P_1, P_2, \ldots\}$, with R a binary relation symbol, and the P_i's unary relation symbols. Let $L(\tau)$ be the set of all first order formulas over τ (with identity). Define a translation $(\cdot)^*$ taking *DML*-formulas to formulas in $L(\tau)$ as in Table 1.

$$\begin{array}{rcl@{\qquad}rcl}
(\top)^* & = & (x = x) & (p)^* & = & P(x) \\
(\neg\varphi)^* & = & \neg\varphi^* & (\varphi \wedge \psi)^* & = & \varphi^* \wedge \psi^* \\
(\mathbf{D}\varphi)^* & = & \exists y\,(x \neq y \wedge [y/x]\varphi^*) & (\mathsf{do}(\alpha))^* & = & \exists y\,(\alpha^*) \\
(\mathsf{ra}(\alpha))^* & = & \exists y\,[y/x, x/y](\alpha^*) & (\mathsf{fix}(\alpha))^* & = & [x/y](\alpha^*) \\
\hline\hline
\delta & = & (x = y) & (-\alpha)^* & = & \neg(\alpha^*) \\
(\alpha \cap \beta)^* & = & \alpha^* \wedge \beta^* & (\alpha;\beta)^* & = & \exists z\,([z/y]\alpha^* \wedge [z/x]\beta^*) \\
(\alpha^{\smile})^* & = & [y/x, x/y]\alpha^* & (\varphi?)^* & = & (x = y) \wedge \varphi^* \\
(\mathsf{exp}(\varphi))^* & = & (xRy) \wedge [y/x]\varphi^* & (\mathsf{con}(\varphi))^* & = & (yRx) \wedge \neg[y/x]\varphi^*
\end{array}$$

Table 1: The standard translation.

Proposition 4.2 *Let θ be an expression in DML(Φ). Then, for any $\mathfrak{A}$, and for any $x, y \in A$, we have $\mathfrak{A}, x \models \theta$ iff $\mathfrak{A} \models \theta^*[x]$, if $\theta \in Form(\Phi)$, and $(x, y) \in [\![\theta]\!]_{\mathfrak{A}}$ iff $\mathfrak{A} \models \theta^*[x, y]$, in case $\theta \in Proc(\Phi)$.*

The $(\cdot)^*$-translations of *DML*-formulas can be described exactly using the following definition.

Definition 4.3 Fix individual variables x_1, x_2, x_3 as before, and let $\tau = \{R, P_1, P_2, \ldots\}$ be as before. Let x, y range over $\{x_1, x_2\}$, with the understanding that $x \not\equiv y$. The set of first order formulas (with identity) $L_3^{1,2}(\tau)$ is the smallest set X such that

1. $x_1 = x_1, P_i x_1 \in X$;
2. if $\varphi(x_1), \psi(x_1) \in X$, then so are their conjunction, disjunction, and negations;
3. $Rx_1x_2, (x_1 = x_2) \in X$;
4. if $\varphi(x,y), \psi(x,y) \in X$, then so are their conjunction, disjunction, and negations;
5. if $\varphi(x,y) \in X$, then so is $\varphi(y,x)$;
6. if $\varphi(x,y) \in X$, then so is $\exists x_2\, \varphi(x,y)$;
7. if $\varphi(x,y), \psi(x,y) \in X$, then so is $\exists x_3\, (\varphi(x,x_3) \wedge \psi(x_3,y))$;
8. if $\varphi(x,y), \psi(x_1) \in X$, then so is $\varphi(x,y) \wedge \psi(x_2)$.

Proposition 4.4 *Every expression in the DML-language translates into a formula in $L_3^{1,2}(\tau)$ via $(\cdot)^*$. And conversely, for every $\varphi \in L_3^{1,2}(\tau)$ there is an expression $\theta \in DML(\Phi)$ such that $\models \theta^* \leftrightarrow \varphi$.*

Proof. One may use an inductive argument to see that every expression in $DML(\Phi)$ translates into a formula in $L_3^{1,2}(\tau)$ via the mapping $(\cdot)^*$. For the converse, define a mapping $(\cdot)^\dagger : L_3^{1,2}(\tau) \to DML(\Phi)$ as follows:

$$
\begin{array}{rcl@{\qquad}rcl}
(x_1 = x_1)^\dagger & = & \top & (Px_1)^\dagger & = & p \\
(\varphi(x_1) \wedge \psi(x_1))^\dagger & = & \varphi(x_1)^\dagger \wedge \psi(x_1)^\dagger & (\varphi(x_1) \vee \psi(x_1))^\dagger & = & \varphi(x_1)^\dagger \vee \psi(x_1)^\dagger \\
(\neg\varphi(x_1))^\dagger & = & \neg\varphi(x_1)^\dagger & (Rx_1x_2)^\dagger & = & \mathsf{exp}(\top) \\
(Rx_2x_1)^\dagger & = & \mathsf{con}(\bot) & (x_1 = x_2)^\dagger & = & \top? \\
(\varphi(x,y) \wedge \psi(x,y))^\dagger & = & \varphi(x,y)^\dagger \cap \psi(x,y)^\dagger & (\neg\varphi(x,y))^\dagger & = & -\varphi(x,y)^\dagger \\
(\varphi(x,y) \vee \psi(x,y))^\dagger & = & \varphi(x,y)^\dagger \cup \psi(x,y)^\dagger & (\exists x_2\, \varphi(x_1,x_2))^\dagger & = & \mathsf{do}(\varphi(x_1,x_2)^\dagger) \\
(\exists x_2\, \varphi(x_2,x_1))^\dagger & = & \mathsf{ra}(\varphi(x_1,x_2)^\dagger) & & & \\
(\exists x_3\, (\varphi(x,x_3) \wedge \psi(x_3,y)))^\dagger & = & \varphi(x,y)^\dagger ; \psi(x,y)^\dagger & & & \\
(\varphi(x,y) \wedge \psi(x_2))^\dagger & = & \varphi^\dagger ; (\top? \cap (\psi(x_1)^\dagger ?)). & & &
\end{array}
$$

Then, for all $\varphi \in L_3^{1,2}(\tau)$, and all $\mathfrak{M}, \vec{x}$, we have $\mathfrak{M}, \vec{x} \models \varphi$ iff $\mathfrak{M}, \vec{x} \models \varphi^\dagger$. $\dashv$

Let X be a set of (first order) formulas, and let K be a class of models. Then the *DML*-language is called *expressively complete with respect to X over* K if for all $\chi \in X$ there is a *DML*-expression φ such that $\mathsf{K} \models \varphi^* \leftrightarrow \chi$. If K is the class of all models I will suppress 'over K'.

The *two-variable fragment* $L_2(\tau)$ is the set of all first order formulas over τ using only two variables.

Corollary 4.5 *The DML-language is expressively complete with respect to the two-variable fragment $L_2(\tau)$ of first order logic over $\{ R, P_1, \ldots \}$ with identity.*

Proof. This is immediate from 4.4: since the two-variable fragment $L_2(\tau)$ over $\{ R, P_1, \dots \}$ (with identity) is contained in $L_3^{1,2}(\tau)$, it follows that $DML(\Phi)$ is expressively complete for that fragment.

Alternatively, one can give an explicit algorithm for transforming $L_2(\tau)$ in *DML*-expressions. Since this would take up too much space here without yielding additional insights, I will only give an illustrative example. Consider

$$x \neq y \wedge \forall x \exists y \, (xRy \wedge \forall x \, (yRx \rightarrow \neg Px)) \wedge (\neg Qx \rightarrow \forall y \, (Qy \rightarrow \neg Ryx)).$$

Abbreviating $\mathsf{exp}(\top)$ by R, this formula may be given the following modal transcription:

$$-\delta \cap \Big[\mathbf{A}\,\mathsf{do}\Big(R; \neg\mathsf{do}(R^{\smile}; p?)?\Big)?; \delta \cup -\delta\Big] \cap -\Big[\neg q? \cap \mathsf{do}(R^{\smile}; q?)?; \delta \cup -\delta\Big]. \quad \dashv$$

What about expressive completeness of the *DML*-language with respect to the full first order language? It may amuse the reader to check that the temporal operator *UNTIL*, whose truth definition is

$$\mathfrak{M}, x \models UNTIL(p,q) \text{ iff } \exists y \, (xRy \wedge Py \wedge \neg\exists z \, (xRzRy \wedge z \neq y \wedge \neg Qz)),$$

can be defined by

$$\mathsf{do}(\mathsf{exp}(p) \cap -[\mathsf{exp}(\neg q); (R \cap -\delta)]).$$

(And similarly for *SINCE*, the backward-looking version of *UNTIL*.) Hence, by Kamp's Theorem (cf. Kamp [17]), the *DML*-language is expressively complete with respect to the full first order language over continuous linear orders.

An obvious question here is whether the Stavi connectives *SINCE'* and *UNTIL'* are definable in the *DML*-language, and, thus, by a result of J. Stavi, whether the *DML*-language is expressively complete with respect to the language of first order logic over *all* linear orders (cf. Gabbay [9]). Here, $UNTIL'(p,q)$ is defined by

$$\exists y \, \Big(xRy \wedge \forall z \, (xRzRy \rightarrow Qz)\Big) \wedge \tag{1}$$

$$\forall y \, \Big(xRy \wedge \forall z \, (xRzRy \rightarrow Qz) \rightarrow \Big(Qy \wedge \exists x \, (yRx \wedge \forall z \, (yRzRx \rightarrow Qz))\Big)\Big) \wedge \tag{2}$$

$$\exists y \, \Big(xRy \wedge \neg Qy \wedge Py \wedge \forall z \, (xRzRy \wedge \exists y \, (xRyRz \wedge \neg Qy) \rightarrow Pz)\Big). \tag{3}$$

Of course $SINCE'(p,q)$ is the 'backward-looking' version of $UNTIL'(p,q)$. In the *DML*-language the operator $UNTIL'(p,q)$ can be defined as follows:

$$\mathsf{do}\Big(R \cap -[\mathsf{exp}(\neg q); R]\Big) \wedge \tag{4}$$

$$\neg\mathsf{do}\Big(R \cap -[\mathsf{exp}(\neg q); R] \cap -\Big[\mathsf{exp}(q) \cap \mathsf{do}\Big(R \cap -[\mathsf{exp}(\neg q); R]\Big)?\Big]\Big) \wedge \tag{5}$$

$$\mathsf{do}\Big(\mathsf{exp}(\neg q \wedge p) \cap -\Big[\Big(\mathsf{exp}(\neg p) \cap (\mathsf{exp}(\neg q); R)\Big); R\Big]\Big). \tag{6}$$

I leave it as an exercise to check that (1), (2) and (3) are defined by the *DML*-formulas (4), (5) and (6), respectively.

4.2 Bisimulations

I will now characterize $L_3^{1,2}(\tau)$, and hence, by 4.4, the *DML*-language, semantically. The key notion here will be an appropriate kind of bisimulations, generalizing the so-called *p-relations* of [1, Theorem 3.9] and [25, Theorem 4.7].

A *2-partial isomorphism* f from $\mathfrak{M}$ to $\mathfrak{N}$ is simply an isomorphism $f : \mathfrak{M}_0 \cong \mathfrak{N}_0$, where $\mathfrak{M}_0$, $\mathfrak{N}_0$ are substructures of $\mathfrak{M}$ and $\mathfrak{N}$, respectively, whose domains have cardinality at most 2. A set I of 2-partial isomorphisms from $\mathfrak{M}$ into $\mathfrak{N}$ has the *back and forth property* if

> for every $f \in I$ with $|f| \leq 1$, and every $x \in \mathfrak{M}$ (or $y \in \mathfrak{N}$) there is a $g \in I$ with $f \subseteq g$ and $x \in \mathrm{domain}(g)$ (or $y \in \mathrm{range}(g)$).

I write $I : \mathfrak{M} \cong_2 \mathfrak{N}$ if I is a non-empty set of 2-partial isomorphisms and I has the back and forth property.

By 4.5 the full 2-variable fragment of $L(\tau)$ is contained in the *DML*-language. Hence any relation between models that is to preserve truth of *DML*-formulas should 'act' like a (partial) isomorphism on sequences of length at most 2. Indeed, modulo one additional requirement the latter completely characterizes the *DML*-language (cf. 4.11).

Definition 4.6 A *bisimulation* between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is a relation $\mathcal{B} \subseteq (W_1 \times W_2) \cup (W_1^2 \times W_2^2)$ such that

1. $\mathcal{B} \neq \emptyset$,
2. $\vec{x}\mathcal{B}\vec{y}$ implies $\mathrm{lh}(\vec{x}) = \mathrm{lh}(\vec{y})$, where $\mathrm{lh}(\vec{x})$ is the length of $\vec{x}$,
3. if $x_1x_2\mathcal{B}y_1y_2$ then $x_1\mathcal{B}y_1$ and $x_2\mathcal{B}y_2$,
4. if $x_1x_2\mathcal{B}y_1y_2$ and $x_3 \in \mathfrak{M}_1$ then there is a $y_3 \in \mathfrak{M}_2$ such that $x_1x_3\mathcal{B}y_1y_3$ and $x_3x_2\mathcal{B}y_3y_2$, and similarly in the opposite direction,
5. for $I = \{\emptyset\} \cup \{(\vec{x}, \vec{y}) : \vec{x}\mathcal{B}\vec{y}\}$ we have $I : \mathfrak{M}_1 \cong_2 \mathfrak{M}_2$.

Example 4.7 The conditions in Definition 4.6 are rather strong, as is witnessed, for instance, by the fact that two finite linear models are isomorphic iff they are bisimular. in the sense of 4.6. The truth of this claim may be seen as follows: any two finite linear models that have the same first order theory are isomorphic, and on linear models the two notions of first order equivalence and of being equivalent for all *DMl*-formulas coincide, by my remarks in §4.1; furthermore, by Proposition 4.10 two models that are bisimular verify the same *DML*-formulas.

However, on the class of all finite models bisimilarity and isomorphism do not coincide. Here are two models establishing this claim:

$$\mathfrak{M}: \boxed{\bullet_1 \quad \bullet_2 \quad \bullet_3 \quad \bullet_4} \qquad\qquad \mathfrak{M}': \boxed{\bullet_{1'} \quad \bullet_{2'} \quad \bullet_{3'}},$$

where for *all* points (in $\mathfrak{M}$ and $\mathfrak{M}'$) have the same valuation. Define $\mathcal{B} \subseteq (W \times W') \cup (W^2 \times W'^2)$ by putting

$$\begin{aligned}
\mathcal{B} \;=\; & \{(i,i') : 1 \leq i = i' \leq 3\} \cup \{(4,i') : 1 \leq i' \leq 3\} \\
& \cup \{(ij, i'j') : 1 \leq i = i' \leq 3, 1 \leq j = j' \leq 3\} \\
& \cup \{(i4, j'k'), (4i, j'k') : 1 \leq i \leq 3, 1 \leq j' \neq k' \leq 3\} \\
& \cup \{(44, i'i') : 1 \leq i' \leq 3\}.
\end{aligned}$$

The reader may verify that this is indeed a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$; hence, bisimulations and isomorphisms do not coincide on all finite models.

Example 4.8 Given two finite models $\mathfrak{M}_1, \mathfrak{M}_2$ with $\vec{x} \in \mathfrak{M}_1$, $\vec{y} \in \mathfrak{M}_2$ such that for all $\varphi \in L(\tau)$, $\mathfrak{M}_1, \vec{x} \models \varphi$ iff $\mathfrak{M}_2, \vec{y} \models \varphi$, one may define a 'canonical' bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ that connects $\vec{x}$ and $\vec{y}$, by putting

$$\vec{u}\mathcal{B}\vec{v} \text{ iff for all } \varphi \in DML(\Phi), \mathfrak{M}_1, \vec{u} \models \varphi \text{ iff } \mathfrak{M}_2, \vec{v} \models \varphi.$$

(That this does indeed define a bisimulation is essentially because $\mathfrak{M}_1$ and $\mathfrak{M}_2$, being finite, are saturated, cf. the proof of 4.11.) It follows that two finite *DML*-models are bisimular iff they satisfy the same formulas.

Proposition 4.9 *Let $\mathcal{B}$ be a bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$. Then*

1. domain($\mathcal{B}$) = $\mathfrak{M}$, range($\mathcal{B}$) = $\mathfrak{N}$
2. *if $x \in \mathfrak{M}$, $y \in \mathfrak{N}$, $x\mathcal{B}y$ and $x' \in \mathfrak{M}$, then there is a $y' \in \mathfrak{N}$ with $xx'\mathcal{B}yy'$, and similarly in the opposite direction.*

An $L(\tau)$-formula $\varphi(\vec{x})$ is *invariant for bisimulations* if for all models $\mathfrak{M}_1, \mathfrak{M}_2$ and all bisimulations $\mathcal{B}$ between $\mathfrak{M}_1$ and $\mathfrak{M}_2$, and all $\vec{x} \in W_1, \vec{y} \in W_2$ such that $\vec{x}\mathcal{B}\vec{y}$, we have $\mathfrak{M}_1, \vec{x} \models \varphi$ iff $\mathfrak{M}_2, \vec{y} \models \varphi$.

Proposition 4.10 *$L_3^{1,2}(\tau)$-formulas are invariant for bisimulations.*

Proof. By induction on *DMl*-expressions plus an application of 4.4. Here are some cases in the inductive proof. Let $\mathcal{B}$ be a bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$.

$\mathsf{do}(\alpha)$. Suppose $x\mathcal{B}y$. Then

$$\begin{aligned}\mathfrak{M}, x \models \mathsf{do}(\alpha) &\implies \text{for some } x'\,((x,x') \in [\![\alpha]\!]_{\mathfrak{M}}),\\ &\implies \exists y'\,(xx'\mathcal{B}yy' \wedge (y,y') \in [\![\alpha]\!]_{\mathfrak{N}}), \text{ by 4.9(2) and IH},\\ &\implies \mathfrak{N}, y \models \mathsf{do}(\alpha).\end{aligned}$$

$\mathsf{exp}(\varphi)$. Suppose $xx'\mathcal{B}yy'$. Then

$$\begin{aligned}(x,x') \in [\![\mathsf{exp}(\varphi)]\!]_{\mathfrak{M}} &\implies (x,x') \in R_{\mathfrak{M}} \models Rxy \text{ and } \mathfrak{M}, x' \models \varphi,\\ &\implies (y,y') \in R_{\mathfrak{N}} \text{ and } \mathfrak{N}, y' \models \varphi \text{ by 4.6(5) and IH},\\ &\implies (y,y') \in [\![\mathsf{exp}(\varphi)]\!]_{\mathfrak{N}}.\end{aligned}$$

$\alpha;\beta$. Suppose $xx'\mathcal{B}yy'$. Then

$$\begin{aligned}(x,x') \in [\![\alpha;\beta]\!]_{\mathfrak{M}} &\implies \text{for some } x''\,((x,x'') \in [\![\alpha]\!]_{\mathfrak{M}} \text{ and } (x'',x') \in [\![\beta]\!]_{\mathfrak{M}}),\\ &\implies \exists y''\,(xx''\mathcal{B}yy'' \wedge x''x'\mathcal{B}y''y' \wedge (y,y'') \in [\![\alpha]\!]_{\mathfrak{N}} \wedge (y'',y') \in [\![\beta]\!]_{\mathfrak{N}}),\\ &\qquad \text{by 4.6(4) and IH},\\ &\implies (y,y') \in [\![\alpha;\beta]\!]_{\mathfrak{N}}. \quad \dashv\end{aligned}$$

It's the converse of 4.10 that is more interesting:

Theorem 4.11 *A first order formula $\varphi(\vec{x})$ in $L(\tau)$ is equivalent to an $L_3^{1,2}(\tau)$-formula only if it is invariant for bisimulations.*

Proof. The proof is an extension of [25, Theorem 4.7]. Define

$$E(\varphi) = \{\, \psi \in L_3^{1,2}(\tau) : \varphi \models \psi \text{ and } FV(\psi) \subseteq FV(\varphi)\,\}.$$

We show that $E(\varphi) \models \varphi$. Then, by compactness the result follows.

So assume $\mathfrak{M}, \vec{w} \models E(\varphi)$. Introduce new constants $\underline{\vec{w}}$ to stand for the objects $\vec{w}$. Set $L^*(\tau) = L(\tau) \cup \{\,\underline{\vec{w}}\,\}$, and expand $\mathfrak{M}$ to an $L^*(\tau)$-model in the obvious way. For $\{\,\psi\,\} \cup T \subseteq L(\tau)$, ψ^* and T^* have the obvious meaning.

Define $T = \{\, \psi \in L_3^{1,2}(\tau) : \mathfrak{M}, \vec{w} \models \psi, FV(\psi) \subseteq FV(\varphi)\,\}$. By compactness there is an $L^*(\tau)$-model $\mathfrak{N}^*$ such that $\mathfrak{N}^* \models T^* \cup \{\,\varphi^*\,\}$. By standard model theory there are ω-saturated extensions $\mathfrak{M}_1^* = (\,W_1, \ldots, \vec{w}_1) \succ \mathfrak{M}^*$ and $\mathfrak{N}_1^* = (\,W_2, \ldots, \vec{w}_2) \succ \mathfrak{N}^*$ such that $\vec{w}_1$ and $\vec{w}_2$ both realize T, and $\mathfrak{N}_1^* \models \varphi^*$.

Define a relation $\mathcal{B} \subseteq (\,W_1 \times W_2) \cup (\,W_1^2 \times W_2^2)$ between (the $L(\tau)$-reducts of) $\mathfrak{M}_1^*$ and $\mathfrak{N}_1^*$, by putting

$$\begin{aligned} x_1\mathcal{B}y_1 \quad &\text{iff} \quad \text{for all } \psi(x) \in L_3^{1,2}(\tau), \text{ and } \mathfrak{M}_1, x_1 \models \psi \text{ iff } \mathfrak{N}_1, y_1 \models \psi, \text{ and}\\ x_1x_2\mathcal{B}y_1y_2 \quad &\text{iff} \quad \text{for all } \psi(x,y) \in L_3^{1,2}(\tau), \mathfrak{M}_1, x_1, x_2 \models \psi \text{ iff } \mathfrak{N}_1, y_1, y_2 \models \psi.\end{aligned}$$

I claim that $\mathcal{B}$ is in fact a bisimulation between $\mathfrak{M}_1$ and $\mathfrak{N}_1$. To see this, let us check that the conditions of 4.6 hold. Firstly, we have $\mathcal{B} \neq \emptyset$ because $\vec{w}_1\mathcal{B}\vec{w}_2$ holds. For, suppose that $\psi(\vec{x}) \in L_3^{1,2}(\tau)$; then $\psi \in T$, hence $\mathfrak{M}, \vec{w}_2 \models \psi$; and similarly in the opposite direction.

Conditions 2 and 3 are trivial, and to see that 4 is fulfilled, assume that $x_1x_2\mathcal{B}y_1y_2$ and $x_3 \in \mathfrak{M}_1$. What I need to show is: $\exists y_3\,(x_1x_3\mathcal{B}y_1y_3 \wedge x_3x_2\mathcal{B}y_3y_2)$. To this end set

$$\begin{aligned}\Psi(x,y) &= \{\,\psi(x,y) \in L_3^{1,2}(\tau) : \mathfrak{M}_1^*, x_1, x_3 \models \psi\,\},\\ \Xi(x,y) &= \{\,\psi(x,y) \in L_3^{1,2}(\tau) : \mathfrak{M}_1^*, x_3, x_2 \models \psi\,\}.\end{aligned}$$

Then $\Psi(\underline{y}_1, y) \cup \Xi(y, \underline{y}_2)$ is finitely satisfiable in $(\mathfrak{N}_1^*, \underline{y}_1, \underline{y}_2)$. Hence, since $\mathfrak{N}_1^*$ is ω-saturated, it is satisfiable in $(\mathfrak{N}_1^*, \underline{y}_1, \underline{y}_2)$. But this means that for some $y_3 \in W_2$, $x_1x_3\mathcal{B}y_1y_3$ and $x_3x_2\mathcal{B}y_3y_2$, as required. The other half of Condition 4 may be established in a similar way.

Next, we have to check that for $I = \{\,\emptyset\,\} \cup \{\,(\vec{x}, \vec{y}) : \vec{x}\mathcal{B}\vec{y}\,\}$ we have $I : \mathfrak{M}_1 \cong_2 \mathfrak{N}_1$. Now obviously, since each of $(\neg)P_ix, (\neg)x = y, (\neg)Rxy$ and $(\neg)Ryx$ is in $L_3^{1,2}(\tau)$, any $f \in I$ must be a 2-partial isomorphism. So all we have left to do, is show that I has the back and forth property. But this may done along the lines of the proof that Condition 4 is satisfied.

To conclude, $\mathcal{B}$ is a bisimulation between $\mathfrak{M}_1$ and $\mathfrak{N}_1$. So, by invariance for bisimulations $\mathfrak{N}_1^* \models \varphi^*$ implies $\mathfrak{M}_1^* \models \varphi$. Since $\mathfrak{M}_1^* \succ \mathfrak{M}^*$ it follows that $\mathfrak{M}^* \models \varphi^*$, and so $\mathfrak{M}, \vec{w} \models \varphi$. $\dashv$

Using 4.11 some results about definability of classes of *DML*-models can easily be derived. For an elegant formulation of these results it is convenient to consider so-called pointed models as our fundamental structures (as in Kripke's original publications). Here, a *pointed* model is a structure of the form $(W, \sqsubseteq, [\![\cdot]\!], V, w)$, where $(W, \sqsubseteq, [\![\cdot]\!], V)$ is an ordinary *DML*-model, and $w \in W$.

Corollary 4.12 *Let* M *be a class of pointed models. Then* M *is definable by means of a DML-formula iff it is closed under bisimulations and ultraproducts, while its complement is closed under ultraproducts.*

Proof. Similar to [25, Theorem 4.8]. $\dashv$

5 Decidability

In the preceding sections I have given several examples demonstrating that the *DML*-language is a powerful one as far as expressiveness is concerned. Of course, this power does not come without a price: I will show that satisfiability in the *DML*-language is not decidable. After that I show that decidability may be restored either by restricting the language, or by restricting the class of structures used to interpret the *DML*-language.

5.1 The full language interpreted on pre-orders

As a language, *DML* is somewhere in between the language of $S4_t$, the temporal analogue of the modal system $S4$, and full relational algebra. It is well-known that the latter is undecidable. Since in the intermediate case of *DML* we only have the operations of relation algebra on top of a single relation, it may be hoped that we are closer to $S4_t$ than to relational algebra, and hence that *DML* is decidable.

But here is already an important difference between the two: $S4_t$ enjoys the finite model property, while *DML* does not. To see this, define

- $R := \mathsf{exp}(\top)$,
- $\infty := \neg\mathbf{E}\mathsf{do}((R \cap -\delta) \cap R^{\smile})$.

Then, since ∞ forces the absence of loops, the formula $\mathbf{A}\mathsf{do}(R \cap -\delta) \wedge \infty$ is satisfiable only on infinite *DML*-models. And in fact we have the following result:

Theorem 5.1 *Satisfiability in DML is* Π^0_1*-hard.*

Proof. This is a reduction of a known Π^0_1-complete problem, a so-called *unbounded tiling problem* (UTP), to satisfiability in *DML*. The version of the UTP that I will use here is given by the following data. Given a set of tiles $T = \{ d_0, \ldots, d_m \}$, each having 4 sides whose colors are in $C = \{ c_0, \ldots, c_k \}$, is there a tiling of $\mathbb{N} \times \mathbb{N}$? The rules of the tiling game are

1. every point in the grid is associated with a single tile,
2. adjacent edges have the same color.

Now, the version of the UTP presented here is known to be Π^0_1-complete (cf. Harel [15]). So to prove the theorem it suffices to define, for a given set of tiles T, a formula φ_T in the *DML*-language such that

1. its models look like grids,
2. it says that every point is covered by a tile from T,
3. and that colors match right and above neighbors,

and show that φ_T is satisfiable iff T can tile $\mathbb{N} \times \mathbb{N}$. Let's get to work now. To make a grid, define

- $\text{LEAVE}(\varphi) := (\varphi?; R)$,
- $\text{ONE} := (R \cap -\delta) \cap -[(R \cap -\delta); (R \cap -\delta)]$; then, for all $\mathfrak{M}$, and for all $x, y \in \mathfrak{M}$,

$$(x, y) \in [\![\text{ONE}]\!]_{\mathfrak{M}} \text{ iff } xRy \wedge x \neq y \wedge \neg\exists z\, (xRz \wedge x \neq z \wedge zRy \wedge z \neq y),$$

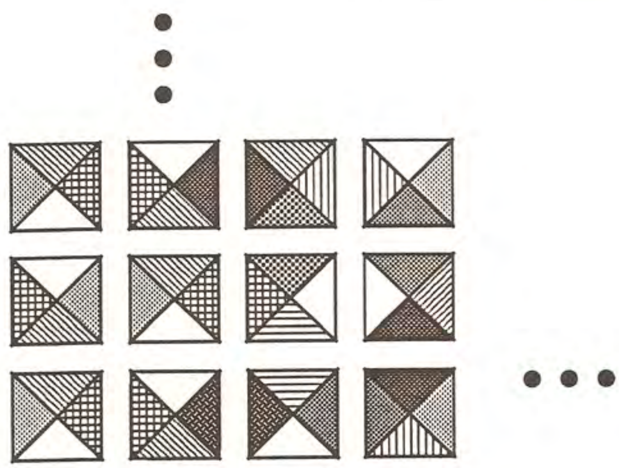

Figure 2: The unbounded tiling problem.

- $\mathrm{UP} := [\mathrm{ONE} \cap \mathrm{LEAVE}(p \wedge q) \cap \mathsf{exp}(p \wedge \neg q)]$
  $\cup\, [\mathrm{ONE} \cap \mathrm{LEAVE}(p \wedge \neg q) \cap \mathsf{exp}(p \wedge q)]$
  $\cup\, [\mathrm{ONE} \cap \mathrm{LEAVE}(\neg p \wedge q) \cap \mathsf{exp}(\neg p \wedge \neg q)]$
  $\cup\, [\mathrm{ONE} \cap \mathrm{LEAVE}(\neg p \wedge \neg q) \cap \mathsf{exp}(\neg p \wedge q)]$,
- $\mathrm{RIGHT} := [\mathrm{ONE} \cap \mathrm{LEAVE}(p \wedge q) \cap \mathsf{exp}(\neg p \wedge q)]$
  $\cup\, [\mathrm{ONE} \cap \mathrm{LEAVE}(\neg p \wedge q) \cap \mathsf{exp}(p \wedge q)]$
  $\cup\, [\mathrm{ONE} \cap \mathrm{LEAVE}(p \wedge \neg q) \cap \mathsf{exp}(\neg p \wedge \neg q)]$
  $\cup\, [\mathrm{ONE} \cap \mathrm{LEAVE}(\neg p \wedge \neg q) \cap \mathsf{exp}(p \wedge \neg q)]$,
- $\mathrm{EQUAL}(\alpha, \beta) := \neg \mathbf{E}\mathsf{do}(\alpha \cap -\beta) \wedge \neg \mathbf{E}\mathsf{do}(\beta \cap -\alpha)$,
- $\mathrm{CR} := \mathrm{EQUAL}\Big((\mathrm{UP}; \mathrm{RIGHT}), (\mathrm{RIGHT}; \mathrm{UP})\Big)$.

Here, finally, is the formula that will force our models to contain a copy of $\mathbb{N} \times \mathbb{N}$:

- $\mathrm{GRID} := (p \wedge q) \wedge \mathbf{A}\mathsf{do}(\mathrm{UP}) \wedge \mathbf{A}\mathsf{do}(\mathrm{RIGHT}) \wedge \mathrm{CR} \wedge \infty$.

Next we have to define formulas that force 2 and 3. Let $T = \{ d_0, \ldots, d_m \}$ and $C = \{ c_0, \ldots, c_k \}$ be given. For each color $c_i \in C$ introduce four proposition letters, suggestively denoted by $up = c_i, right = c_i, down = c_i, left = c_i$. Identifying each tile $d \in T$ with its four sides I assume that each tile d is represented as

$$(up = c_{i_1} \wedge right = c_{i_2} \wedge down = c_{i_3} \wedge left = c_{i_4}) \\ \wedge \Big(\bigwedge_{c \in T \setminus \{ c_{i_1} \}} \neg up = c \wedge \ldots \wedge \bigwedge_{c \in T \setminus \{ c_{i_4} \}} \neg left = c \Big).$$

Then, put

- $\text{COVER} := \mathbf{A} \bigvee_{d \in T} d$,

and

- $\text{MATCH} := \mathbf{A}\Big(\bigwedge_{c \in C} (up = c \rightarrow [\text{UP}]down = c)$
$$\wedge \bigwedge_{c \in C} (right = c \rightarrow [\text{RIGHT}]left = c)\Big).$$

Put $\varphi_T := \text{GRID} \wedge \text{COVER} \wedge \text{MATCH}$. Then φ_T is satisfiable in an *DML*-model iff T can tile $\mathbb{N} \times \mathbb{N}$. The *if*-direction is trivial, since if a tiling exists φ_T is satisfiable in $\mathbb{N} \times \mathbb{N}$, simply by verifying $(p \wedge q)$ in $(0,0)$, switching the truth values of p and q while going right and up through the grid, respectively, while the tiling will tell you how to satisfy COVER and MATCH. Conversely, the domain of any *DML*-model in which φ_T is satisfied in some point x, must contain a copy of $\mathbb{N} \times \mathbb{N}$ with x as its origin; as COVER and MATCH are satisfied in x there must be a tiling of this copy of $\mathbb{N} \times \mathbb{N}$. $\dashv$

Corollary 5.2 *Satisfiability in DML is* Π^0_1*-hard.*

One may get a reduction of the UTP to *DML*-satisfiability with somewhat less than what I have used in the proof of 5.1. For instance, it is not necessary to actually have a real grid inside models satisfying the 'reduction formula' φ_T; instead it suffices to have structures satisfying a Church-Rosser like property like $\forall yz\,(Rxy \wedge Rxz \rightarrow \exists u\,(Ryu \wedge Rzu))$.

5.2 Fragments and special frame classes

A natural move at this point is to try and find reasonably large fragments of the *DML*-language that are decidable. To this end, let's step back a second and see what made the proof of 5.1 work. Essentially, we were able to build a grid there, thanks to the availability of ;, $\cap$ and $-$. Thus, when looking for reasonably large *decidable* fragments of the *DML*-language, giving up some of these three might get us results. Indeed, giving up ; (and $\breve{}$, by the way) restricts *DML* to the Boolean modal logic of Gargov and Passy as mentioned in §2, and this is a decidable system (cf. [12]). Alternatively, giving up $-$ again yields decidability by Danecki [6]. Of course, in these fragments some of the more complex operators like $\mathbf{D}$ and $\mu\text{-}\mathsf{exp}(\cdot), \mu\text{-}\mathsf{con}(\cdot)$ will no longer be definable, thus it remains to be seen whether adding any of these to the above fragments preserves decidability.

Another approach towards obtaining decidability is not to restrict the Language, but to restrict the structures used to interpret the language. As an example I will consider the class of all trees. Just to be precise, by a *tree* is a meant a structure $(W, \sqsubseteq)$ with $\sqsubset\, \subseteq W^2$

a transitive, asymmetric relation such that for each $x \in W$ the set of $\sqsubseteq$-predecessors of x is linearly ordered by $\sqsubseteq$.

Let $\mathrm{Th}_{DML}(\mathsf{TREES})$ denote the set of *DML*-formulas valid on the class TREES of all trees. In §6.3 I will axiomatize $\mathrm{Th}_{DML}(\mathsf{TREES})$, but for the moment all we need to know about it, is that it lacks the finite model property. (To see this simply consider the formula $\mathbf{Ado}(R) \wedge \infty$ ¿From §5.1.) Thus, to establish decidability of this theory some other tools will have to be employed. Of course, one obvious candidate is Rabin's Theorem [24]; to apply this result the semantics of $\mathrm{Th}_{DML}(\mathsf{TREES})$ has to be embedded in $S\omega S$, the monadic second order theory of infinitely many successor functions. Here, I will take an easier way out by appealing to a result by Gurevich and Shelah [14]. Let L_{GS} be the language of monadic second order with additional unary predicates, i.e. it has individual variables and unary predicate variables (ranging over branches) as well as a binary relation symbol $<$ and unary predicate constants $P_0, P_1, \ldots$. And let $\mathrm{Th}_{GS}(\mathsf{TREES})$ be the set of L_{GS} formulas valid on all trees. Then obviously, the question whether a given *DML* formulas φ is valid on all trees, boils down to the question whether its standard translation φ^* is a theorem of $\mathrm{Th}_{GS}(\mathsf{TREES})$. But by [14] the latter question is decidable.

Theorem 5.3 *Given a DML formula φ, the question "Is $\mathrm{Th}_{DML}(\mathsf{TREES}) \cup \{\varphi\}$ satisfiable?" is decidable.*

Several variations on the above, variations, moreover, that will still yield decidable theories, are quite natural and worth considering. They include, for example, the set of *DML* formulas valid on all trees of finite depth, or the *DML* formulas valid on all well-founded trees.

6 Completeness

Before presenting a complete calculus *DML* for the *DML*-language in §6.2, I will sketch a construction for a completeness proof involving the $\mathbf{D}$ operator; this construction is presented in more detail in Venema [29], and generalizes some constructions that may be found in Gabbay and Hodkinson [10]. Then, in §6.3, this construction will be used to prove *DML* complete; §6.3 also contains a finite axiomatization for the set $\mathrm{Th}_{DML}(\mathsf{TREES}) \cup \{\varphi\}$ of *DML* formulas valid on all trees.

6.1 How to use the D operator

Let me first present the logic governing the $\mathbf{D}$ operator:

Definition 6.1 Let $\overline{\mathbf{D}}$ abbreviate $\neg\mathbf{D}\neg$. Besides the classical tautologies *DL* has the following axioms

(D1) $\overline{\mathbf{D}}(p \to q) \to (\overline{\mathbf{D}}p \to \overline{\mathbf{D}}q)$,
(D2) $p \to \overline{\mathbf{D}}\mathbf{D}p$,
(D3) $\mathbf{D}\mathbf{D}p \to p \vee \mathbf{D}q$.

Its rules of inference are:

(MP) $\varphi \to \psi, \varphi/\psi$,
($\mathrm{UG}_{\neq}$) $\varphi/\overline{\mathbf{D}}\varphi$,
($\mathrm{IR}_{\mathbf{D}}$) $p \wedge \neg\mathbf{D}p \to \varphi/\varphi$, provided p does not occur in φ.

Let $\mathcal{O} = \{\, \mathbf{D} \,\} \cup \{\, \Diamond_1, \Diamond_2, \ldots, \Diamond_1{}^{\smile}, \Diamond_2{}^{\smile}, \ldots \}$ be a collection of unary modal operators. I will write $\Box_i$ for the dual $\neg\Diamond_i\neg$ of $\Diamond_i$, and I suppose that for every $\Diamond \in \mathcal{O}$ we have its converse $\Diamond^{\smile}$ available in $\mathcal{O}$ (the converse of $\mathbf{D}$ is $\mathbf{D}$ itself). For the time being I assume the language does not contain any operators like do, ra, or fix.

Let Λ be a logic which contains the axioms of *DL* plus $\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$, $\varphi \to \Box_i\Diamond_i{}^{\smile}\varphi$, $\varphi \to \Box_i{}^{\smile}\Diamond_i\varphi$, and $\Diamond_i\varphi \to \varphi \vee \mathbf{D}\varphi$, for every $\Diamond_i \in \mathcal{O}$, and which has MP, $\mathrm{UG}_{\Box_i}$, $\mathrm{IR}_{\mathbf{D}}$, and SUB as its rules of inference.

Definition 6.2 Let Λ be a logic as specified above. A theory Δ is Λ-*consistent* if $\Delta \nvdash_\Lambda \bot$.

Let Φ be a collection of proposition letters. A theory Δ is called a Φ-*theory* if all proposition letters occurring in formulas in Δ are elements of Φ. Δ is called a *complete* Φ-theory if $\varphi \in \Delta$ or $\neg\varphi \in \Delta$, for all formulas built up using proposition letters in Φ.

Δ is a *distinguishing* Φ-theory if (i) for some proposition letter p, $p \wedge \neg\mathbf{D}p \in \Delta$, and (ii) whenever $\Diamond_1(\varphi_1 \wedge \Diamond_2(\varphi_2 \wedge \ldots \wedge \Diamond_m\varphi_m)\ldots) \in \Delta$, then for some proposition letter p, $\Diamond_1(\varphi_1 \wedge \Diamond_2(\varphi_2 \wedge \ldots \wedge \Diamond_m(\varphi_m \wedge p \wedge \neg\mathbf{D}p))\ldots) \in \Delta$.

Lemma 6.3 *Let Σ be a consistent theory in Λ, and let p be a proposition letter not occurring in any formula in Σ. Suppose $\varphi_1 \wedge \Diamond_1(\varphi_2 \wedge \Diamond_2(\ldots \wedge \Diamond_{m-1}\varphi_m)\ldots) \in \Sigma$. Then the union of Σ and $\{\, \varphi_1 \wedge \Diamond_1(\varphi_2 \wedge \Diamond_2(\ldots \wedge \Diamond_{m-1}[\varphi_m \wedge p \wedge \neg\mathbf{D}p])\ldots) \,\}$ is consistent.*

Proof. Cf. [10, Corollary 2.2.3]. ⊣

Lemma 6.4 (Extension Lemma) *Let Σ be a Λ-consistent Φ-theory. Let $\Phi' \supseteq \Phi$ be an extension of Φ by a countably infinite set of proposition letters. Then there is a complete, Λ-consistent, distinguishing Φ'-theory Δ containing Σ.*

Proof. This is similar to the proof of [10, Theorem 2.3.1] or [29, Lemma 4.6]. Nevertheless, it is short enough to be included here.

Define $\Delta = \bigcup_{n<\omega} \Delta_n$, where each Δ_n is a consistent Φ'-theory, satisfying $|\Delta_n \backslash \Delta_1| < \omega$, for all n. To define these Δ_ns, let $p \in \Phi' \backslash \Phi$. Then, by 6.3, $\Sigma \cup \{\, p \wedge \neg\mathbf{D}p \,\}$ is Λ-consistent. Set $\Delta_1 = \Sigma \cup \{\, p \wedge \neg\mathbf{D}p \,\}$.

Let $\varphi_2, \varphi_3, \ldots$ be an enumeration of all Φ'-formulas, and suppose that Δ_n has been defined and has the properties cited. Define $\Delta_{n+1} = \Delta_n \cup E_n$, where

(*) $E_n = \{ \neg\varphi_n \}$, if $\Delta_n \cup \{ \varphi_n \}$ is not Λ-consistent,

(**) $E_n = \{ \varphi_n \}$, if $\Delta_n \cup \{ \varphi_n \}$ is Λ-consistent, and φ is not of the form $\Diamond_1(\psi_1 \wedge \Diamond(\psi_2 \wedge \ldots \wedge \Diamond_m \psi_m) \ldots)$,

(***) if φ_n does have this form, then, since $|\Delta_n \setminus \Delta_1| < \omega$, there are proposition letters $p_1, \ldots, p_m \in \Phi' \setminus \Phi$ that do not occur in Δ_n. Set

$$E_n = \{ \varphi_n, \Diamond_1(\psi_1 \wedge p_1 \wedge \neg \mathbf{D} p_1 \wedge \Diamond_2(\ldots \wedge \Diamond_m(\psi_m \wedge p_m \wedge \neg \mathbf{D} p_m)) \ldots) \}.$$

It is obvious that Δ_{n+1} is Λ-consistent if it has been defined according to (*) or (**). But, by repeated applications of 6.3 it is also consistent when defined according to (***). I leave it to the reader to check that Δ is complete, Λ-consistent and distinguishing. $\dashv$

Definition 6.5 Let Φ be a countably infinite set of proposition letters. Let W_c be the set of all complete, Λ-consistent, distinguishing Φ-theories. On W_c we define relations $R_{c\Diamond}$, for $\Diamond \in \mathcal{O}$, by putting $\Delta_1 R_{c\Diamond} \Delta_2$ iff for all Φ-formulas φ, if $\Box\varphi \in \Delta_1$, then $\varphi \in \Delta_2$ (or equivalently: if $\varphi \in \Delta_2$ then $\Diamond\varphi \in \Delta_1$, or equivalently: if $\Box^{\smile}\varphi \in \Delta_2$ then $\varphi \in \Delta_1$).

I use $R_{c\mathbf{D}}$ to denote the relation defined using the $\mathbf{D}$ operator.

Lemma 6.6 (Successor Lemma) *Let $\Delta_1 \in W_c$. Assume $\Diamond_i\varphi \in \Delta_1$. Then there is a $\Delta_2 \in W_c$ with $\Delta_1 R_{c\Diamond_i} \Delta_2$ and $\varphi \in \Delta_2$.*

Proof. Cf. [10, Proposition 2.3.2] or [29, Lemma 4.7]. $\dashv$

I now turn to defining a model in which the interpretation of the $\mathbf{D}$ operator is real inequality.

Define $\Delta \sim_{\mathbf{D}} \Delta_2$ if $\Delta_1 = \Delta_2$ or $\Delta_1 R_{c\mathbf{D}} \Delta_2$. By [25, Theorem 3.2] or [29, Lemma 4.9] $\sim_{\mathbf{D}}$ is an equivalence relation. A subset of W_c is called *connected* if it is a $\sim_{\mathbf{D}}$-equivalence class. By an easy argument one can show that $R_{c\mathbf{D}}$ is *real* inequality when restricted to a connected subset of W_c (cf. [25, Theorem 3.2] or [29, Lemma 4.11]). Also, since Λ contains the axioms $\Diamond_i\varphi \to \varphi \vee \mathbf{D}\varphi$, any connected subset of W_c must be closed under $R_{c\Diamond_i}$.

Definition 6.7 A *d-canonical frame* for Λ is a tuple $\mathfrak{F}_d = (W_d, \{ R_{d\Diamond} : \Diamond \in \mathcal{O} \})$, where W_d is a connected subset of W_c, and $R_{d\Diamond} = R_{c\Diamond} \restriction (W_c \times W_c)$.

A *d-canonical model* for Λ is a tuple $\mathfrak{M}_d = (\mathfrak{F}_d, V_d)$, where $\mathfrak{F}_d$ is d-canonical frame, and V_d is given by $\Delta \in V_d(p)$ iff $p \in \Delta$.

Lemma 6.8 (Truth Lemma) *For all formulas φ in the language containing the modal operators in $\mathcal{O}$, and all $\Delta \in \mathfrak{M}_d$, we have $\mathfrak{M}_d, \Delta \models \varphi$ iff $\varphi \in \Delta$.*

Proof. I argue by induction on φ, and only treat the case $\varphi \equiv \Diamond\psi$. If $\mathfrak{M}_d, \Delta_1 \models \Diamond\psi$, then there is a $\Delta_2 \in W_d$ with $\Delta_1 R_{d\Diamond} \Delta_2$ and $\mathfrak{M}_d, \Delta_2 \models \psi$. By the induction hypothesis $\psi \in \Delta_2$. Since $\Delta_1 R_{d\Diamond} \Delta_2$ this implies $\Diamond\psi \in \Delta_1$.

Conversely, by the Successor Lemma and the remarks preceding Definition 6.7 $\Diamond\psi \in \Delta_1$ implies that for some $\Delta_2 \in W_d$, $\Delta_1 R_{d\Diamond} \Delta_2$ and $\psi \in \Delta_2$. By the induction hypothesis this gives $\mathfrak{M}_d, \Delta \models \psi$, and hence $\mathfrak{M}_d, \Delta_1 \models \Diamond\psi$. $\dashv$

6.2 Axioms

We may as well start with a definition:

Definition 6.9 Let $\langle\alpha\rangle\varphi$ abbreviate $\mathsf{do}(\alpha; \varphi?)$, let $[\alpha]\varphi \equiv \neg\langle\alpha\rangle\neg\varphi$, and let $\langle\sqsubseteq\rangle\varphi$ be short for $\langle\mathsf{exp}(\top)\rangle\varphi$. Besides enough classical tautologies, and the axioms of *DL* (taken as axioms over $DML(\Phi)$) the system *DML* has the following axioms:

(D4) $\mathbf{D}p \leftrightarrow \mathsf{do}(-\delta; p?)$,

(A1) $[\alpha](p \to q) \to ([\alpha]p \to [\alpha]q)$,
(A2) $\langle\alpha\rangle p \to p \vee \mathbf{D}p$,
(A3) $\langle\sqsubseteq\rangle\langle\sqsubseteq\rangle p \to \langle\sqsubseteq\rangle p$,
(A4) $p \to \langle\sqsubseteq\rangle p$,

(A5) $\mathsf{do}(\mathsf{exp}(p)) \leftrightarrow \langle\sqsubseteq\rangle p$,
(A6) $\mathsf{do}(\mathsf{con}(\varphi)) \leftrightarrow \langle\sqsupseteq\rangle p$,
(A7) $\mathsf{ra}(\mathsf{exp}(p)) \leftrightarrow p$,
(A8) $\mathsf{ra}(\mathsf{con}(p)) \leftrightarrow \neg p$,
(A9) $\mathsf{fix}(\mathsf{exp}(p)) \leftrightarrow p$,
(A10) $\mathsf{fix}(\mathsf{con}(p)) \leftrightarrow \neg p$,

(A11) $\langle\alpha \cap \beta\rangle p \to \langle\alpha\rangle p \wedge \langle\beta\rangle p$,
(A12) $\mathbf{E}(p \wedge \neg\mathbf{D}p) \to (\langle\alpha\rangle p \wedge \langle\beta\rangle p \to \langle\alpha \cap \beta\rangle p)$,
(A13) $\mathsf{do}(\alpha \cap \beta) \leftrightarrow \langle\alpha \cap \beta\rangle\top$,
(A14) $\mathsf{ra}(\alpha \cap \beta) \leftrightarrow \langle(\alpha \cap \beta)^{\smallsmile}\rangle\top$,
(A15) $p \wedge \neg\mathbf{D}p \to (\mathsf{fix}(\alpha \cap \beta) \leftrightarrow \langle\alpha \cap \beta\rangle p)$

(A16) $\langle\alpha; \beta\rangle p \leftrightarrow \langle\alpha\rangle\langle\beta\rangle p$,
(A17) $\mathsf{do}(\alpha; \beta) \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\top$,
(A18) $\mathsf{ra}(\alpha; \beta) \leftrightarrow \langle\beta^{\smallsmile}\rangle\langle\alpha^{\smallsmile}\rangle\top$,
(A19) $p \wedge \neg\mathbf{D}p \to (\mathsf{fix}(\alpha; \beta) \leftrightarrow \langle\alpha\rangle\langle\beta\rangle p)$,

(A20) $\mathbf{E}(p \wedge \neg\mathbf{D}p) \rightarrow (\langle\alpha\rangle p \leftrightarrow \neg\langle-\alpha\rangle p)$,
(A21) $\mathsf{do}(-\alpha) \leftrightarrow \langle-\alpha\rangle\top$,
(A22) $\mathsf{ra}(-\alpha) \leftrightarrow \langle(-\alpha)^{\check{}}\rangle\top$,
(A23) $p \wedge \neg\mathbf{D}p \rightarrow (\mathsf{fix}(-\alpha) \leftrightarrow \langle-\alpha\rangle p)$,

(A24) $p \rightarrow [\alpha]\langle\alpha^{\check{}}\rangle p$,
(A25) $p \rightarrow [\alpha^{\check{}}]\langle\alpha\rangle p$,
(A26) $\mathsf{do}(\alpha^{\check{}}) \leftrightarrow \langle\alpha^{\check{}}\rangle\top$,
(A27) $\mathsf{ra}(\alpha^{\check{}}) \leftrightarrow \langle\alpha\rangle\top$,
(A28) $p \wedge \neg\mathbf{D}p \rightarrow (\mathsf{fix}(\alpha^{\check{}}) \leftrightarrow \langle\alpha\rangle p)$,

(A29) $\langle p?\rangle q \leftrightarrow (p \wedge q)$,
(A30) $\mathsf{do}(p?) \leftrightarrow p$,
(A31) $\mathsf{ra}(p?) \leftrightarrow p$,
(A32) $\mathsf{fix}(p?) \leftrightarrow p$.

Besides those of *DL*, the rules of inference of *DML* are:

(UG_α) $\varphi/[\alpha]\varphi$, for $\alpha \in \mathit{Proc}(\Phi)$.

Remark 6.10 Obviously there are various redundancies and dependencies in the above list of axioms. For instance, as pointed out earlier, the modes ra and fix are definable via $\mathsf{ra}(\alpha) \leftrightarrow \mathsf{do}(\alpha^{\check{}})$ and $\mathsf{fix}(\alpha) \leftrightarrow \mathsf{do}(\alpha \cap \delta)$. Hence, all axioms involving ra and fix can be left out when these axioms are added. But I am not after the most economic set of axioms here.

Note by the way how the above list is organized. Apart from some initial bookkeeping axioms and axioms ensuring that the canonical structure will be transitive and reflexive, there are, for every relational connective $\bullet$, one or two axioms to make sure that the interpretation of $\bullet$ in the canonical model is the intended one, plus three more axioms to make the Truth Lemma work for $\bullet$.

Definition 6.11 The system DML_T is defined as the extension of *DML* with the following axioms. Let $\langle\sqsubset\rangle\varphi$ abbreviate $\mathsf{do}(\mathsf{exp}(\top) \cap -\delta; \varphi?)$, and let $[\sqsubset]$ be the dual of $\langle\sqsubset\rangle$. (And similarly for $\langle\sqsupset\rangle$ and $[\sqsupset]$.)

(T1) $p \wedge \neg\mathbf{D}p \rightarrow [\sqsubset][\sqsupset]\neg p$,
(T2) $\langle\sqsupseteq\rangle(p \wedge \neg\mathbf{D}p) \rightarrow [\sqsupset](\langle\sqsubset\rangle p \vee p \vee \langle\sqsubset\rangle p)$.

Axiom T1 will make sure that in the canonical model the relation $\sqsubset$ is asymmetric, while axiom T2 will guarantee that sets of predecessors in the canonical model are linearly ordered by $\sqsubset$.

6.3 Theorems

Theorem 6.12 *The system DML is sound and complete with respect to its standard models.*

Proof. Proving soundness is left to the reader. To prove completeness, define $\mathcal{O}$ to be the union of $\{\mathbf{D}, \langle\sqsubseteq\rangle, \langle\sqsupseteq\rangle\}$ and $\{\langle\alpha\rangle : \alpha \in \mathit{Proc}(\Phi')\}$, where $\langle\sqsubseteq\rangle$ and $\langle\sqsupseteq\rangle$ abbreviate $\mathsf{do}(\mathsf{exp}(\top);\cdot)$ and $\mathsf{do}(\mathsf{con}(\bot);\cdot)$, and repeat the construction of §6.1.

Define $\mathfrak{M} = (W_d, \sqsubseteq, [\![\cdot]\!], V_d)$, where W_d, V_d are as in 6.7, $\sqsubseteq$ denotes $R_{d\langle\sqsubseteq\rangle}$, and for $\alpha \in \mathit{Proc}(\Phi')$, $[\![\cdot]\!] = R_{d\langle\alpha\rangle}$. Then, by axioms A3 and A4 $\sqsubseteq$ is both transitive and reflexive.

First I check that $\mathfrak{M}$ is a standard model. Checking that $\mathfrak{M}$ is standard as far as exp and con is concerned, is left to the reader (use A5–A10).

A useful thing to note for dealing with the relational operators of *DML* is that, by 6.4 for any $\Delta \in \mathfrak{M}$ there is a (unique) proposition letter p_Δ such that $p_\Delta \wedge \neg\mathbf{D}p_\Delta \in \Delta$. Let's first consider $[\![\alpha \cap \beta]\!]$. By A11 $[\![\alpha \cap \beta]\!] \subseteq [\![\alpha]\!] \cap [\![\beta]\!]$. To prove the converse, assume $(\Delta_1, \Delta_2) \in [\![\alpha]\!] \cap [\![\beta]\!]$. Let p be a proposition letter such that $p \wedge \neg\mathbf{D}p \in \Delta_2$. Then $\mathbf{E}(p \wedge \neg\mathbf{D}p), \langle\alpha\rangle p, \langle\beta\rangle p \in \Delta_1$. Hence, by A12, $\langle\alpha \cap \beta\rangle p \in \Delta_1$. But this is possible only if $(\Delta_1, \Delta_2) \in [\![\alpha \cap \beta]\!]$, as required.

It is easily verified that $[\![\alpha;\beta]\!] = [\![\alpha]\!];[\![\beta]\!]$, by using A16. To prove that $[\![-\alpha]\!] = -[\![\alpha]\!]$ argue as follows. Assume that $(\Delta_1, \Delta_2) \in [\![-\alpha]\!]$. Let p be a proposition letter such that $p \wedge \neg\mathbf{D}p \in \Delta_2$. Then $\mathbf{E}(p \wedge \neg\mathbf{D}p), \langle-\alpha\rangle p \in \Delta_1$. Ergo, $\neg\langle\alpha\rangle p \in \Delta_1$, by A20. Hence $(\Delta_1, \Delta_2) \notin [\![\alpha]\!]$. This establishes $[\![-\alpha]\!] \subseteq -[\![\alpha]\!]$. The converse inclusion may be established in a similar way.

I leave it to the reader to use A24 and A25 to verify that $(\cdot)^\smallsmile$ does indeed act as the converse operator. The final case we have to check, $[\![\varphi?]\!]$, is easy: simply use A31.

We are now in a position to prove a Truth Lemma: for all formulas $\varphi \in \mathit{DML}(\Phi')$, and all $\Delta \in \mathfrak{M}$, we have $\mathfrak{M}, \Delta \models \varphi$ iff $\varphi \in \Delta$. Of course the proof is by induction on φ, and the only interesting cases are $\mathbf{D}\varphi$ (but this case is covered by the construction in §6.1), and $\mathsf{xx}(\mathsf{exp}(\varphi), \mathsf{xx}(\mathsf{con}(\varphi), \mathsf{xx}(\alpha \cap \beta), \mathsf{xx}(\alpha;\beta), \mathsf{xx}(-\alpha), \mathsf{xx}(\alpha^\smallsmile)$ and $\mathsf{xx}(\varphi?)$, where $\mathsf{xx} \in \{\mathsf{do}, \mathsf{ra}, \mathsf{fix}\}$.

Here we go: the cases $\mathsf{xx}(\mathsf{exp}(\varphi)$ and $\mathsf{xx}(\mathsf{con}(\varphi)$ (for $\mathsf{xx} \in \{\mathsf{do}, \mathsf{ra}, \mathsf{fix}\}$) can either be established by standard modal arguments, or are immediate from axioms A5–A10.

By A13 $\mathsf{do}(\alpha \cap \beta) \in \Delta$ iff $\langle\alpha \cap \beta\rangle\top \in \Delta$. By the Extension Lemma and the definition of $[\![\cdot]\!]$ this is equivalent to $\exists\Delta_1\,((\Delta, \Delta_1) \in [\![\alpha \cap \beta]\!])$, which is the case iff $\mathfrak{M}, \Delta \models \mathsf{do}(\alpha \cap \beta)$. Next, by a similar argument, using A14 plus the fact that $\mathfrak{M}$ is standard, we have that $\mathsf{ra}(\alpha \cap \beta) \in \Delta$ iff $\mathfrak{M}, \Delta \models \mathsf{ra}(\alpha \cap \beta)$. Furthermore, let p be a proposition letter such that $p \wedge \neg\mathbf{D}p \in \Delta$; then $\mathsf{fix}(\alpha \cap \beta) \in \Delta$ iff $\langle\alpha \cap \beta\rangle p \in \Delta$ (by A15) iff $(\Delta, \Delta) \in [\![\alpha \cap \beta]\!]$ iff $\mathfrak{M}, \Delta \models \mathsf{fix}(\alpha \cap \beta)$.

The next procedure operator to consider is ;. But this one may be treated completely analogous to $\cap$, using A17–A19. We move on to '$-$'. By A21 we have $\mathsf{do}(-\alpha) \in \Delta$ iff $\langle -\alpha \rangle \top \in \Delta$ iff for some Δ_1, $(\Delta, \Delta_1) \in [\![-\alpha]\!]$ (by the Extension Lemma and the definition of $[\![\cdot]\!]$) iff $\mathfrak{M}, \Delta \models \mathsf{do}(-\alpha)$. The case $\mathsf{ra}(-\alpha)$ is entirely analogous and uses A22, while the case $\mathsf{fix}(-\alpha)$ is similar to the earlier 'fix-cases' (use A23). A similar line of reasoning (using A26–A28) establishes all cases for the converse operator $(\cdot)^{\smile}$, so I'll skip that one, leaving only procedures of the form $\varphi?$ to consider. But here one may use the induction hypothesis together with axioms A30–A32.

Of course, from the Truth Lemma the completeness of *DML* follows by a standard argument. $\dashv$

Theorem 6.13 $\mathrm{Th}_{DML}(\mathsf{TREES})$, *the set of DML formulas valid on all trees, is axiomatized by* DML_T.

Proof. Obviously, if $\vdash_{DML_T} \varphi$ then $\varphi \in \mathrm{Th}_{DML}(\mathsf{TREES})$. To prove the converse, assume $\not\vdash_{DML_T} \varphi$. Repeat the construction used in the proof of 6.12, and let $\mathfrak{M} = (W, \sqsubseteq, [\![\cdot]\!], V)$ be a d-canonical structure such that for some $\Delta \in \mathfrak{M}$, $\varphi \notin \Delta$. All we have to do to complete the proof of the theorem, is show that $(W, \sqsubseteq)$ is a tree.

First, $\sqsubseteq$ is obviously transitive, and it is asymmetric by axiom T1. To see that $\sqsubseteq$-predecessors are linearly ordered by $\sqsubseteq$, use T2: let $\Delta_1, \Delta_2 \sqsubseteq \Delta$. By construction there is a proposition letter p such that $p \wedge \neg \mathbf{D}p \in \Delta_1$. By T2 this yields $\langle \sqsubset \rangle p \vee p \vee \langle \sqsupset \rangle p \in \Delta_2$, implying that either $\Delta_2 \sqsubset \Delta_1$, $\Delta_2 = \Delta_1$ or $\Delta_1 \sqsubset \Delta_2$, as required. $\dashv$

7 Which algebras?

In this section I will define modal algebras appropriate for the *DML*-language. I will need one or two preliminary definitions. First, a *Boolean module* is a structure $\mathfrak{M} = (\mathfrak{B}, \mathfrak{R}, \diamond)$, where $\mathfrak{B}$ is a Boolean algebra, $\mathfrak{R}$ is a relation algebra and $\diamond$ is a mapping $\mathfrak{R} \times \mathfrak{B} \to \mathfrak{B}$ such that

M1 $\diamond(r, a + b) = \diamond(r, a) + \diamond(r, b)$,
M2 $\diamond(r + s, a) = \diamond(r, a) + \diamond(s, a)$,
M3 $\diamond(r, \diamond(s, a)) = \diamond((r; s), a)$,
M4 $\diamond(\delta, a) = a$,
M5 $\diamond(0, a) = 0$,
M6 $\diamond(r^{\smile}, \diamond(r, a)') \leq a'$.

Just as Boolean algebras formalize reasoning about sets, and relation algebras formalize reasoning about relations, Boolean modules formalizes reasoning about sets interacting with relations through $\diamond$. In the full Boolean module $\mathfrak{M}(U) = (\mathfrak{B}(U), \mathfrak{R}(U), \diamond)$ over a set

$U \neq \emptyset$ the operation $\diamond$ is defined by

$$\diamond(R, A) = \langle R \rangle A = \{ x : \exists y ((x, y) \in R \wedge y \in A) \}.$$

(See Brink [4] for a formal definition of Boolean modules and some examples.)

Now, Boolean modules are almost, but not quite, the modal algebras of the *DML*-language. To obtain a perfect match, what we need in addition to the *set* forming operation or *projection* $\diamond$, is an operation that forms new *relations*, i.e., a *mode*. This brings us to the notion of a *Peirce algebra*, which is a two-sorted algebra $\mathfrak{P} = (\mathfrak{B}, \mathfrak{R}, \diamond, {}^c)$ with $(\mathfrak{B}, \mathfrak{R}, \diamond)$ a Boolean module, and $(\cdot)^c : \mathfrak{B} \to \mathfrak{R}$ a mapping, called *(left) cylindrification*, such that for every $a \in \mathfrak{B}$, $r \in \mathfrak{R}$ we have

- $\diamond(a^c, 1) = a$, and
- $\diamond(r, 1)^c = r; 1$.

In the full Peirce algebra $\mathfrak{P}(U)$ over a set $U \neq \emptyset$, $(\cdot)^c$ is defined as $A^c = \{ (x, y) : x \in A \}$. The algebraic apparatus of Peirce algebras has been used as an inference mechanism in terminological representation (cf. Brink and Schmidt [5]).

The precise connection between the *DML*-language and Peirce algebras is:

> the modal algebras for the *DML*-language $DML(\Phi)$ are the Peirce algebras generated by a single relation R and the 'propositions' Φ.

To see this, it suffices to show that $\diamond$ and ^{c} are definable in the *DML*-language, and that $\mathsf{do}, \mathsf{ra}, \mathsf{fix}$ and $\mathsf{exp}, \mathsf{con}, ?$ are definable in Peirce algebras generated by R and Φ:

$$\begin{array}{rcccl} \diamond(\alpha, \varphi) & = & \{ x : \exists y ((x, y) \in [\![\alpha]\!]_{\mathfrak{M}} \wedge \mathfrak{M}, y \models \varphi) \} & = & \mathsf{do}(\alpha; \varphi?), \\ \varphi^c & = & \{ (x, y) : \mathfrak{M}, x \models \varphi \} & = & \varphi?; (\delta \cup -\delta), \end{array}$$

and

$$\begin{array}{rclrcl} \mathsf{do}(\alpha) & = & \diamond(\alpha, 1), & \mathsf{exp}(\varphi) & = & R \cap \varphi^{c\smile}, \\ \mathsf{ra}(\alpha) & = & \diamond(\alpha^\smile, 1), & \mathsf{con}(\varphi) & = & R^\smile \cap (\neg\varphi)^{c\smile}, \\ \mathsf{fix}(\alpha) & = & \diamond((\alpha \cap \delta), 1), & \varphi? & = & \delta \cap \varphi^c. \end{array}$$

Given Theorem 6.12, the connection between *DML* and Peirce algebras established here may be interpreted as saying that (the obvious algebraic counterpart of) *DML* completely axiomatizes the identities valid in all representable Peirce algebras over a single relation R.

Of course, as *DML* is closely related to propositional dynamic logic *PDL* (cf. §2), its modal algebras for *DML* are closely related to the *dynamic algebras* $\mathfrak{D} = (\mathfrak{B}, \mathfrak{K}, \diamond)$ of Kozen [20], Pratt [23]. These too are structures that serve to interpret a two-sorted language: propositions are represented in a Boolean algebra $\mathfrak{B}$ as in our case, but relations (or programs) are represented in a *Kleenean algebra* $\mathfrak{K} = (K, \cup, ;, 0, ^*)$, where * is the Kleene star. However Kleenean algebras need not be Boolean ones, and in most definitions they don't include a converse operation $^\smallsmile$. Like Boolean modules dynamic algebras have a projection $\diamond : \mathfrak{K} \times \mathfrak{B} \to \mathfrak{B}$; but in most definitions they are not equipped with any modes.

8 What's next?

Here are some afterthoughts and suggestions for further work.

As I mentioned in §§2, 7, there is a close connection between the *DML*-language and dynamic algebras/logic, with an obvious difference being the absence of the Kleene star * in the *DML*-language. Adding the * to the *DML*-language seems a natural move, but as a corollary to a result of Gargov's (cf. [21]) this will bring up the undecidability to (at least) Σ^1_1.

Of course, another possible extension of the *DML*-language would be to have multiple 'base' relations in the procedural component. This would bring the system closer to the 'general' Peircean algebras of §7 based on multiple relations, and it would also allow us to analyze the interaction between expansions and contractions 'performed' by different agents, and it might even be of some use in modeling the *exchange* of information.

And finally, as pointed out in §3, for some application it may be necessary to be more precise about the structure of the states in *DML*-models, rather than treating them as some kind of 'black boxes'. Using a result by Finger and Gabbay [8], if we have a complete 'local' logic governing what happens inside these boxes, this local system can be amalgamated with the *DML*-language as a global system on top of it—while preserving such properties as completeness. In a similar fashion it may be useful to add (more) structure to the transitions or changes as well. One can think of formalisms involving intricate plans or processes here as an area where this could be of use. Although at this stage I have no clear idea on how this should be implemented, I do think that interesting applications and technical questions will arise from implementations of this idea, and that results similar to the Finger and Gabbay result should be aimed for.

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