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Prefixed Resolution
A Resolution Method for Modal and Description Logics

Carlos Areces, Hans de Nivelle, and Maarten de Rijke*

ILLC, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands. E-mail: {carlos,nivelle,mdr}@wins.uva.nl

Abstract. We provide a resolution-based proof procedure for modal and description logics that improves on previous proposals in a number of important ways. First, it avoids translations into large undecidable logics, and works directly on modal or description logic formulas instead. Second, by using labeled formulas it avoids the complexities of earlier propositional resolution-based methods for modal logic. Third, it provides a method for manipulating so-called assertional information in the description logic setting. And fourth, we believe that it combines ideas from the method of prefixes used in tableaux and resolution in such a way that some of the heuristics and optimizations devised in either field are applicable.

1 Introduction

In this paper we develop a novel *direct* resolution method for modal logics and description logics. Designing resolution methods that can directly (that is, without having to perform translations) be applied to modal logics, received quite a bit of attention in the late 1980s and early 1990s, cf. [12, 17, 8]. In contrast, recent years have witnessed an increase of attention for translation-based resolution calculi for modal (and modal-like) logics; here, one translates modal languages into a large background language (typically first-order logic), and devises strategies that guarantee termination for the fragment corresponding to the original modal language; see [14, 16, 9].

In parallel with these developments, the description logic community has been very active in devising tableaux-based methods. There is some work on devising translation-based resolution methods for description logics [20, 16], but we are not aware of any work on *direct* resolution-based methods for description logics. This is surprising for at least two reasons. First, description logics are closely related to modal logics (see [18, 10]), and, hence, tools in one field can easily be used in the other. Secondly, and more importantly, in contrast with modal logic, the field of description logic has a very strong focus on decision

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methods and computational tools, and, surely, resolution-based methods rather than tableaux-based methods provide the basis for most of today's powerful computational logic tools, and so it seems natural to try and apply the former in the setting of description logics.

Now, *translation-based* approaches to resolution for modal and description logics suffer from the drawback that termination becomes a highly non-trivial task as we are now working within full first-order logic. Existing *direct* resolution methods for modal logics, on the other hand, lack the elegance and efficiency of the original resolution method because they need to perform 'cuts' inside modal operators to achieve completeness.

In this paper we develop a direct resolution method for modal and description logics that retains as much as possible of the lean 'one-rule' character of traditional resolution methods. The key idea introduced here is to use labels to decorate formulas with additional information. These labels encode accessibility relations (or, in description logic terms, roles) as well as worlds (or, objects). The use of labels allows us to avoid the complexities involved in previous proposals for direct resolution methods for modal logics. The intuition is that labels make information explicit as we need it, so that the basic resolution rule only needs to be used 'at the top level.'

The main achievements of this paper can be summarized as follows:

- it proposes a resolution method that does not involve skolemization beyond the use of constants;
- it presents an elegant and direct propositional resolution calculus for classical modal and description logic;
- description logics split information in two kinds: A-boxes which contain assertional information (facts about a particular domain), and T-boxes with terminological knowledge (definitions of derived notions). As far as we know, our proposal is the first one to account for assertional information with a propositional resolution approach;
- our method is hybrid and conservative in more than one sense: it allows one to adopt ideas from different fields and amalgamate them together.

The rest of the paper is organized as follows. Because of space limitations we will restrict our attention to the description logic $\mathcal{ALC}$ and its extension $\mathcal{ALCR}$; but the similarities between $\mathcal{ALC}$ and the basic multi-modal logic $\mathbf{K}_m$ are well-known [18], and they should allow anyone to transfer our results to the modal setting without problems. In Section 2 we provide some basics on description logic, and in Section 3 we present a resolution method for the description logic $\mathcal{ALCR}$. Then, in Section 4 we discuss various extensions of our results, covering both modal and description logics, and in Section 5 we point out links with related work. We conclude with a summary and further questions in Section 6.

2 Basic Issues in Description Logic

In this section we provide some background information on description logics, as well as some basic definitions.

Description logics are a family of specialized languages for the representation and structuring of knowledge, together with efficient methods to perform different 'reasoning tasks.' They are specialized languages related to the KL-ONE system of Brachman and Schmolze [6]. Nowadays, description logics are generally considered to be "variations" of first-order logic—either restrictions or restrictions plus some added operators. On the one hand these variations are motivated by the undecidability of the inference problem for first-order logic, and on the other by a desire to preserve the structure of the knowledge being represented. The main tools used for providing decision methods and studying complexity-theoretic aspects in the area of description logic are based on labeled tableaux.

Let us make things more precise now.

Definition 1 (Signature). Let $\mathcal{L} = \{C_i\} \cup \{R_i\} \cup \{a_i\}$ be a denumerable set of symbols. We will call C_i *atomic concepts*, R_i *atomic roles* and a_i *constants.* $\mathcal{L}$ is called a *signature.*

Definition 2 (Interpretation). Given a signature $\mathcal{L} = \{C_i\} \cup \{R_i\} \cup \{a_i\}$, an *interpretation* $\mathcal{I}$ for $\mathcal{L}$ is a tuple $\mathcal{I} = \langle \Delta, \cdot^{\mathcal{I}} \rangle$, where

- Δ is a non empty set.
- $\cdot^{\mathcal{I}}$ is a function assigning an element $a_i^{\mathcal{I}} \in \Delta$ to each constant a_i; a subset $C_i^{\mathcal{I}} \subseteq \Delta$ to each atomic concept C_i; and a relation $R_i^{\mathcal{I}} \subseteq \Delta \times \Delta$ to each atomic role R_i.

Definition 3 (Concepts and Roles). Given a signature $\mathcal{L}$, each description logic will define a set of *defined concepts* and a set of *defined roles* (usually just called concepts and roles). Table 1 below defines the standard connectives together with their usual names and semantics.

Constructor	Syntax	Semantics
concept name	C	$C^{\mathcal{I}}$
top	$\top$	$\Delta^{\mathcal{I}}$
bottom	$\bot$	$\emptyset$
conjunction	$C_1 \sqcap C_2$	$C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$
disjunction ($\mathcal{U}$)	$C_1 \sqcup C_2$	$C_1^{\mathcal{I}} \cup C_2^{\mathcal{I}}$
negation ($\mathcal{C}$)	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
univ. quant.	$\forall R.C$	$\{d_1 \mid \forall d_2 \in \Delta((d_1, d_2) \in R^{\mathcal{I}} \rightarrow d_2 \in C^{\mathcal{I}})\}$
exist. quant. ($\mathcal{E}$)	$\exists R.C$	$\{d_1 \mid \exists d_2 \in \Delta((d_1, d_2) \in R^{\mathcal{I}} \wedge d_2 \in C^{\mathcal{I}})\}$
role name	R	$R^{\mathcal{I}}$
role conj. ($\mathcal{R}$)	$R_1 \sqcap R_2$	$R_1^{\mathcal{I}} \cap R_2^{\mathcal{I}}$

Table 1. Common operators of description logics

The above semantic definition of $\forall R.C$ and $\exists R.C$ matches the semantic definition of the modal operators $\Box$ and $\Diamond$; the connection was made explicit in [18].

Historically, a number of description logics received a special name; it is customary to define systems by postfixing the names of these original systems with the added operators. The logic $\mathcal{FL}^-$ [5] is defined as the description logic allowing universal quantification, conjunction and unqualified existential quantifications of the form $\exists R.\top$. The logic $\mathcal{AL}$ [19] extends $\mathcal{FL}^-$ with negation of atomic concept names. The names in parentheses in Table 1 are the usual ones for defining extensions. Hence, $\mathcal{ALC}$ is $\mathcal{AL}$ extended with full negation. In the system $\mathcal{ALCR}$ all the other operators in Table 1 can be defined.

In description logics we are interested in performing inferences given certain background knowledge.

Definition 4 (Knowledge Bases). A *knowledge base* Σ is a pair $\Sigma = \langle T, A \rangle$ such that

- T is the T(erminological)-Box, a (possibly empty) set of expressions of the forms $C_1 \sqsubseteq C_2$ or $R_1 \sqsubseteq R_2$ ($C_1, C_2 \in$ Concepts, $R_1, R_2 \in$ Roles)
- A is the A(ssertional)-Box, a (possibly empty) set of expressions of the forms $a{:}C$ or $(a,b){:}R$ ($C \in$ Concepts, $R \in$ Roles, $a, b \in$ Constants).

Definition 5 (Models). Let $\mathcal{I}$ be an interpretation and φ an expression of the kind specified below. Then $\mathcal{I}$ *models* φ (notation: $\mathcal{I} \models \varphi$) if

- $\varphi = C_1 \sqsubseteq C_2$ and $C_1^{\mathcal{I}} \subseteq {C_2}^{\mathcal{I}}$, or
- $\varphi = R_1 \sqsubseteq R_2$ and $R_1^{\mathcal{I}} \subseteq {R_2}^{\mathcal{I}}$, or
- $\varphi = a{:}C$ and $a^{\mathcal{I}} \in C^{\mathcal{I}}$, or
- $\varphi = (a,b){:}R$ and $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$.

Let $\Sigma = \langle T, A \rangle$ be a knowledge base and $\mathcal{I}$ an interpretation, then $\mathcal{I}$ *models* Σ (notation: $\mathcal{I} \models \Sigma$) if for all $\varphi \in T \cup A, \mathcal{I} \models \varphi$.

Definition 6 (Reasoning Tasks). The following are some of the standard reasoning tasks considered for description logic. Let Σ be a knowledge base:

- Subsumption ($\Sigma \models C_1 \sqsubseteq C_2$): check whether for all interpretations $\mathcal{I}$ such that $\mathcal{I} \models \Sigma$ we have $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$.
- Instance Checking ($\Sigma \models a : C$): check whether for all interpretations $\mathcal{I}$ such that $\mathcal{I} \models \Sigma$ we have $a^{\mathcal{I}} \in C^{\mathcal{I}}$.
- Concept Consistency ($\Sigma \not\models C \doteq \bot$): check whether for some interpretation $\mathcal{I}$ such that $\mathcal{I} \models \Sigma$ we have $C^{\mathcal{I}} \neq \{\}$.
- Knowledge Base Consistency ($\Sigma \not\models \bot$): check whether there exists $\mathcal{I}$ such that $\mathcal{I} \models \Sigma$.

Similar tasks can obviously also be defined for roles whenever role definitions have a richer structure than we have considered here.

In this paper we will be concerned with knowledge base consistency, which, in sufficiently strong description logics like $\mathcal{ALC}$ and its extensions, can decide all the other reasoning tasks.

3 Decision Methods for Description Logics

Weak logics like $\mathcal{FL}^-$ have very effective decision methods. For some of the standard reasoning tasks mentioned in Definition 6 these methods are polynomial and only need to perform a structural analysis of the concepts involved (i.e., no "real deduction" is performed). It is interesting to note that at this (low) level of expressive power the different reasoning tasks cannot be mapped into each other. But, of course, once we have full boolean expressive power as in $\mathcal{ALC}$, reasoning tasks like subsumption can be translated into satisfaction queries.

However, even at the level of $\mathcal{ALC}$ there is another dimension which matters: the difference between dealing with assertional information and terminological information. More precisely, assertions increase the expressive power of description logics. The standard connection between description logics and K_m is established at the terminological level. To account for the assertional information the notion of *nominal* or name for a world is needed. See [4] for a recent study of this topic and Section 5 for further comments.

Below we give a resolution proof procedure for the description logic $\mathcal{ALCR}$ that is able to cope with assertional information.

Definition 7 (Weak Negation Form). Define the following rewriting procedure WNF on concepts:

1. $\neg\neg C \overset{\text{WNF}}{\leadsto} C$
2. $\exists R.C \overset{\text{WNF}}{\leadsto} \neg(\forall R.\neg C)$
3. $(C_1 \sqcup \cdots \sqcup C_j) \overset{\text{WNF}}{\leadsto} \neg(\neg C_1 \sqcap \cdots \sqcap \neg C_j)$
4. $\top \overset{\text{WNF}}{\leadsto} \neg(C \sqcap \neg C)$, for C arbitrary
5. $\bot \overset{\text{WNF}}{\leadsto} \neg\top$

For any concept C, WNF converges to a unique normal form which we denote as WNF(C). WNF(C) is logically equivalent to C. WNF can trivially be extended to expressions $a : C_1$ by setting $\text{WNF}(a : C_1) = a : \text{WNF}(C_1)$. If we interpret $\sqcup$, $\exists R.C$, $\top$ and $\bot$ as defined operators, then WNF is slightly more than an expansion of definitions.

Definition 8 (Clause). Given an infinite set of labels L disjoint from Constants, a *clause* is a set Cl such that each element of Cl is either

- a concept assertion of the form $t{:}C$ where t is either a constant or a label in L.
- a role assertion of the form $(t_1, t_2) : R$, where t_1, t_2 are either constants or labels in L.

We will use the notation $t{:}C$ (with possible subindices) for concept assertions and $(t_1, t_2){:}R$ (with subindices) for role assertions, and $\bar{t}{:}N$ for both of them.

A formula in a clause is a *literal* if it is either a role assertion, a concept or negated concept assertion on an atomic concept, or a universal or negated universal concept assertion.

Definition 9 (Model for a Clause and a Set of Clauses). Notice that formulas in a clause are simply assertions over an expanded set of constants. Let Cl be a clause, and $\mathcal{I} = \langle \Delta, \cdot^{\mathcal{I}} \rangle$ a model in the expanded signature; we put $\mathcal{I} \models Cl$ if $\mathcal{I} \models \bigvee Cl$. A set of clauses S *has a model* if there is model $\mathcal{I}$ such that for all $Cl \in S$, $\mathcal{I} \models Cl$.

Definition 10 (Set of Clauses of a Knowledge Base). Let $\Sigma = \langle T, A \rangle$ be a knowledge base (with non-cyclic definitions). It is known that Σ can be transformed into an "unfolded" equivalent knowledge base $\Sigma' = \langle \emptyset, A' \rangle$ where all concept and role assertions use only atomic concept and role symbols [11].

The set S_Σ of clauses corresponding to Σ is the smallest set such that

- if $a : C_1 \sqcap \cdots \sqcap C_n = \mathrm{WNF}(a : C)$ $(n \geq 1)$ for $a : C \in A'$ then $\{a : C_i\} \in S_\Sigma$, for $1 \leq i \leq n$.
- if $(a, b) : R_1 \sqcap \cdots \sqcap R_n \in A'$ then $\{(a, b) : R_i\} \in S_\Sigma$, for $1 \leq i \leq n$.

Notice that the "unfolded" assertions of A' are used in this translation. Furthermore, in S_Σ we can identify a (possibly empty) subset of clauses RA of the form $\{(a, b) : R\}$ which we call *role assertions*, and for each constant a a (possibly empty) subset CA_a of clauses of the form $\{a : C\}$ which we call *concept assertions*.

Because of the format of a knowledge base it is impossible to find in S_Σ *mixed* clauses containing both (in disjunction) concept and role assertions. Furthermore there are no disjunctive concept assertions on different constants, i.e., there is no clause Cl in S_Σ such that $Cl = Cl' \cup \{a : C_1\} \cup \{b : C_2\}$ for $a \neq b$. These properties will be relevant in the first steps of the completeness proof.

Proposition 1. *Let Σ be a knowledge base and S_Σ its corresponding set of clauses. Then Σ is satisfiable iff S_Σ is satisfiable.*

$$(\sqcap)\ \frac{Cl \cup \{\bar{t} : N_1 \sqcap N_2\}}{\begin{array}{c} Cl \cup \{\bar{t} : N_1\} \\ Cl \cup \{\bar{t} : N_2\} \end{array}} \qquad (\neg\sqcap)\ \frac{Cl \cup \{t : \neg(C_1 \sqcap C_2)\}}{Cl \cup \{t : \mathrm{WNF}(\neg C_1), t : \mathrm{WNF}(\neg C_2)\}}$$

$$(\mathrm{RES}) \qquad \frac{Cl_1 \cup \{\bar{t} : N\} \quad Cl_2 \cup \{\bar{t} : \neg N\}}{Cl_1 \cup Cl_2}$$

$$(\forall) \qquad \frac{Cl_1 \cup \{t_1 : \forall R.C\} \quad Cl_2 \cup \{(t_1, t_2) : R\}}{Cl_1 \cup Cl_2 \cup \{t_2 : C\}}$$

$$(\neg\forall) \qquad \frac{Cl \cup \{t : \neg\forall R.C\}}{\begin{array}{c} Cl \cup \{(t, n) : R\} \\ Cl \cup \{n : \mathrm{WNF}(\neg C)\} \end{array}}, \text{ where } n \text{ is new.}$$

(notice that $(\sqcap)$ also covers role conjunction and that $(\neg\forall)$ is a mild kind of skolemnization which only involves the introduction of constants).

Table 2: The Resolution Rules

Table 2 shows the resolution rules we will consider.

Definition 11 (Deduction). A *deduction* of a clause Cl from a set of clauses S is a finite sequence $S_1, \ldots, S_n$ of sets of clauses such that $S = S_1$, $Cl \in S_n$ and each S_i (for $i > 1$) is obtained from S_{i-1} by adding the consequent clauses of the application of one of the resolution rules in Table 2 to clauses in S_{i-1}. Cl is a *consequence* of S if there is a deduction of Cl from S. A deduction of $\{\}$ from S is a *refutation* of S.

Before proving soundness, completeness and termination we present a simple example of resolution in our system.

Example 1. Consider the following description. Ignoring some fundamental genetic laws, suppose that children of tall people are blond (1). Furthermore, all Tom's daughters are tall (2), but he has a non-blond grandchild (3). Can we infer that Tom has a son (4)?

$$
\begin{array}{ll}
(0) & \mathsf{female} \doteq \neg\mathsf{male} \\
(1) & \mathsf{tall} \sqsubseteq \forall\mathsf{CHILD}.\mathsf{blond} \\
(2) & \mathsf{t}:\forall\mathsf{CHILD}.(\neg\mathsf{female} \sqcup \mathsf{tall}) \\
(3) & \mathsf{t}:\exists\mathsf{CHILD}.\exists\mathsf{CHILD}.\neg\mathsf{blond} \\
\hline
(4) & \mathsf{t}:\exists\mathsf{CHILD}.\mathsf{male}
\end{array}
$$

As is standard, we use a new proposition letter $\mathsf{rest\text{-}tall}$ to complete the partial definition in (1) and we resolve with the negation of the formula we want to infer. After unfolding and applying WNF we obtain the following three clauses

1. $\{\mathsf{t}:\forall\mathsf{CHILD}.\neg(\neg\mathsf{male} \sqcap \neg((\forall\mathsf{CHILD}.\mathsf{blond}) \sqcap \mathsf{rest\text{-}tall}))\}$
2. $\{\mathsf{t}:\neg\forall\mathsf{CHILD}.\forall\mathsf{CHILD}.\mathsf{blond}\}$
3. $\{\mathsf{t}:\forall\mathsf{CHILD}.\neg\mathsf{male}\}$

Now we start resolving,

4. $\{\mathsf{t}':\neg\forall\mathsf{CHILD}.\mathsf{blond}\}$	by $(\neg\forall)$ in 2.
5. $\{(\mathsf{t},\mathsf{t}'):\mathsf{CHILD}\}$	by $(\neg\forall)$ in 2.
6. $\{\mathsf{t}':\neg\mathsf{male}\}$	by $(\forall)$ in 3.
7. $\{\mathsf{t}':\neg(\neg\mathsf{male} \sqcap \neg((\forall\mathsf{CHILD}.\mathsf{blond}) \sqcap \mathsf{rest\text{-}tall})\}$	by $(\forall)$ in 1.
8. $\{\mathsf{t}':\mathsf{male}, \mathsf{t}':((\forall\mathsf{CHILD}.\mathsf{blond}) \sqcap \mathsf{rest\text{-}tall})\}$	by $(\neg\sqcap)$ in 7.
9. $\{\mathsf{t}':((\forall\mathsf{CHILD}.\mathsf{blond}) \sqcap \mathsf{rest\text{-}tall})\}$	by (RES) in 6 and 8.
10. $\{\mathsf{t}':\forall\mathsf{CHILD}.\mathsf{blond}\}$	by $(\sqcap)$ in 9.
11 $\{\mathsf{t}':\mathsf{rest\text{-}tall}\}$.	by $(\sqcap)$ in 9.
12. $\{\}$	by (RES) in 4 and 10.

Theorem 1 (Soundness). *The resolution rules described in Table 2 are sound. That is, if Σ is a knowledge base, then S_Σ has a refutation only if Σ is unsatisfiable.*

Proof. We will prove that the resolution rules we introduced preserve satisfiability. That is, given a rule, if the premises are satisfiable, then so are the conclusions. We only discuss $(\neg\forall)$.

Let $\mathcal{I}$ be a model of the antecedent. If $\mathcal{I}$ is a model of Cl we are done. If $\mathcal{I}$ is a model of $t:\neg\forall R.C$, then there exists d in the domain, such that $(t^{\mathcal{I}}, d) \in R^{\mathcal{I}}$ and

$d \in \neg C^{\mathcal{I}}$. Let $\mathcal{I}'$ be identical to $\mathcal{I}$ except perhaps in the interpretation of n where $n^{\mathcal{I}'} = d$. As n is a new label, also $\mathcal{I}' \models t{:}\neg\forall R.C$. But now $\mathcal{I}' \models Cl \cup \{(t,n){:}R\}$ and $\mathcal{I}' \models Cl \cup \{n{:}\mathrm{WNF}(\neg C)\}$. □

Our next aim is to prove completeness. We follow the approach used in [12]: given a set of clauses S we aim to define a structure T_S such that

(†) if S is satisfiable, a model can be effectively constructed from T_S; and
(††) if S is unsatisfiable, a refutation can be effectively constructed from T_S.

But in our case this proves to be more difficult than in [12] because we have to deal with A-Box information, that is, with named objects or worlds (concept assertions) and fixed constraints on relations (role assertions). We will proceed in stages. To begin, we will obtain a first structure to account for named worlds and their fixed relation constraints. After that we can use a simple generalization of results in [12]. We base our construction on trees which will help in guiding the construction of the corresponding refutation proof.

Let Σ be a knowledge base and S_Σ its corresponding set of clauses. Let a be a constant and CA_a the subset of CA of concept assertions concerning the constant a. Define the following operation to be performed on CA_a.

We construct for each CA_a a binary tree T_a inductively. Let the original tree u consist of the single node CA_a and repeat the following operations in an alternating fashion.

Operation A1. Repeat the following steps as long as possible:
- choose a leaf w. Replace any clause of the form $\{a : \neg(C_1 \sqcap C_2)\}$ by $\{a : \mathrm{WNF}(\neg C_1),\ a : \mathrm{WNF}(\neg C_2)\}$; and any clause of the form $\{a : C_1 \sqcap C_2\}$ by $\{a{:}C_1\}$ and $\{a{:}C_2\}$.

Operation A2. Repeat the following steps as long as possible:
- choose a leaf w of u and a clause Cl in w of the form $Cl = \{a{:}C_1, a{:}C_2\} \cup Cl'$;
- add two children w_1 and w_2 to w, where $w_1 = w \backslash \{Cl\} \cup \{\{a : C_1\}\}$ and $w_2 = w \backslash \{Cl\} \cup \{\{a{:}C_2\} \cup Cl'\}$.

The leaves of T_a give us the possibilities for "named worlds" in our model (remember that concept prefixes act as names for worlds/objects). We can view each leaf as a set S_a^j, representing a possible configuration for world a.

Proposition 2. *Operation A (the combination of A1 and A2) terminates, and upon termination*

1. *all the leaves S_a^1 to S_a^n of the tree are sets of unit literal clauses;*
2. *if all $S_a^1, \ldots, S_a^n$ are refutable, then CA_a is refutable;*
3. *if one S_a^j is satisfiable, then CA_a is satisfiable.*

Proof. Termination is trivial. Item 1 holds by virtue of the construction, and item 2 is proved by induction on the depth of the tree. We need only realize that by simple propositional resolution if the two children of a node w are refutable, then so is w. Item 3 is also easy. Informally, Operation A "splits" disjunctions and "carries along" conjunctions. Hence if some S_a^j has a model we have a model satisfying all conjuncts in CA_a and at least one of each disjuncts. □

We should now consider the set RA of role assertions. Let NAMES be the set of constants which appear in Σ. If a is in NAMES but CA_a is empty in S_Σ, define $S_a^1 = \{\{a:C, a:\neg C\}\}$ for some concept C. We will construct a set of sets of nodes $\mathcal{N} = \{N_i \mid N_i$ contains exactly one leaf of each $T_a\}$. Each N_i is a possible set of constraints for the named worlds in a model of S_Σ.

Proposition 3. *If for all i, $\bigcup N_i \cup RA$ is refutable, then so is S_Σ.*

Proof. If for all i, $\bigcup N_i \cup RA$ is refutable, then for some constant a we have that for all S_a^j obtained from CA_a, $S_a^j \cup RA$ is refutable. Hence by Proposition 2, $CA_a \cup RA$ is refutable, and so is S_Σ. □

For all i, we will now extend each set in N_i with further constraints. For each $S_a \in N_i$, start with a node w_a labeled by S_a .

Operation B1. Equal to *Operation A1.* *Operation B2.* Repeat the following steps as long as possible: – choose nodes w_a, w_b such that $\{(a,b):R\}$ in RA, $\{a:\forall R_i.C_i\} \in w_a$, $\{b:C_i\} \notin w_b$, where w_b is without children; – add a child to w_b, $w'_b = w_b \cup \{\{b:C_i\}\}$.

Call N_i^* the set of all leaves obtained from the forest constructed in B.

Proposition 4. *Operation B terminates, and upon termination*

1. *all nodes created are derivable from $\bigcup N_i \cup RA$, and hence if a leaf is refutable so is $\bigcup N_i \cup RA$;*
2. *if some $\bigcup N_i^*$ is satisfiable, then S_Σ is satisfiable.*

Proof. To prove termination, notice that in each cycle the quantifier depth of the formulas considered decreases. Furthermore, it is not possible to apply twice the operation to a node named by a and b and a formula $a:\forall R_i.C_i$.

As to item 1, each node is created by an application of the $(\forall)$ rule to members of $N_i \cup RA$ or clauses previously derived by such applications. To prove item 2, let $\mathcal{I}$ be a model of N_i^*. Define a new model $\mathcal{I}' = \langle \Delta', \cdot^{\mathcal{I}'} \rangle$ as follows.

- $\Delta' = \Delta$;
- $a^{\mathcal{I}'} = a^{\mathcal{I}}$ for all constants a;
- $C^{\mathcal{I}'} = C^{\mathcal{I}}$ for all atomic concepts C; and
- $R^{\mathcal{I}'} = R^{\mathcal{I}} \cup \{(a^{\mathcal{I}}, b^{\mathcal{I}}) \mid \{(a,b):R\} \in RA\}$.

Observe that $\mathcal{I}'$ differs from $\mathcal{I}$ only in an extended interpretation of role symbols. By definition, $\mathcal{I}' \models RA$. It remains to prove that $\mathcal{I}' \models CA$. By Proposition 2, we are done if we prove that $\mathcal{I}' \models \bigcup N_i^*$. Now, since we only expanded the interpretation of relations, $\mathcal{I}$ and $\mathcal{I}'$ can only disagree on universal concepts of the form $a:\forall R.C$. By induction on the quantifier depth we prove this to be false.

Assume that $\mathcal{I}$ and $\mathcal{I}'$ agree on all formulas of quantifier depth less than n, and let $a:\forall R.C$ be of quantifier depth n, for $\{a:\forall R.C\} \in S_a^*$. Suppose

$\mathcal{I}' \not\models \forall R.C$. This holds iff there exists b such that $(a^{\mathcal{I}'}, b^{\mathcal{I}'}) \in R^{\mathcal{I}'}$ and $\mathcal{I}' \not\models b\!:\!C$. By the inductive hypothesis, $\mathcal{I} \not\models b : C$. Now, if $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$ we are done. Otherwise, by definition $\{(a,b)\!:\!R\} \in RA$. But then $\{b\!:\!C\} \in S_b^*$ by construction and as $\mathcal{I} \models S_b^*$, we also have $\mathcal{I} \models b\!:\!C$—a contradiction. □

As we said above, each N_i^* represents the "named core" of a model of S. The final step is to define the non-named part of the model. The following operations are performed to each set in each of the N_i^* obtaining in such a way a forest F_i.

Fix N_i^*, and a. We construct a tree "hanging" from the corresponding $S_a^* \in N_i^*$. The condition that each node of the tree is named by either a constant of a new label (that is, all the formulas have the same prefix) will be preserved as an invariant during the construction. Set the original tree u to S_a^* and repeat the following operations C1, C2 and C3 in succession until the end-condition holds.

Operation C1. Equal to *Operation A1.*

Operation C2. Equal to *Operation A2.*

Operation C3. For each leaf w of u,
- if for some concept we have $\{C\}, \{\neg C\} \in w$, do nothing;
- otherwise, since w is a set of unit clauses, we can write $w = \{\{t : C_1\}, \ldots, \{t\!:\!C_m\}, \{t\!:\!\forall R_{k_1}.A_1\}, \ldots, \{t\!:\!\forall R_{k_n}.A_n\}, \{t\!:\!\neg\forall R_{l_1}.P_1\}, \ldots, \{t\!:\!\neg\forall R_{l_q}.P_q\}\}$. Form the sets $w_i = \{\{\mathrm{WNF}(t' : \neg P_i)\}\} \cup S_i$, where t' is a new label, and $S_i = \{\{t'\!:\!A_h\} \mid \{t\!:\!\forall R_i.A_h\} \in w\}$, and append each of them to w as children marking the edges as R_i links. The nodes w_i are called the *projections* of w.

End-condition. Operation C3 is inapplicable.

Proposition 5. *Operation C cannot be applied indefinitely.*

Definition 12. We call nodes to which Operation C1 or C2 has been applied of type 1, and those to which Operation C3 has been applied of type 2. The set of *closed nodes* is recursively defined as follows,

- if for some concept $\{t\!:\!C\}, \{t\!:\!\neg C\}$ are in w then w is closed.
- if w is of type 1 and all its children are closed, w is closed.
- if w is of type 2 and some of its children is closed, w is closed.

Let F_i be a forest that is obtained by applying Operations C1, C2, and C3 to N_i^* as often as possible. Then F_i is *closed* if any of its roots is closed.

Lemma 1. *If one of the forest F_i obtained from S_Σ is non-closed, then S_Σ has a model.*

Proof. Let F_i be a non-closed forest. By a simple generalization of the results in [12, Lemma 2.7] we can obtain a model $\mathcal{I} = \langle \Delta, \cdot^{\mathcal{I}} \rangle$ of all roots S_a^* in F_i, from the trees "hanging" from them, ie., a model of $\bigcup N_i^*$. By Proposition 4, S_Σ has a model. □

Lemma 1 establishes the property (†) we wanted in our structure T_S. To establish (††) we need a further auxiliary result.

Proposition 6. *Let w be a node of type 2. If one of its projections w_i is refutable, then so is w.*

Proof. Let w be a set of unit clauses $w = \{\{t\!:\!C_1\}, \ldots, \{t\!:\!C_m\}, \{t\!:\!\forall R_{k_1}.A_1\}, \ldots, \{t\!:\!\forall R_{k_n}.A_n\}, \{t\!:\!\neg\forall R_{l_1}.P_1\}, \ldots, \{t\!:\!\neg\forall R_{l_q}.P_q\}\}$. And let w_i be its refutable projection: $w_i = \{\{\mathrm{WNF}(t'\!:\!\neg P_i)\}\} \cup S_i$, where t' is a new label, and $S_i = \{\{t'\!:\!A_h\} \mid \{t\!:\!\forall R_i.A_h\} \in w\}$. We use resolution on w to arrive to the clauses in w_i from which the refutation can carried out: Apply $(\neg\forall)$ to $\{t\!:\!\neg\forall R_i.P_i\}$ in w to obtain $\{t' : \mathrm{WNF}(t' : \neg P_i)\}$ and $\{(t,t') : R_i\}$. Now apply $(\forall)$ to all the clauses $\{t\!:\!\forall R_i.A_h\}$ in w to obtain $\{t'\!:\!A_h\}$. □

Lemma 2. *In a forest F_i, every closed node is refutable.*

Proof. For w a node in F_i, let $d(w)$ be the longest distance from w to a leaf.

If $d(w) = 0$, then w is a leaf, thus for some concept C, $\{t\!:\!C\}$ and $\{t\!:\!\neg C\}$ are in w. Using (RES) we immediately derive $\{\}$.

For the induction step, suppose the proposition holds for all w' such that $d(w') < n$ and that $d(w) = n$. If w is of type 1, let $w_1 = w \setminus \{Cl\} \cup \{Cl_1\}$ and $w_2 = w \setminus \{Cl\} \cup \{Cl_2\}$ be its children. By the inductive hypothesis there is a refutation for w_1 and w_2. By propositional resolution there is a refutation of w: repeat the refutation proof for w_2 but starting with w, instead of the empty clause we should obtain a derivation of Cl_2, now use the refutation of w_2.

Suppose w is of type 2. Because w is closed, one of its projections is closed. Hence, by the inductive hypothesis it has a refutation. By Proposition 6, w itself has a refutation. □

Theorem 2 (Completeness). *The resolution method described above is complete: if Σ is a knowledge base, then S_Σ is refutable whenever Σ is unsatisfiable.*

Proof. It is only necessary to put together the previous pieces. If Σ does not have a model then, by Proposition 1, there is no model for S_Σ. Hence by Lemma 1 all the forests F_i obtained from S_Σ are closed, and by Lemma 2, for each N_i^*, one of the sets $S_{a_j}^*$ is refutable. By Proposition 4, for all i, $\bigcup N_i \cup RA$ is refutable. By Proposition 3, S_Σ is refutable. □

Because we have shown how to *effectively* obtain a refutation from an inconsistent set of clauses we have also established termination. Notice that during the completeness proof we have used a *specific strategy* in the application of the resolution rules (for example, the $(\neg\forall)$ rule is never applied twice to the same formula).

Theorem 3 (Termination). *Given a knowledge base Σ, the resolution method (with the strategy described above) terminates with answer **YES** if Σ is inconsistent and with answer **NO** otherwise.*

As a corollary of the results above, we obtain soundness, completeness and termination of our resolution method for $\mathbf{K}_m$. Notice that this is really a weaker result than the ones proved above, since we don't have to bother about assertional A-Box information when dealing with $\mathbf{K}_m$. When using our resolution method for $\mathbf{K}_m$ the prefix labels are really metalogical entities and not part of the logic. We will discuss this matter further in Section 5.

4 Extensions and Variations

In addition to the basic results in Section 3, we will now discuss some extensions and variations. Because of space constraints we provide few details.

Modal Extensions. The natural step, from a classical modal point of view, is to consider systems above $\mathbf{K}_m$. We choose systems **T**, **D**, and **4** as examples. Each system is defined as an extension of the basic system **K** by the addition of an axiom scheme which characterizes certain properties of the accessibility relation:

Name	Axiom scheme	Accessibility Relation
T	$p \to \Diamond p$	reflexivity: $\forall x.xRx$
D	$\Box p \to \Diamond p$	seriality: $\forall x \exists y.xRy$
4	$\Diamond\Diamond p \to \Diamond p$	transitivity: $\forall xyz.(xRy \wedge yRz \to xRz)$

Corresponding to each of the axioms we add a new resolution rule.

$$(\mathbf{T}_i) \qquad \frac{Cl \cup \{t : \forall R_i.C\}}{Cl \cup \{t : C\}}$$

$$(\mathbf{D}_i) \qquad \frac{Cl \cup \{t : \forall R_i.C\}}{Cl \cup \{t : \neg\forall R_i.\mathrm{WNF}(\neg C)\}}$$

$$(\mathbf{4}_i) \qquad \frac{Cl_1 \cup \{t_1 : \forall R_i.C\} \quad Cl_2 \cup \{(t_1, t_2) : R_i\}}{Cl_1 \cup Cl_2 \cup \{t_2 : \forall R_i.C\}}$$

Of course, because we are in a multi-modal formalism, these rules can be specified for any particular relation R_i. From the description logic point of view these extensions can be understood as forcing certain properties on a specific relation. There exist description logics which permit the definition of the reflexive-transitive closure of a relation (R^*). Seriality is related to functionality of roles, another feature common in description logic formalisms.

Soundness for these systems is immediate:

Theorem 4. *The resolution methods obtained by adding the rules* (**T**), (**D**) *and* (**4**) *for a particular relation R_i, are sound with respect to the class of knowledge bases where the relation R_i is always interpreted as reflexive, serial and transitive, respectively.*

For completeness and termination we should modify the construction we defined previously (in particular ($\mathbf{4}_i$) needs a mechanism of cycle detection); this can be done again using methods from [12].

Theorem 5. *The resolution methods obtained by adding the rules* (**T**), (**D**) *and* (**4**) *for a particular relation R_i, are complete and terminate with respect to the class of knowledge bases where the relation R_i is always interpreted as reflexive, serial and transitive, respectively.*

DL Extensions. In the description logic community one considers a kind of extensions of the language that is different from the ones we already introduced. For instance, recently in [7] some attention has been given to n-ary relations in description logics (in modal logic terms, n-dimensional modal operators). Our approach seems to generalize without further problems to account for this.

Finally, another direction for extensions is to consider additional structure on roles. We have limited ourselves to conjunction, but disjunction, negation, composition, etc. can be considered. Description logics allowing these operations are known as *very expressive description logics*, and their worst case complexity is high, even though they perform well in some limited cases; their modal logic counterparts are related to dynamic logics based on PDL. For a translation based resolution treatment of these, see [16].

5 Related Work

The Connection with Resolution for Modal Logics. Resolution methods for modal logics (without translation) have been investigated before [13, 17, 12, 8]. The innovation introduced in this paper is in the use of labels. We think this is the key to simplify the complexities involved in previous proposals. Previously, resolution had to be performed "inside" modalities (in a similar way as how new tableaux had to be started in non-prefixed tableaux systems). Labels allow us to make information explicit and resolution can then always be performed at the "top level." Because we have labels available, we can also deal with properties on relations—like reflexivity, seriality or transitivity—in a tableaux-like fashion, and a single new rule is all that is necessary to account for them.

Comparison with the Tableaux Method. Once labels are introduced the resolution method is very close to the tableaux approach, but we are still doing resolution. The rules ($\sqcap$), ($\neg\sqcap$) and ($\neg\forall$), prepare formulas to be "fed" into the resolution rules (RES) and ($\forall$).[1] And the aim is still to derive the empty clause instead of finding a model by exhausting a branch.

But, is this method any better than tableaux? We don't think this is the correct question to ask. We believe that we learn different things from studying different methods. For example, [15] studies a number of interesting optimizations of the tableaux implementation which were tested on the tableaux based theorem prover DLP. Some of their ideas were already incorporated in our resolution method (lexical normalization and early detection of clashes), and others might perhaps be used in implementations of our method. But what is perhaps more interesting to the description logic community, is that new optimizations, specific to the resolution approach, can now be exploited.

Strategies for Modal Resolution. In implementations of the resolution algorithm, strategies for selecting the resolving pairs are critical. Heuristics for the case of modal logics have been investigated in [1]. Some of their results extend to our framework, and in certain cases proofs are simpler because of our explicit use of resolution via labels. We cannot give a full description of this issue here.

[1] ($\forall$) is added to account for the "hidden" negation in the guard of the quantifier.

Assertional Information and Hybrid Logics. There is a final topic on which we would like to comment: the relation between nominals and assertional information. The similarity between $\mathcal{ALC}$ and the basic multi-modal logic $\mathbf{K}_m$ is well-known. But this connection concerns the terminological part of $\mathcal{ALC}$. Recent work on nominals and hybrid languages [4] explains how assertions enter the picture. This paper investigates $\mathcal{ALCNO}$, $\mathcal{ALC}$ plus counting, plus the set formation operator in terms of individuals: $\mathcal{O}(a_1, \ldots, a_n)^{\mathcal{I}} = \{a_1^{\mathcal{I}}, \ldots, a_n^{\mathcal{I}}\}$, embedding this logic in a very expressive hybrid formalism $\mathcal{H}(\forall)$. To account for $\mathcal{ALC}$ (including assertions) a subset of a weaker system called $\mathcal{H}(@)$ is enough. For this language, labeled tableaux appear as a very natural choice [3]; however, by using our labeled calculus, resolution has become just as natural a choice.

6 Conclusions and Further Work

In this paper we have provided a propositional resolution method for deciding knowledge base consistency for $\mathcal{ALCR}$. This result is further extended to account for reflexive, serial and transitive relations. Because of the connection between $\mathcal{ALC}$ and $\mathbf{K}_m$, our methods can also be used as resolution methods for deciding theoremhood of modal logics. Due to space limitations, only the basics of related issues such as more expressive description logics, and optimized strategies where discussed. These issues are being dealt with in the full version of the paper.

There is a number of important questions which are still open at this stage of our research. First, up to now we have no implementation, but this issue is high on our agenda. We believe that the ideas behind our resolution method are simple enough so that even adapting already available provers should not prove to be a very difficult task.

Further, a very attractive idea which matches nicely with the resolution approach is to incorporate a limited kind of subsumption on universal prefixes to account for "on the fly" unfolding of terminological definitions. The use of such "universal labels" should make it unnecessary to perform a complete unfolding of the knowledge base as a pre-processing step: The leitmotiv would be "perform expansion by definitions only when needed in deduction." On the fly unfolding has already been implemented in tableaux based systems like KRIS [2].

As to the complexity of resolution: we have not attempted to formally establish the complexity of our resolution method so far. We conjecture that a PSPACE heuristic for prefixed resolution exists, even though in this first account the naïve heuristic we have introduced requires exponential space.

Finally, our completeness proof is constructive: when a refutation cannot be found we can actually define a model for the knowledge base. Hence, our method can also be used for model extraction. How does this method perform in comparison with traditional model extraction from tableaux systems?

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