Interpolation and Bisimulation in Temporal Logic

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Abstract

Building on recent model theoretic results for Since-Until logics we define an adequate notion of bisimulation and establish general theorems concerning the interpolation property. Using these general results we prove that the basic SU-logic and any SU-logic whose class of frames can be defined by universal Horn formulas have interpolation. In particular, the SU-logic of branching time has interpolation, while linear time fails to have this property.

1 Introduction

For many years, modal logic was viewed as an extension of propositional logic by the addition of new modalities $\Diamond$ and $\Box$. Nowadays, the picture has changed in many ways. First, modal logic is no longer seen as just an extension of propositional logic but also as a restriction of first-order logic (FOL) — when formulas are interpreted over models, or second-order logic (SOL) — when formulas are considered on frames. Furthermore, $\Diamond$ and $\Box$ have lost their privileged position as a wide variety of new modalities have been introduced in the last years, witness for instance the work on Since-Until Logics [7], the universal modality [8], the difference operator [16], counting modalities [9], arrow logics [20].[1]

The source of the first change is probably Johan van Benthem [5] and his introduction of the standard translation mapping classical modal formulas to FOL (or SOL) formulas, and the corresponding notion of bisimulation as the modal equivalent of the notion of partial isomorphism known from classical model theory. With the aid of bisimulations a number of important model theoretic results for classical modal logic were derived [17, 10].

[1] In this paper we will refer to the modal logic of $\Diamond$ and $\Box$ as classical modal logic.

The natural next step is to try and reproduce these results for the other modal logics we mentioned. The work for SU-logics was started by Kurtonina and de Rijke in [11]. In that paper the authors provide a notion of bisimulation for SU-logics with the standard relational semantics, and they establish a characterization result and a number of separation and preservation results. It turns out, though, that the standard relational semantics is not completely adequate (witness the absence of canonical models and of an appropriate algebraic counterpart). Bellisima and Cittadini [4] have introduced a new semantics with a dual algebraic construction, and a Stone-like representation theorem and a canonicity result have been proved. Much of the present paper is based on this new semantics. We start by defining the appropriate notion of bisimulation and then focus on the use of bisimulations to prove general results about the interpolation property, leaving other model theoretic considerations for further research. Recent results concerning interpolation by means of bisimulations can be found in [2, 14] for classical modal logic and in [3, 19] for infinitary classical modal logic. In contrast, surprisingly little is known about interpolation (and meta-logical properties in general) for SU-logics.

In what follows, we assume basic knowledge of modal logic. In Section 2 we summarize the required definitions and results about the interpolation property. In Section 3 we recall two general results about interpolation and failure of interpolation for classical modal logic, and we extend these results to SU-logics in Section 4. In Section 5 we apply these two theorems to special cases and establish interpolation for the basic SU-logic K_{SU} and for any SU-logic whose class of frames can be defined by universal Horn clauses in FOL. An important case covered by this result is branching time (the class of frames where the accessibility relation is a partial order). We also prove that linear time fails to have interpolation. Finally, in Section 6 we comment on the results obtained and discuss further directions of research.

2 The Interpolation Property

The interpolation property (IP) is an important meta-logical notion. Originally, IP was a syntactic property of a given deductive system. Syntactically, a deductive system has the IP if whenever $A \vdash B$ then there exists a formula C in the common language of A and B such that $A \vdash C$ and $C \vdash B$. As a syntactic property, the IP is a sign of a well behaved deductive system. It amounts to the fact that when proving B from A, intermediate lemmas can be restricted only to the common language. Obviously, once a soundness and completeness result for a given logic is obtained, the IP can also be established by semantic means and this is perhaps the standard approach nowadays [6].

Besides its wide use in the area of automated theorem proving (see for instance [1, 18]), the interpolation property has turned to be interesting in fields like software engineering where it can be used to proved certain modularity properties of the specification of a system [12, 15].

Our approach to the IP is purely semantical. For K a class of models (say of FOL), let $\models_{\mathsf{K}}$ be the standard semantic consequence relation for K: for $\Phi \cup \{\psi\}$ a set of sentences, $\Phi \models_{\mathsf{K}} \psi$ if all models of Φ in K are also models of ψ. As usual, for $\Phi = \{\varphi\}$ we use $\varphi \models_{\mathsf{K}} \psi$, and for $\emptyset \models_{\mathsf{K}} \psi$ we use $\models_{\mathsf{K}} \psi$.

Among others, the following definitions of interpolation can be found in the literature. Let $\mathbb{P}(\varphi)$ be the set of atomic symbols occurring in φ (proposition variables in modal logic, relation symbols in FOL) and let $\mathcal{L} = \mathrm{Th}(\mathsf{K})$ be the theory of a class of models K.

AIP $\mathcal{L}$ has the *Arrow Interpolation Property* (AIP) if, whenever $\models_{\mathsf{K}} \varphi \to \psi$, there exists a formula θ such that $\models_{\mathsf{K}} \varphi \to \theta$, $\models_{\mathsf{K}} \theta \to \psi$ and $\mathbb{P}(\theta) \subseteq \mathbb{P}(\varphi) \cap \mathbb{P}(\psi)$.

TIP $\mathcal{L}$ has the *Turnstile Interpolation Property* (TIP) if, whenever $\varphi \models_{\mathsf{K}} \psi$, there exists a formula θ such that $\varphi \models_{\mathsf{K}} \theta$, $\theta \models_{\mathsf{K}} \psi$ and $\mathbb{P}(\theta) \subseteq \mathbb{P}(\varphi) \cap \mathbb{P}(\psi)$.

SIP $\mathcal{L}$ has the *Splitting Interpolation Property* (SIP) if, whenever $\varphi_0 \wedge \varphi_1 \models_{\mathsf{K}} \psi$, there exists a formula θ such that $\varphi_0 \models_{\mathsf{K}} \theta$, $\varphi_1 \wedge \theta \models_{\mathsf{K}} \psi$ and $\mathbb{P}(\theta) \subseteq \mathbb{P}(\varphi_0) \cap (\mathbb{P}(\varphi_1) \cup \mathbb{P}(\psi))$.

For FOL the above definitions are all equivalent but in general this is not the case (depending on both compactness and the availability of a deduction theorem in the logic). The meaning of TIP and SIP in modal logic depends on the way we define the consequence relation $\varphi \models_{\mathsf{K}} \psi$. There are two options: a local and a global one (cf. [5, 14] for a discussion of their relative merits.) Let K be a class of frames.

- The *local consequence relation* $\Phi \models^{loc}_{\mathsf{K}} \psi$ holds if for every $\mathcal{F} \in \mathsf{K}$, every valuation V and every world w in F, $(\mathcal{F}, V), w \models \Gamma$ implies $(\mathcal{F}, V), w \models \psi$,
- the *global consequence relation* $\Gamma \models^{glo}_{\mathsf{K}} \psi$ holds if for every $\mathcal{F} \in \mathsf{K}$ and every valuation V, $(\mathcal{F}, V) \models \Gamma$ implies $(\mathcal{F}, V) \models \psi$.

The global relation is the one familiar from first-order logic, but it is always defined for sets of *sentences* Φ (if they are formulas, the universal closure is considered.) If we view the world w as an assignment, then for sentences as premises, the two notions are equivalent. Indeed, when Φ is a set of formulas —and they are treated as formulas— the local definition becomes the more interesting one (cf. the definition just before Proposition 2.3.6 in [6]).

Proposition 2.1

1. *With the local consequence relation, AIP, TIP and SIP are all equivalent.*

2. *If* $\models^{glo}_{\mathsf{K}}$ *is compact, then AIP implies TIP, and TIP and SIP are equivalent.*

Proof. For item 1 use the fact that with the local relation the deduction theorem $\varphi \models^{loc} \psi$ iff $\models \varphi \to \psi$ holds.

We prove item 2 for the uni-modal case only. The proof easily extends to any modal similarity type. We use the fact that we can switch from the global to the local perspective by $\varphi \models^{glo} \psi$ iff $\{\Box^n \varphi \mid n < \omega\} \models^{loc} \psi$ (see, for example, [5, Lemma 2.33]).

(AIP implies TIP.) Assume $\varphi \models^{glo} \psi$. This holds iff $\{\Box^n \varphi \mid n < \omega\} \models^{loc} \psi$, iff (by compactness) $\Box^{m*}\varphi \models^{loc} \psi$ for some m, where

$$\Box^{m*}\varphi = \varphi \wedge \Box\varphi \wedge \Box\Box\varphi \wedge \cdots \wedge \Box^m\varphi.$$

By the deduction theorem this is equivalent to $\models \Box^{m*}\varphi \to \psi$. But then, by AIP, there is an interpolant θ such that $\models \Box^{m*}\varphi \to \theta$ and $\models \theta \to \psi$. Whence $\varphi \models^{glo} \theta$ and $\theta \models^{glo} \psi$.

(SIP is equivalent to TIP.) The direction from SIP to TIP is trivial. For the other direction, assume $\varphi_0 \wedge \varphi_1 \models^{glo} \psi$. Reasoning as before, we find

$$\Box^{m*}\varphi_0 \wedge \Box^{k*}\varphi_1 \models^{loc} \psi.$$

By the deduction theorem, $\Box^{m*}\varphi_0 \models^{loc} \Box^{k*}\varphi \to \psi$. Whence, $\Box^{m*}\varphi_0 \models^{glo} \Box^{k*}\varphi \to \psi$. By TIP, we find an interpolant θ such that $\Box^{m*}\varphi_0 \models^{glo} \theta$ and $\theta \models \Box^{k*}\varphi \to \psi$. Whence, $\varphi_0 \models^{glo} \theta$ and $\varphi_1 \wedge \psi$. $\dashv$

Notice that, by Proposition 2.1, if a logic is compact (which is always the case if its class of frames is elementary), a proof of AIP implies that all the other kinds of interpolation also hold. Conversely, disproving SIP (or TIP) implies the failure of all of them. In the remainder of the paper $\models$ refers always to the global consequence relation.

3 Interpolation in Classical Modal Logic

Important general results concerning interpolation for standard modal logics are known, witness [13]. These results are a byproduct of the strong connections between the interpolation property and the algebraic property of amalgamation. We will now discuss two recent results providing, respectively, a method to prove AIP and a method to disprove SIP in classical modal logic; in Section 4 we will extend these results to SU-logics. First, the following notions should be introduced.

Definition 3.1 Let $\mathcal{F}^c_{\mathcal{L}}$ be the canonical frame for the logic $\mathcal{L}$. Then $\mathcal{L}$ is *canonical* if $\mathcal{F}^c_{\mathcal{L}} \models \mathcal{L}$.

Definition 3.2 Let $\mathcal{G} = (G, R_{\mathcal{G}})$ and $\mathcal{H} = (H, R_{\mathcal{H}})$ be two frames. Let $B \subseteq G \times H$ be nonempty.

1. We say that B is a *bisimulation* between $\mathcal{G}$ and $\mathcal{H}$ if whenever gBh and $gR_{\mathcal{G}}g'$, then there exists $h' \in H$ such that $hR_{\mathcal{H}}h'$ and $g'Bh'$, and similarly in the other direction. If gBh holds we will call g and h *bisimilar*.

2. If B is a total function g, then it is called a *zigzag morphism*. If g is also surjective we use the notation $\mathcal{G} \overset{g}{\twoheadrightarrow} \mathcal{H}$, and call $\mathcal{H}$ the *zigzag morphic image* of $\mathcal{G}$ by g.

3. The notions of bisimulation and zigzag morphism can also be defined for models $\mathfrak{M}_{\mathcal{G}} = (\mathcal{G}, v_{\mathcal{G}})$ and $\mathfrak{M}_{\mathcal{H}} = (\mathcal{H}, v_{\mathcal{H}})$, relative to a given set of propositional variables $\mathbb{P}$ by adding the condition: if gBh then for all $p_i \in \mathbb{P}$, $\mathfrak{M}_{\mathcal{G}}$, $g \models p_i$ iff $\mathfrak{M}_{\mathcal{H}}, h \models p_i$. We will say in this case that B is a $\mathbb{P}$-*bisimulation* or a $\mathbb{P}$-*zigzag morphism*.

Definition 3.3 Let $\mathcal{F}$ and $\mathcal{G}$ be two frames. Let $\mathcal{H}$ be a submodel of the direct product $\mathcal{F} \times \mathcal{G}$. $\mathcal{H}$ is called a *zigzag product* of $\mathcal{F}$ and $\mathcal{G}$ if the projections are surjective zigzag morphisms. We say that a class K of frames is *closed under zigzag products* if every zigzag product of two frames in K is also in K.

Theorem 3.4 ([14]) *Let $\mathcal{L}$ be a canonical classical modal logic. If the class of frames of $\mathcal{L}$ is closed under zigzag products, then $\mathcal{L}$ has the AIP.*

Theorem 3.5 ([2]) *Let K be a class of frames, and $\mathcal{L}_{\mathsf{K}}$ be the classical modal logic of K. Then SIP fails in $\mathcal{L}_{\mathsf{K}}$ whenever there are finite frames $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{F}$ such the following conditions are satisfied*

1. *$\mathcal{G}$, $\mathcal{H} \in \mathsf{K}$;*

2. *there are surjective zigzag morphisms m, n such that $\mathcal{G} \overset{m}{\twoheadrightarrow} \mathcal{F} \overset{n}{\twoheadleftarrow} \mathcal{H}$;*

3. *$\mathcal{F}$ is generated by a single point w*

4. *every m-pre-image of w in G generates $\mathcal{G}$, and similarly for $\mathcal{H}$; and*

5. *there is no frame $\mathcal{J} \in \mathsf{K}$ with commuting surjective zigzag morphisms g and h from $\mathcal{J}$ onto $\mathcal{G}$ and $\mathcal{H}$ (i.e., $\mathcal{G} \overset{g}{\twoheadrightarrow} \mathcal{J} \overset{h}{\twoheadleftarrow} \mathcal{H}$ and $m \circ g = n \circ h$.)*

Moreover, an explicit counterexample for SIP can be algorithmically constructed from the frames and functions $\mathcal{G} \overset{m}{\twoheadrightarrow} \mathcal{F} \overset{n}{\twoheadleftarrow} \mathcal{H}$.

Theorem 3.4 provides us with a tool for proving interpolation. We need only verify that the class of frames of our logic is closed under a given model theoretic construction. As a corollary all (canonical) classical modal logics whose class of frames can be defined by universal Horn sentences have interpolation [14] as such formulas are preserved under zigzag products. Theorem 3.5 shows how in some special cases, failure of closure under zigzag products produces failure of interpolation. Using this method, failure of interpolation for finite variable fragments of FOL, the difference modality, Humberstone's inaccessibility operator, and various product logics and union logics has been proved (or re-proved) in [2]. We will now extend these results to SU-logics.

4 General Results for Since-Until Logics

What do we need to extend the main results of Section 3 to SU-logics? We need canonical frames and bisimulations. Until recently neither was available. In [4] Bellisima and Cittadini introduce a new semantics for SU-logics and prove it to be strongly adequate by providing a Stone-like duality theorem between the "new" general frames (called e-frames) and algebras with operators u and s defined in the obvious way. We will not discuss the algebraic counterpart in this paper as we are mainly concerned with a modal approach. But our results are based on their semantics and their results concerning the existence of canonical models.

Definition 4.1 ([4]) An *e-frame* is a Kripke frame (W, R) together with a function β from R into $\mathcal{P}(\mathcal{P}(W))$ such that for every x and y with xRy we have $\beta(x,y) \neq \emptyset$ and, if $Z \in \beta(x,y)$, then $Z \subseteq \{z \mid xRzRy\}$. $\beta(x,y)$ can be thought of as the sets of relevant points situated between x and y.

An *e-model* is an e-frame together with a valuation. The truth definition for e-models is standard for propositional variables and Boolean connectives. Furthermore, $x \models U(\varphi,\psi)$ if there exists a point y such that xRy and $y \models \varphi$ and there exists $Z \in \beta(x,y)$ such that $z \models \psi$ for each $z \in Z$, and analogously for S.

For this semantics, appropriate canonical models and frames are defined as follows.

Definition 4.2 ([4]) Given any US-logic $\mathcal{L}$ we define its *canonical model* $\mathfrak{M}_{\mathcal{L}}$ as the e-model $(W_{\mathcal{L}}, R_{\mathcal{L}}, \beta_{\mathcal{L}}, V_{\mathcal{L}})$ where:

1. $W_{\mathcal{L}}$ is the set of all maximal consistent extensions of $\mathcal{L}$.
2. $xR_{\mathcal{L}}y$ if $U(\varphi,\top)$ belongs to x for any $\varphi \in y$.
3. Let $xR_{\mathcal{L}}y$ and $Z \subseteq \{z \mid xR_{\mathcal{L}}xR_{\mathcal{L}}y\}$. Then $Z \in \beta_{\mathcal{L}}(x,y)$ if for any φ, ψ such that $\varphi \in y$ and, for each $z \in Z$, $\psi \in z$, we have that $U(\varphi,\psi) \in x$.

4. $V_{\mathcal{L}}(p) = \{x \in W_{\mathcal{L}} \mid p \in x\}$, for any variable p.

Next come bisimulations. For the standard relational semantics for SU-logics, an apprpriate notion of bisimulation we introduced by Kurtonina and de Rijke [11]. We now adapt it to e-frames and e-models.

Definition 4.3 Let $\mathcal{G} = (G, R_{\mathcal{G}}, \beta_{\mathcal{G}})$ and $\mathcal{H} = (H, R_{\mathcal{H}}, \beta_{\mathcal{G}})$ be two frames. Let $B \subseteq G \times H$ be non-empty.

1. We say that B is an *e-bisimulation* between $\mathcal{G}$ and $\mathcal{H}$ if the following clauses hold:

   (a) If gBh and $gR_{\mathcal{G}}g'$, then there exists $h' \in H$ such that $hR_{\mathcal{H}}h'$ and $g'Bh'$, while for all $Z \in \beta_{\mathcal{G}}(g, g')$ there exists $Z' \in \beta_{\mathcal{H}}(h, h')$ such that for all $h'' \in Z'$ there is $g'' \in Z$ and $g''Bh''$.

   (b) Clause (a) with the converse $R_{\mathcal{G}}^{-1}$ and $R_{\mathcal{H}}^{-1}$ of $R_{\mathcal{G}}$ and $R_{\mathcal{H}}$.

   (c) Clauses (a) and (b) but going from $\mathcal{H}$ to $\mathcal{G}$.

   If gBh holds we will call g and h *e-bisimilar*.

2. If B is a total function f, then it is called an *e-zigzag morphism*. If f is also surjective we use the notation $\mathcal{G} \overset{B}{\twoheadrightarrow} \mathcal{H}$), and call $\mathcal{H}$ the *e-zigzag morphic image* of $\mathcal{G}$ by f.

3. The notions of e-bisimulation and e-zigzag morphism can also be defined for models $\mathfrak{M}_{\mathcal{G}} = (\mathcal{G}, v_{\mathcal{G}})$ and $\mathfrak{M}_{\mathcal{H}} = (\mathcal{H}, v_{\mathcal{H}})$, relative to a given set of propositional variables $\mathbb{P}$ by adding the condition: if gBh then for all $p_i \in \mathbb{P}$, $\mathfrak{M}_{\mathcal{G}}, g \models p_i$ iff $\mathfrak{M}_{\mathcal{H}}, h \models p_i$. We will say in this case that it is a $\mathbb{P}$-*e-bisimulation* or a $\mathbb{P}$-*e-zigzag morphism*.

E-bisimulations are defined to make the following result true.

Proposition 4.4 *Let $\mathfrak{M}_{\mathcal{G}}$ and $\mathfrak{M}_{\mathcal{H}}$ be two e-models and B a $\mathbb{P}$-e-bisimulation between them. Then, for every SU-formula φ constructed from variables in $\mathbb{P}$, gBh implies that $g \models \varphi$ iff $h \models \varphi$.*

Once the correct definition of bisimulation is obtained and we have a "nice" semantics, Theorems 3.4 and 3.5 can be re-proved for SU-logic. The only important step is to check the required lemmas in [14] and [2].

First, some notation. Let $\mathcal{L}_{SU}$ be the set of all formulas in a SU-logic. For $\varphi \in \mathcal{L}_{SU}$, let $\mathcal{L}_{\varphi}$ be the restriction of $\mathcal{L}_{SU}$ to the propositional symbols in φ. $\mathfrak{M}_{\varphi} = (W_{\varphi}, R_{\varphi}, \beta_{\varphi}, v_{\varphi})$ is the canonical model over $\mathcal{L}_{\varphi}$.

For the positive characterization the following result is vital.

Lemma 4.5 *Let $\varphi, \psi \in \mathcal{L}_{SU}$. Let $\mathfrak{M}_\varphi$, $\mathfrak{M}_\psi$ be the corresponding canonical models. Then*

$$B = \{(w, v) \in W_\varphi \times W_\psi \mid \mathcal{L}_\varphi \cap \mathcal{L}_\psi \cap w = \mathcal{L}_\varphi \cap \mathcal{L}_\psi \cap v\}$$

is a total, surjective $(\mathcal{L}_\varphi \cap \mathcal{L}_\psi)$-e-bisimulation between $\mathfrak{M}_\varphi$ and $\mathfrak{M}_\psi$.

Proof. By definition B is a relation on $W_\varphi \times W_\psi$. That B is total (surjective) can be proved from the fact that if $w \in W_\varphi$ ($v \in W_\psi$) then $\mathcal{L}_\varphi \cap \mathcal{L}_\psi \cap w$ ($\mathcal{L}_\varphi \cap \mathcal{L}_\psi \cap v$) is $(\mathcal{L}_\varphi \cap \mathcal{L}_\psi)$-consistent, and hence can be extended to a set in W_ψ (W_φ). To prove that B satisfies the conditions in the definition of e-bisimulation it is enough to notice the following.

Let $\mathfrak{M}_{\varphi,\psi} = (W_{\varphi,\psi}, R_{\varphi,\psi}, \beta_{\varphi,\psi}, v_{\varphi,\psi})$ be the canonical model on $\mathcal{L}_\varphi \cap \mathcal{L}_\psi$. Before proceeding, let us make two observations:

1. Let $x, y \in W_\varphi$ and $Z \in \beta_\varphi(x, y)$, then

   $$Z'' = \{z \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi \mid z \in Z\} \in \beta_{\varphi,\psi}(x \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi, y \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi).$$

2. Let $x'', y'' \in W_{\varphi,\psi}$ and $Z'' \in \beta_{\varphi,\psi}(x'', y'')$, then for all $x', y' \in W_\psi$ and $Z' \in \beta_\psi(x', y')$ such that $x' \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi = x''$, $y' \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi = y''$ we have $\{z' \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi \mid z' \in Z'\} = Z''$.

Now suppose $Z \in \beta(g, g')$. We need to prove that there is $Z' \in \beta(h, h')$ such that for all $h'' \in Z'$ there is $g'' \in Z$ and $g'' B h''$. Define $Z'' = \{z \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi \mid z \in Z\}$. Then by observation 1 $Z'' \in \beta_{\varphi,\psi}(g \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi, g' \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi)$. By item 2, as $h \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi = g \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi$ and $h' \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi = g' \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi$ and for Z'' defined as above, there is $Z' \in \beta(h, h')$ such that $Z'' = \{z \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi \mid z \in Z'\}$. Then, if $h'' \in Z'$, $h'' \cap \mathcal{L}_\varphi \cap \mathcal{L}_\psi \in Z'$ and there exists $g'' \in Z$ such that $g'' B h''$. $\dashv$

The main argument in [14] now establishes the theorem.

Theorem 4.6 *Let $\mathcal{L}$ be a canonical SU-modal logic. If the class of frames of $\mathcal{L}$ is closed under e-zigzag products, then $\mathcal{L}$ has the arrow interpolation property.*

Proof. The outline of the proof in [14] is as follows. Reason by contraposition. Suppose there is no interpolant for $\varphi \to \psi$. We will prove that $\varphi \wedge \neg\psi$ is satisfiable.

Define B as in Lemma 4.5. We claim that there is $(w, v) \in B$ such that $\mathfrak{M}_\varphi, w \models \varphi$ and $\mathfrak{M}_\psi, v \models \neg\psi$. Consider the set

$$\{\theta \in \mathcal{L}_\varphi \cap \mathcal{L}_\psi \mid \models \varphi \to \theta\} \cup \{\neg\theta \mid \mathcal{L}_\varphi \cap \mathcal{L}_\psi\} \models \theta \to \psi.$$

Since there is no interpolant the set is consistent and can be extended to an element u of $W_{\varphi,\psi}$. Now, $u \cup \{\varphi\}$ and $u \cup \{\neg\psi\}$ are also consistent and can be extended to elements $w \in W_\varphi$ and $v \in W_\psi$. This fact, plus Lemma 4.5, together with the hypothesis that the class of frames is closed under e-zigzag products, yields the model we need. $\dashv$

What we need to establish an analog to Theorem 3.5 for SU-logic, is another result involving e-bisimulations. Now we need to describe finite e-frames up to e-bisimulation.

Lemma 4.7 *Let $\mathcal{F} = (F, R_\mathcal{F}, \beta_\mathcal{F})$ be a finite frame generated by f_1 and let $|F| = n$; say, $F = \{f_1, \dots, f_n\}$. Let $\mathfrak{M}_\mathcal{F} = (\mathcal{F}, V_\mathcal{F})$ be a model such that $V_\mathcal{F}(p_i) = \{f_i\}$ for $p_1, \dots, p_n$.*

Then there exists an SU-formula $\Sigma_\mathcal{F}$ such that for any model $\mathfrak{M}_\mathcal{G} = (G, R_\mathcal{G}, \beta_\mathcal{G}, V_\mathcal{G})$ with $\mathfrak{M}_\mathcal{G} \models \Sigma_\mathcal{F}$, the relation $B \subseteq G \times F$ defined by

$$gBf \text{ iff } g \text{ and } f \text{ agree in the truth value assigned to } \{p_1, \dots, p_n\}$$

is a surjective $\{p_1, \dots, p_n\}$-e-zigzag morphism from $\mathfrak{M}_\mathcal{G}$ onto $\mathfrak{M}_\mathcal{F}$.

Proof. Define $\Sigma_\mathcal{F}$ as the conjunction of

$$
\begin{aligned}
A_1 &= \bigvee_{1\le k\le n} p_k, \\
A_2 &= \bigwedge_{1\le k\le n} (p_k \to \bigwedge\{\neg p_l \mid k \neq l\}) \\
A_3 &= \bigwedge_{1\le k\le n} (p_k \to \bigwedge\{\neg F p_l \mid \text{not } f_k R_\mathcal{F} f_l\}) \\
A_4 &= \bigwedge_{1\le k\le n} (p_k \to \bigwedge\{U(p_l, \bigvee_I p_i) \mid f_k R_\mathcal{F} f_l \text{ and} \\
&\qquad I = \{i \mid Z \in \beta_\mathcal{F}(f_k, f_l), f_i \in Z\}\}) \\
A_5 &= \bigwedge_{1\le k\le n} (p_k \to \bigwedge\{\neg U(p_l, \bigvee_I p_i) \mid f_k R_\mathcal{F} f_l \text{ and} \\
&\qquad I = \{i \mid Z \subseteq F, f_i \in Z, (\forall Z' \in \beta_\mathcal{F}(f_k, f_l))(Z' \not\subseteq Z)\}\}) \\
A_6 &= \bigwedge_{1\le k\le n} (p_k \to \bigwedge\{\neg P p_l \mid \text{not } f_l R_\mathcal{F} f_k\}) \\
A_7 &= \bigwedge_{1\le k\le n} (p_k \to \bigwedge\{S(p_l, \bigvee_I p_i) \mid f_l R_\mathcal{F} f_k \text{ and} \\
&\qquad I = \{i \mid Z \in \beta_\mathcal{F}(f_l, f_k), f_i \in Z\}\}) \\
A_8 &= \bigwedge_{1\le k\le n} (p_k \to \bigwedge\{\neg S(p_l, \bigvee_I p_i) \mid f_l R_\mathcal{F} f_k \text{ and} \\
&\qquad I = \{i \mid Z \subseteq F, f_i \in Z, (\forall Z' \in \beta_\mathcal{F}(f_l, f_k))(Z' \not\subseteq Z)\}\})
\end{aligned}
$$

Using $\Sigma_\mathcal{F}$, we will prove that B is a surjective $\{p_1, \dots, p_n\}$-e-zigzag morphism. We only prove the conditions for e-bisimulations, leaving the others conditions to the reader.

Assume gBf and $gR_\mathcal{G}g'$. We need to prove two things. First, we should show that there exists f' such that $fR_\mathcal{F}f'$. Let $g \models p_i$, $g' \models p_j$, $i, j \in n$

as given by A_1 and A_2. Hence $g \models Fp_j$, and $f_i R_{\mathcal{F}} f_j$. Furthermore, by definition of B

$$(*) \qquad g' B f_j.$$

Secondly, we need to show the following: if $Z \in \beta_{\mathcal{G}}(g, g')$, then there exists $Z' \in \beta_{\mathcal{F}}(f_i, f_j)$ such that for all $f'' \in Z'$ there is $g'' \in Z$ and $g'' B f''$. Consider the set $X = \{f \in F \mid zBf \text{ and } z \in Z\}$. If we can show that for some $Z' \subseteq X$, $Z' \in \beta_{\mathcal{F}}(f_i, f_j)$, then we are done. Observe first that for any $f \in X$, $f_i R_{\mathcal{F}} f R_{\mathcal{F}} f_j$. Next, to arrie at a contradiction, assume that for all $Z' \subseteq X$ we have got $Z' \notin \beta_{\mathcal{F}}(f_i, f_j)$. Then

$$g \models U(p_j, \bigvee\{p_k \mid f_k \in X\}),$$

as $Z \in \beta_{\mathcal{G}}(g, g')$, $g R_{\mathcal{G}} g'$ and for all $g'' \in Z$, $g'' \models \bigvee\{p_k \mid f_k \in X\}$. On the other hand, by A_5 we have

$$g \models \bigwedge\{\neg U(p_j, \bigvee_L p_l) \mid L = \{l \mid Z \subseteq F, f_l \in Z, \forall Z' \in \beta_{\mathcal{F}}(f_j, f_i)\, Z' \not\subseteq Z\}\},$$

and hence

$$g \models \neg U(p_j, \bigvee\{p_k \mid f_k \in X\}),$$

a contradiction.

To complete the argument, let $Z' \subseteq X$ be such that $Z' \in \beta_{\mathcal{F}}(f_i, f_j)$. By A_4 it follows that

$$g \models \bigwedge\{S(p_j, \bigvee_L p_l) \mid L = \{l \mid Y \in \beta_{\mathcal{F}}(f_i, f_j), f_i \in Y\}\}.$$

Now, take Y to be Z'. Then, there exists y with $g R_{\mathcal{G}} y$ and $y \models p_j$, while for some $W \in \beta_{\mathcal{G}}(g, y)$ we find that all its elements w have $w \models \bigvee\{p_l \mid f_l \in Z'\}$. Hence, for all $z' \in Z'$ there exists $w \in Z$ such that wBz', as required. $\dashv$

Theorem 4.8 *Let K be a class of e-frames and let $\mathcal{L}_{\mathsf{K}}$ be the SU-logic of K.*

SIP fails in $\mathcal{L}_{\mathsf{K}}$ if there are finite frames $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ such that the following hold:

1. *there are surjective zigzag SU-morphisms m, n such that $\mathcal{G} \overset{m}{\twoheadrightarrow} \mathcal{F} \overset{n}{\twoheadleftarrow} \mathcal{H}$;*
2. *$\mathcal{F}$ is generated by one point w;*
3. *every m-pre-image of w in G generates $\mathcal{G}$, and similarly for $\mathcal{H}$; and*
4. *there is no frame $\mathcal{J} \in \mathsf{K}$ with commuting surjective zigzag SU-morphisms g and h from $\mathcal{J}$ onto $\mathcal{G}$ and $\mathcal{H}$ (i.e., $\mathcal{G} \overset{m}{\twoheadleftarrow} \mathcal{J} \overset{n}{\twoheadrightarrow} \mathcal{H}$.)*

Moreover, an explicit counterexample for SIP can be algorithmically constructed from the frames and functions $\mathcal{G} \overset{m}{\twoheadrightarrow} \mathcal{F} \overset{n}{\twoheadleftarrow} \mathcal{H}$.

Proof. We give the outline of the proof in [2] for completeness. Assume $\mathcal{G} \overset{m}{\twoheadrightarrow} \mathcal{F} \overset{n}{\twoheadleftarrow} \mathcal{H}$. Obtain models from the frames by providing a valuation which assigns a unique propositional symbol to the elements of the domain. Use disjoint vocabularies $\{f_1, \ldots, f_{|F|}\}$, $\{g_1, \ldots, g_{|G|}\}$ and $\{h_1, \ldots, h_{|H|}\}$. Assume f_1 is the propositional symbol true at the world generating $\mathcal{F}$. The formulas

$$\Gamma_m = \bigwedge_{1 \le i \le |F|} (f_i \leftrightarrow \bigvee \{g_j \mid m(w_j) = w_i\})$$

$$\Gamma_n = \bigwedge_{1 \le i \le |F|} (f_i \leftrightarrow \bigvee \{h_j \mid n(w_j) = w_i\})$$

describe the functions m and n.

Now the following can be established:

(1) $$(\Sigma_{\mathcal{G}} \wedge \Gamma_m) \wedge (\Sigma_{\mathcal{H}} \wedge \Gamma_n) \models \neg f_1,$$

(2) there is no splitting interpolant.

The proof of (1) and (2) proceeds by contradiction. Assuming the negation of (1) forces $(\mathcal{G}, v_{\mathcal{H}})$ to satisfy $\neg f_1$ and at the same time there is a state in H which is mapped by n to f_1. Assuming the negation of (2) let us create in K a frame $\mathcal{J}$ with commuting surjective e-zigzag morphisms onto $\mathcal{G}$ and $\mathcal{H}$ contradicting hypothesis. $\dashv$

As a final remark, we notice that once a good definition of bisimulation for a logic is obtained, the results above are easily deduced. We conjecture that this is an instance of a more general fact. Our ongoing research is to explore this phenomenon in greater detail.

5 Applications

In this section we discuss how the general results we just proved we can put to work. We only mention a couple of specific instances where Theorems 4.6 and 4.8 apply, and further research is needed here.

Positive Results. The first positive result is immediate. The class of all e-frames is canonical [4] and trivially closed under e-zigzag-products. Hence

Proposition 5.1 *The basic SU-logic K_{SU} has AIP, TIP and SIP.*

Next, the result concerning universal Horn formulas mentioned in [14, Corollary B.4.] transfers.

Corollary 5.2 *Let $\mathcal{L}$ be a canonical SU-logic. If $\mathsf{Fr}_{\mathcal{L}}$ can be defined by universal Horn sentences, then $\mathcal{L}$ has the AIP, TIP and SIP.*

In the temporal interpretation of Since and Until, an important class of frames which is covered by the above corollary is the class of frames where R is a partial order, i.e., transitive, antisymmetric and reflexive. It is immediate to check that these conditions are universal Horn.

Proposition 5.3 *Let K_{bran} be the SU-logic of the class of e-frames where the accessibility relation is a partial order. Then AIP, TIP and SIP hold for K_{bran}.*

Negative Results. We will only instantiate Theorem 4.8 for one case: linear time. By adding the condition of totality for the accessibility relation, i.e., $(\forall x, y)(xRy \vee yRx)$, we end up outside the universal Horn fragment. We can actually prove that the SU-logic of this class of frames does not have interpolation.

Proposition 5.4 *Let K_{lin} be the SU-logic of the class of e-frames where the accessibility relation is a linear order. Then AIP, TIP and SIP all fail for K_{lin}.*

Proof. To use Theorem 4.8 we should provide three finite frames $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{F}$. We propose the following

The sets of sets labeling the accessibility relations define the β function for the e-frames. It is not difficult to check that the e-frames satisfy the conditions in Theorem 4.8. Also, the functions m and n that map all elements to f_1 are surjective e-zigzag morphisms. We will now prove that no e-frame $\mathcal{J}$ exists in K_{lin} with surjective e-zigzag morphism g and h onto $\mathcal{G}$ and $\mathcal{H}$ respectively.

When we say that (g_k, h_l) is an element of $\mathcal{J}$, we mean that there is an element $j \in J$ such that $g(j) = g_k$ and $h(j) = h_l$. As g, h should be surjective, $(g_1, h_i) \in J$. By the zigzag conditions it follows that $(g_1, h_i)R_{\mathcal{J}}(g_2, h_j)$. To satisfy the condition on $\beta_{\mathcal{J}}$, it should be the case that $h_i = h_j$. Suppose $i \in \{1, 2\}$; then $(g_2, h_i)R_{\mathcal{J}}(g_k, h_{i+1})$. By definition of $R_{\mathcal{G}}$, $k = 2$. But now, by transitivity, $(g_1, h_i)R_{\mathcal{J}}(g_2, h_{i+1})$ and the condition for $\beta_{\mathcal{J}}$ for this pair cannot be fulfilled. In the case where $i = 3$, we find that $(g_k, h_{i-1})R_{\mathcal{J}}(g_1, h_i)$, and we reason similarly. $\dashv$

6 Conclusions

In this paper we have focused on the use of bisimulation for SU-logics in the context of interpolation. We have proved that general (positive and negative) results hold, similar to those established for classical modal logic [2, 14]. By means of the two main theorems presented in Section 4, interpolation for a rich class of logics including branching time has been proved, while we have shown that linear time fails to have this property.

We conjecture that a number of similar results can be established for other modal logics which are based only on the presence of appropriate notions of bisimulation and canonical frame, as well as the availability of certain model theoretic techniques. Our ongoing research is to investigate the general pattern behind this phenomenon. In addition, we plan to investigate characterization and preservation results for SU-logics. Also, given that a very similar notion of bisimulation was given for Interpretability Logic in [21], we are confident that our results may also extend in that direction.

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