Global vs. Local in Basic Modal Logic

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Abstract. We discuss results on global definability in basic modal logic, and contrast our model-theoretic results and proof techniques with known results about local definability.

1 Introduction

Modal concepts play an important role in many areas of philosophy. While this statement may seem to be a truism, it is not taken for granted by everybody; analytic philosophers like Frege and Quine have a very sceptical attitude towards modality. Nevertheless, modal concepts can be found in fields as different as ontology, philosophy of mind, ethics, philosophy of science, and, more recently, philosophy of mathematics. There are at least two reasons for the fact that we are not able to dispense with modal concepts. First, they are deeply tied to our practicing as intellectuals and language users. Hence, in explicating pretheoretic concepts — which is, of course, one of the main concerns of analytic philosophers — we must take modal concepts into account. Second, it is an essential philosophical task to separate the things which only hold contingently from things that are necessarily the case.

The wide-spread use of modal concepts within analytic philosophy is partly related to the impressive development of the *logic* of modality, though only very few profound results from this area have found their way into the realm of philosophy. Of course, the best-known bridge linking the two fields is provided by possible world semantics, as it was foreshadowed in Carnap's work, hidden in the algebraic work by Tarski and Jónsson, and (re-)invented by Kripke, Kanger and Hintikka at the end of the 1950s. Possible world semantics is both an important formal tool on which the model theory of (normal) modal logic is built, and a very suitable basis for intuitive interpretations: many appealing philosophical explanations can be couched in terms of possible world semantics.

Leaving subtleties aside, it is correct to say that possible world semantics is a formal way to restate the Leibnizian idea that the truth of modalized statements in a world essentially depends on the truth of non-modalized statements in other worlds. For instance, a sentence like "It is necessarily the case that water is H_2O" is true (in our actual world) if and only if the non-modalized sentence "Water is H_2O" is true in every possible world, or, more precisely, in every possible world that coincides with the actual world in view of its laws of nature. When we implement the Leibnizian idea in a formal, model-theoretic framework and add a binary relation between worlds, we end up with models of the form $\mathfrak{M} = (W, R, V)$, which are the kind of structures on which the formulas of modal languages[1] are usually interpreted. Here, W is a set of possible worlds, and R is an accessibility relation on W, where Rwv means that v is considered as a possible world or alternative from the perspective of w. The accessibility relation gives an enormous increase in the flexibility of the framework. It enables us to consider several modalities together, and, moreover, the worlds that are considered as possible alternatives are allowed to change from world to world. As to the third part of the structure, the task of V is to fix the basic facts in each world; formally, it is a function that relates each possible world w to the atomic formulas that are regarded as true in it.

Then, the fundamental semantical relation is 'truth of a formula φ in a pair $(\mathfrak{M}, w)$', where $\mathfrak{M}$ is a model and w is a world chosen from this model. Such a pair is called a *pointed* model [3, 10] or a model-world pair [1]. We use $(\mathfrak{M}, w) \Vdash \varphi$ to say that φ is true in $(\mathfrak{M}, w)$. Based on this relation, further semantical concepts are defined as usual. That a formula φ follows from a set of formulas Γ holds, for instance, if for every pointed model $(\mathfrak{M}, w)$ in which all $\psi \in \Gamma$ are true, we have $(\mathfrak{M}, w) \Vdash \varphi$. Or, in modal terms: it is not possible that Γ is true somewhere without φ being true there as well.

Both the satisfaction relation and the consequence relation reflect a *local* perspective on the modal language and its models. Formulas are evaluated inside models, at some particular world w (the 'actual' world). In contrast, we can take a *global* perspective, under which models, and not pointed models, are regarded as the fundamental semantic units. The global counterparts of the satisfaction and the consequence relation are then defined as follows. A formula φ is said to be *true in a model* $\mathfrak{M}$, abbreviated by $\mathfrak{M} \models \varphi$, if φ is true in every pointed model $(\mathfrak{M}, w)$ based on $\mathfrak{M}$; and φ follows globally from Γ, if for all models $\mathfrak{M}$ with $\mathfrak{M} \models \Gamma$ it holds that $\mathfrak{M} \models \varphi$. As several authors have observed [3, 8, 13], a great number of logical properties, like completeness, canonicity, finite model property, and interpolation, come in two flavors: a local one and a global one. For instance, the notions of local and global consequence do not coincide: $\Box\psi$ follows locally from ψ, but not globally, yet the following general fact holds: a formula φ follows locally from a set Γ iff φ follows globally from the set $\{\Box^n\psi \mid \psi \in \Gamma, n \in \omega\}$. So, in a certain way, the global approach can be simulated within the local setting.[2]

On the other hand, the global perspective is worth exploring for its own sake. Why? First, via a straightforward translation modal formulas φ may be regarded as terms t_φ of an algebraic language, so that φ turns out to be true in a model $\mathfrak{M}$ iff the equality $t_\varphi = 1$ holds in the corresponding algebra $\mathfrak{M}^*$. Hence, from the algebraic point of view the global setting proves the more natural one. Second, sometimes we are interested in certain features of the whole model,

[1]Throughout this paper we only consider the basic modal language, which is the language we obtain from the boolean propositional language by adding two modal operators, $\Box$ and $\Diamond$. Nevertheless most of the things that are said in the introduction apply to richer modal languages as well.

[2]By adding some non-standard modal connectives to our language, the other direction holds as well; see [13, Appendix B].

independent of the fact whether they are enforceable by pointwise reasoning. And third, global constraints on models play a key role in the closely related area of terminological reasoning.

The latter two points raise an important question: what is the expressive power of the given modal language? Conceptually, one can distinguish at least two different answers to this question: According to the first, which completely ignores semantics, the formulas of the formal language are treated as paraphrases or translations of natural language sentences; briefly, the more sentences can be paraphrased the more expressive the formal language is. The second approach measures the expressive power with respect to the properties of structures definable or describable within the formal language. In a model-theoretic framework such as the one adopted in this paper, the second approach is the appropriate one.

The view of properties that underlies this account is purely extensional; roughly, a property is identified with the class of structures which have this property. Therefore, a property, that is, a class K of structures is said to be *definable* iff there is a set of formulas Γ such that K equals the class of structures in which Γ holds. This raises the following important question: is there a general characterization of the classes of structures that are definable in the above sense? More precisely, what this questions asks for is a characterization of the elementary classes of the logic under investigation. The answer is usually given by stating algebraic closure conditions that are necessary and sufficient for a class K to be definable. Clearly, in the case of modal logic the characterization problem has two versions: a global and a local one. Our paper contains solutions for both versions. In Section 3 we deal with the local version, whereas Section 4 is reserved for the global setting.

Semantics-based translations form a valuable tool for investigating the meta-properties of a logic more deeply, by relating it to other logics. In the case of basic modal logic there exists a natural translation into first-order logic. Using this translation we may view modal logic as a fragment of first-order logic. Moreover, this fragment has a nice semantical characterization in terms of preservation. A famous result by van Benthem [2] tells us that a first-order formula lies in this fragment if and only if it is preserved under so-called bisimulations (see Section 3). The global counterpart of this result is proved in Section 4. Then, in Section 5 we briefly mention a few definability and preservation results concerning particular classes of models. The paper concludes with some suggestions for future research.

2 Basic Concepts

Fix a countable set $\mathcal{P} := \{p_n \mid n \in \omega\}$ of proposition letters. The set $\mathcal{ML}$ of modal formulas (over $\mathcal{P}$) is then defined as the least set X such that every proposition letter from $\mathcal{P}$ belongs to X, X, and X is closed under the boolean connectives $\neg$, $\vee$, and $\wedge$ as well as under the modal operators $\Box$ and $\Diamond$.

A *model* for $\mathcal{ML}$ is a triple $\mathfrak{M} = (W, R, V)$, where W is a non-empty set, R a binary relation on W, and V a valuation function from $\mathcal{P}$ into the power set of W. As usual, the truth of modal formulas is defined recursively with respect to pairs $(\mathfrak{M}, w)$, consisting of a model $\mathfrak{M}$ and a distinguished element $w \in W$. The atomic and the boolean cases of the definition are clear. For the modal operators we put

$$\begin{array}{lll} (\mathfrak{M}, w) \Vdash \Diamond\varphi & \text{iff} & \text{there is a } w' \in W \text{ with } Rww' \text{ and } (\mathfrak{M}, w') \Vdash \varphi, \\ (\mathfrak{M}, w) \Vdash \Box\varphi & \text{iff} & \text{for all } w' \in W \text{ with } Rww' \text{ we have } (\mathfrak{M}, w') \Vdash \varphi. \end{array}$$

If $\mathfrak{M}$ is a model and φ a modal formula, we use $\mathfrak{M} \models \varphi$ to say that for all $w \in W$ it holds that $(\mathfrak{M}, w) \Vdash \varphi$.

A pointed model $(\mathfrak{M}, w)$ may also be regarded as a first-order model suitable for a first-order vocabulary, lets call it τ, consisting of a countable set $\{P_n \mid n \in \omega\}$ of predicate symbols, a binary relation symbol S and an individual constant c.[3] This, together with the fact that the truth clauses for modal formulas are stated in a first-order metalanguage, suggests a mapping ST from $\mathcal{ML}$ into the set of first-order sentences over τ:

$$\begin{array}{rcl} ST(p_n) & := & P_n c, \text{for } n \in \omega, \\ ST(\neg\varphi) & := & \neg ST(\varphi), \\ ST(\varphi \vee \psi) & := & ST(\varphi) \vee ST(\psi), \\ ST(\varphi \wedge \psi) & := & ST(\varphi) \wedge ST(\psi), \\ ST(\Diamond\varphi) & := & \exists x(Scx \wedge ST(\varphi)[x/c]),^{4,5} \\ ST(\Box\varphi) & := & \forall x(Scx \rightarrow ST(\varphi)[x/c]). \end{array}$$

The following lemma is then proved by an easy induction. In fact, this is the result which allows us to regard modal logic as a fragment of first-order logic.

Lemma 2.1 *For every modal formula φ, every model $\mathfrak{M} = (W, R, V)$ and every $w \in W$: $(\mathfrak{M}, w) \Vdash \varphi \Leftrightarrow (\mathfrak{M}, w) \Vdash ST(\varphi)$.*

3 Local Definability

At the beginning of this section we introduce a well-known type of equivalence relations between models, so-called bisimulations. What makes them important is the fact that modal formulas cannot distinguish between bisimilar models, that is, if there is a bisimulation between two models $\mathfrak{M}$ and $\mathfrak{N}$ which relates worlds w and v, then w and v satisfy exactly the same modal formulas. This result is stated in Lemma 3.2. Moreover, recent work has shown that bisimulations form an important tool in modal model theory. A central result along this line is Theorem 3.3 below, which provides an algebraic characterization of the elementary classes of pointed models, that is, the classes of models that are

[3] This is possible because, neglecting some harmless notational differences, $(\mathfrak{M}, w)$ may be seen as a convenient way of denoting the first-order model $(W, R, (V(p_n))_{n\in\omega}, w)$.

[4] Here, and in the following clause, the variable x is assumed to be the first variable chosen from a list of variables that do not occur in $ST(\varphi)$.

[5] In general, for a first-order formula α and individual terms t_1 and t_2, $\alpha[t_1/t_2]$ denotes the formula one gets by replacing every occurrence of t_2 in α by t_1.

definable by (sets of) modal formulas. This result was first stated and proved in [10].

Definition 3.1 Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{N} = (W', R', V')$ be models. A relation $Z \subseteq W \times W'$ is a *bisimulation* between $\mathfrak{M}$ and $\mathfrak{N}$, if Z satisfies the following conditions:

B1 For every $w \in W$ and $v \in W'$, if Zwv then $(\mathfrak{M}, w) \Vdash p_n \Leftrightarrow (\mathfrak{N}, v) \Vdash p_n$, for every $n \in \omega$.

B2 For every $w, w' \in W$ and $v \in W'$, if Zwv and Rww', then there is some $v' \in W'$ such that $R'vv'$ and $Zw'v'$.

B3 For every $w \in W$ and $v, v' \in W'$, if Zwv and $R'vv'$ then there is some $w' \in W$ such that Rww' and $Zw'v'$.

If Z is a bisimulation such that for every $v \in W'$ there is some $w \in W$ with Zwv, then Z is called a *surjective* bisimulation from $\mathfrak{M}$ to $\mathfrak{N}$.

Lemma 3.2 *Let Z be a bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ such that Zwv. Then for every modal formula φ, $(\mathfrak{M}, w) \Vdash \varphi$ iff $(\mathfrak{N}, v) \Vdash \varphi$.*

Theorem 3.3 *For a class K of pointed models the following equivalences hold.*
1. K is (locally) definable by a set of modal formulas iff K is closed under bisimulations and ultraproducts, and the complement of K, abbreviated by $\overline{\mathsf{K}}$, is closed under ultrapowers.
2. K is (locally) definable by a single modal formula iff both K and $\overline{\mathsf{K}}$ are closed under bisimulations and ultraproducts.

From this theorem we easily obtain van Benthem's bisimulation theorem:

Corollary 3.4 *A first-order sentence α (over τ) is equivalent to the translation of a modal formula if and only if α is preserved under bisimulations.*[6]

4 Global Definability

In the previous section we took a *local* perspective on modal logic and its corresponding first-order fragment. Below we adopt what we call a global point of view: modal formulas will be considered on the level of *models*. The main result of this section characterizes the classes of models that are (globally) definable by modal formulas. (For a proof see the full version of this paper.) This time the key notions are ultraproducts and ultrapowers, as before, as well as surjective bisimulations and disjoint unions. The latter is introduced below.

Definition 4.1 Let $\{\mathfrak{M}_i \mid i \in I\}$ be a non-empty family of models, where the domains of these models are pairwise disjoint. The *disjoint union* of this family, abbreviated by $\biguplus_{i \in I} \mathfrak{M}_i$, is the following model $\mathfrak{M} = (W, R, V)$:

[6] Here we say that a first-order sentence α is preserved under bisimulations if whenever Z is a bisimulation between models $\mathfrak{M}$ and $\mathfrak{N}$ such that Zwv and $(\mathfrak{M}, w) \Vdash \alpha$, then $(\mathfrak{N}, v) \Vdash \alpha$.

$$W := \bigcup_{i \in I} W_i, R := \bigcup_{i \in I} R_i, \text{ and } V(p_n) := \bigcup_{i \in I} V_i(p_n), \text{ for } n \in \omega.$$

Theorem 4.2 *For a class* K *of models the following equivalences hold.*

1. K *is (globally) definable by a set of modal formulas iff* K *is closed under surjective bisimulations, disjoint unions and ultraproducts, and* $\overline{\mathsf{K}}$ *is closed under ultrapowers.*

2. K *is (globally) definable by means of a single modal formula iff* K *is closed under surjective bisimulations and disjoint unions, and both* K *and* $\overline{\mathsf{K}}$ *are closed under ultraproducts.*

Obviously, by a straightforward adaption of the standard translation from Section 3 we may view $\mathcal{ML}$ as a fragment of first-order logic on the (global) level of models as well: we just have to correlate a modal formula φ with the universal closure of its standard translation, that is, with the formula $\forall x(ST(\varphi)[x/c])$. By making use of Lemma 2.1 it is then easy to see that for every model $\mathfrak{M}$, $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \Vdash \forall x(ST(\varphi)[x/c])$.

As in the local case, the modal fragment of first-order logic has a semantical characterization in terms of preservation behavior. This time we can prove that a first-order sentence α lies in this fragment iff α is preserved under surjective bisimulations and disjoint unions, where α is said to be *preserved under disjoint unions* if for every non-empty family $\{\mathfrak{M}_i \mid i \in I\}$ of models such that for every $i \in I$, $\mathfrak{M}_i \models \varphi$, we have $\biguplus_{i \in I} \mathfrak{M}_i \models \varphi$.

Corollary 4.3 *For a first-order sentence* α *over* $\tau \setminus \{c\}$ *the following are equivalent:*

1. There is a modal formula φ *such that* $\models \alpha \leftrightarrow \forall x(ST(\varphi)[x/c])$.

2. α *is preserved under disjoint unions and surjective bisimulations.*

Before concluding this section, we want to emphasize that our Theorem 4.2 is not the first global definability result with respect to classes of models. In [7] Hansoul gave an alternative characterization.[7] Using our own terminology and putting aside the topological notions used by Hansoul, his result can be stated as follows: A class K of models is globally definable if and only if K is closed under isomorphisms, generated submodels and disjoint unions, and for each model $\mathfrak{M}$, $\mathfrak{M} \in \mathsf{K}$ iff $(\mathfrak{M}^*)_* \in \mathsf{K}$. Here, by $\mathfrak{M}^*$ we mean the least modal subalgebra of the complex algebra of $\mathfrak{M}$ that contains all truth-sets of the form $V(\varphi) := \{w \in W \mid (\mathfrak{M}, w) \Vdash \varphi\}$, and by $(\mathfrak{M}^*)_*$ we denote its canonical structure.[8] The full version of the present paper contains a detailed analysis of how the two definability results, Hansoul's and ours, are related to each other.

5 Universal Classes

Applying arguments and tools similar to the ones that were used in the proofs of the previous section, we can also obtain global definability and preservation

[7] In [1] another definability result was proved with respect to infinitary modal languages.

[8] The experienced reader may realize that Hansoul's result shows great similarity to a famous result by Goldblatt [5], which characterizes modally definable classes of generalized frames.

results for modal formulas satisfying various syntactic constraints. In the following we restrict our attention to universal formulas. By a *universal* formula we mean a modal formula that has been built up from atomic formulas and negated atomic formulas, using $\wedge$, $\vee$ and $\Box$ only.

The main result of this section, Theorem 5.1, provides a precise characterization of the conditions under which classes of models are globally definable by sets of *universal* formulas. ¿From this we easily get a preservation result for universal formulas, stated as Corollary 5.2. For lack of space we only mention these two results; comments, proofs and further results are reserved for the full paper. For the local counterparts of the above results the reader is referred to [12].

Theorem 5.1 *A class* K *of models is definable by a set of universal formulas iff* K *is closed under surjective bisimulations, disjoint unions, submodels and ultraproducts.*

Corollary 5.2 *A modal formula* φ *is globally preserved under submodels iff there is a universal formula* ψ *such that* φ *and* ψ *hold in exactly the same models.*

We hasten to add that similar results also exist for positive, that is negation-free formulas. In the full paper we characterize the positive classes as the classes of models that are closed under surjective bisimulations, disjoint unions, ultraproducts and weak extensions,[9] and whose complements are closed under ultrapowers. In addition, it contains a preservation result for positive formulas.

6 Concluding Remarks

In this paper we have presented a number of definability and preservation results for (basic) modal logic. Contrary to common practice, we have emphasized the global perspective. Of course, the above results can only be considered as a modest beginning; a lot of work remains to be done. Our future research should concentrate on the following questions:

1. What other local results (and tools) can be adapted to the global setting?
2. In [11] van Benthem's bisimulation theorem was proved with respect to finite models. Is it possible to finitize our Corollary 4.3 in the same way?
3. Can we apply our results and methods to more expressive modal languages, like temporal logic with Since and Until, or PDL?
4. Is there a uniform way of connecting the local and the global setting? Indeed, we have a general result with respect to preservation, but it is not completely satisfying. First, it looks slightly proof-generated and, second, it makes use of the notion of ω-saturated models which restricts its applicability to modal languages that lie inside first-order logic.

[9] A model $\mathfrak{N} = (W', R', V')$ is a *weak extension* of a model $\mathfrak{M} = (W, R, V)$, if $W = W'$, $R = R'$, and $V(p_n) \subseteq V'(p_n)$, for every $n \in \omega$.

5. How is our approach related to the work of other authors? In particular, how is it related to work on the universal modality [4, 6], and to Kracht and Wolter's work on transfer results [8, 9]?

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