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The Logic of Peirce Algebras

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Abstract. Peirce algebras combine sets, relations and various operations linking the two in a unifying setting. This paper offers a modal perspective on Peirce algebras. Using modal logic a characterization of the full Peirce algebras is given, as well as a finite axiomatization of their equational theory that uses so-called unorthodox derivation rules. In addition, the expressive power of Peirce algebras is analyzed through their connection with first-order logic, and the fragment of first-order logic corresponding to Peirce algebras is described in terms of bisimulations.

Key words: Peirce algebras, modal logic, algebraic logic, relation algebras, logics of programs, knowledge representation

1. Introduction

This paper is part of an enterprise to relate modal languages, algebraic languages, and fragments of first-order logic. We will take a fragment of first-order logic for reasoning about binary relations, sets and certain interactions between them, and consider the algebraic framework of Peirce algebras that has recently been designed to capture this fragment (Brink (1994)). We will show how Peirce algebras arise as algebraic counterparts of a two-sorted modal language $\mathcal{ML}_2$; this language extends the modal formalism $CC\delta$ that was designed by Venema (1991) to reason about binary relations. In $\mathcal{ML}_2$ one can characterize the 'concrete' modal frames corresponding to full Peirce algebras (see De Rijke (1994)). Using this characterization we obtain a completeness result for 'concrete' frames. The reason why we work in the modal language $\mathcal{ML}_2$ rather than in its algebraic or first-order counterpart is that it allows us to reason with simple pictures and diagrams, and that powerful techniques for proving modal completeness in rich modal languages have recently become available through results of Venema (1993). Moreover, via the notion of bisimulation the modal perspective offers a new method for thinking about the expressive power of certain algebraic theories.

In the paper we move back and forth between algebraic logic, modal logic and first-order logic. Although we focus on Peirce algebras as our starting point, we hope that the general message will become clear: to understand fragments of first-order logic it may be very fruitful to view them either as modal or algebraic

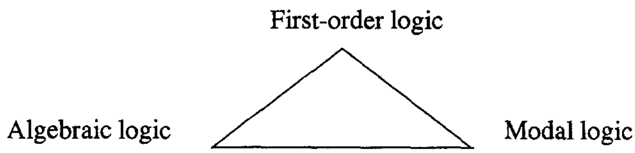


Fig. 1.

languages, and to understand these one can often exploit known results from each of the vertices in Figure 1.

The paper is organized as follows. The next section quickly reviews basic algebraic definitions; it also describes areas where Peirce algebras emerge. Section 3 briefly discusses the relation between relation algebra and Peirce algebras. Section 4 introduces the modal language $\mathcal{ML}_2$ for describing the modal counterparts of Peirce algebras. Section 5 quickly recalls a characterization of the 'real' or 'concrete' modal frames (corresponding to full Peirce algebras) given elsewhere, and Section 6 builds on this characterization to give a finite axiomatization of these concrete frames. Section 7 examines the relation between first-order logic and the modal language $\mathcal{ML}_2$, and Section 8 concludes with some questions.

2. Preliminaries

Peirce algebras have emerged as the common mathematical structures underlying many phenomena being studied in program semantics, AI and natural language analysis; they are also the modal algebras underlying the dynamic modal logic studied in De Rijke (1992b). Peirce algebras are two-sorted algebras in which sets and relations co-exist together with operations between them that model their interaction. The most important such operations considered here are the *Peirce product*: that takes a relation and a set, and returns a set

$$R : A = \{x \mid \exists y\,((x,y) \in R \wedge y \in A)\},$$

and *right cylindrification* ^{c} which takes a set and returns a relation

$$A^c = \{(x,y) \mid x \in A\}.$$

In this section we define Peirce algebras, and list some application areas where they arise. Let U be a set; $Re(U)$ is $\{R \mid R \subseteq U \times U\}$. R, S typically denote elements of $Re(U)$, while A, B typically denote elements of 2^U, the power set of U.

Recall the following operations on elements of $Re(U)$.

top	∇	$\{(r,s) \in (U \times U) \mid r, s \in U\}$
complement	$-R$	$\{(r,s) \in (U \times U) \mid (r,s) \notin R\}$
converse	R^{-1}	$\{(r,s) \in (U \times U) \mid (s,r) \in R\}$
diagonal	Id	$\{(r,s) \in (U \times U) \mid r = s\}$
composition	$R \mid S$	$\{(r,s) \in (U \times U) \mid \exists u\,((r,u) \in R \wedge (u,s) \in S)\}$

We also consider the following operations from $Re(U)$ and $Re(U) \times 2^U$ to 2^U

domain	$\mathsf{do}(R)$	$\{x \in U \mid \exists y \in U\,((x,y) \in R)\}$
range	$\mathsf{ra}(R)$	$\{x \in U \mid \exists y \in U\,((y,x) \in R)\}$
Peirce product	$R : A$	$\{x \in U \mid \exists y \in U\,((x,y) \in R \wedge y \in A)\}$,

as well as the following operations going from 2^U to $Re(U)$

tests	$A?$	$\{(x,y) \in (U \times U) \mid x = y \wedge x \in A)\}$
right cylindrification	A^c	$\{(x,y) \in (U \times U) \mid x \in A\}$.

A *relation type algebra* is a Boolean algebra with a binary operation ;, a unary operation $\breve{}$, and a constant 1'. The class FRA of *full relation algebras* consists of all relation type algebras isomorphic to an algebra of the form $\mathfrak{R}(U) = (Re(U), \cup, -, |, ^{-1}, Id)$. RRA is the class of *representable relation algebras*, that is, the class of subalgebras of products of algebras in FRA; so RRA $= \mathbf{SP}(\mathsf{FRA})$ (by a result due to Birkhoff RRA is also closed under homomorphic images: RRA $= \mathbf{HSP}(\mathsf{FRA})$).* Furthermore, RA is the class of *relation algebras*, that is, of relation type algebras $\mathfrak{A} = (A, +, -, ;, \breve{}, 1')$ satisfying the axioms

(R0)	$(A, +, -)$ is a Boolean algebra	(R5)	$x; 1' = x = 1'; x$
(R1)	$(x + y); z = x; z + y; z$	(R6)	$(x\breve{})\breve{} = x$
(R2)	$(x + y)\breve{} = x\breve{} + y\breve{}$	(R7)	$(x; y)\breve{} = y\breve{}; x\breve{}$
(R4)	$(x; y); z = x; (y; z)$	(R8)	$x\breve{}; -(x; y) \leq -y$.

The reader is referred to Jónsson (1982; 1991) for the essentials on relation algebra.

A *Peirce type algebra* is a two-sorted algebra $(\mathfrak{B}, \mathfrak{R}, :, ^c)$, where $\mathfrak{B}$ is a Boolean algebra, $\mathfrak{R}$ is a relation type algebra, : is a function from $\mathfrak{R} \times \mathfrak{B}$ to $\mathfrak{B}$, and $^c : \mathfrak{B} \to \mathfrak{R}$. The class FPA of *full Peirce algebras* consists of all Peirce type algebras isomorphic to an algebra of the form

$$\mathfrak{P}(U) = ((2^U, \cup, -, \emptyset), (Re(U), \cup, -, ^{-1}, |, Id), :, ^c).$$

* The reader is referred to any standard text on universal algebra for further details, e.g., Burris and Sankappanavar (1981).

The class RPA of *representable Peirce algebras* is defined as $\mathsf{RPA} = \mathbf{HSP}(\mathsf{FPA})$, the variety generated by FPA. PA is the class of *Peirce algebras*, that is of all Peirce type algebras $\mathfrak{A} = (\mathfrak{B}, \mathfrak{R}, :, {}^c)$ where $\mathfrak{B}$ is a Boolean algebra, $\mathfrak{R}$ is a relation algebra, : is a mapping $\mathfrak{R} \times \mathfrak{B} \to \mathfrak{B}$ such that

(P1) $r : (a + b) = (r : a) + (r : b)$ (P4) $1' : a = a$

(P2) $(r + s) : a = (r : a) + (s : a)$ (P5) $0 : a = 0$

(P3) $r : (s : a) = (r; s) : a$ (P6) $r^{\smile} : -(r : a) \leq -a$,

while ^{c} is a mapping $\mathfrak{B} \to \mathfrak{R}$ such that

(P7) $a^c : 1 = a$ (P8) $(r : 1)^c = r; 1$.

Algebras of the form $(\mathfrak{B}, \mathfrak{R}, :)$ were introduced by Brink (1981) as *Boolean modules*. Sources for Peirce algebras are Brink, Britz and Schmidt (1994) and Schmidt (1993).

Unlike the one-sorted language of relation algebras, the algebraic language of Peirce algebras has two sorts of terms: one interpreted in $\mathfrak{B}$, the other in $\mathfrak{R}$. Terms of the first sort are called *set terms*, terms of the second sort *relation terms*. Identities between set terms are called *set identities*; identities between relation terms are *relation identities*.

Brink *et al.* (1994) link Peirce algebras to dynamic algebras. Like Peirce algebras these are two-sorted algebras of sets and relations, but their relations are organized in a Kleene algebra, not in a relation algebra. It may be shown that any join-complete Peirce algebra gives rise to a dynamic algebra.

Another class of algebras closely related to Peirce algebras, is the class of extended relation algebras studied by Suppes (1976). Roughly, these are term-definably equivalent with Peirce algebras in which the sortal distinctions have been dropped.

WHERE PEIRCE ALGEBRAS EMERGE

In a number of areas frameworks are studied that have Peirce algebras in common as their underlying mathematical structures: knowledge representation, natural language analysis, weakest prespecifications, arrow logic, and modal logic.

In *terminological languages* one expresses information about hierarchies of concepts. They allow the definition of concepts and roles built out of primitive concepts and roles using various language constructs. *Concepts* are interpreted as sets, and *roles* as binary relations. Brink *et al.* (1994) propose a terminological language $\mathcal{U}^-$ whose operations are a notational variant of the operations of (full) Peirce algebras. For instance, $\mathcal{U}^-$ has an operation restrict that takes a relation and a set and returns a relation: (restrict R C) $= \{(x, y) \mid (x, y) \in \mathsf{R} \wedge y \in \mathsf{C}\}$. As $\mathcal{U}^-$ and (full) Peirce algebras share the same ontology, and the same operations, Peirce algebras supply a semantic interpretation for the terminological language

$\mathcal{U}^-$, in which the basic terminological concerns, viz. subsumption and satisfiability problems, re-appear as derivability issues in equational logic.

The next example involves the extended relation algebras mentioned earlier; as Suppes (1976) and Böttner (1992) show, those structures arise in attempts to equip fragments of *natural language* with variable free semantics. I will illustrate the main point with an example from Schmidt (1993). Consider a natural language fragment described by a phrase structure grammar G as in the left-hand side of (1), where S, NP, VP, TV, PN have their usual meaning: 'sentence,' 'noun phrase,' 'verb phrase,' 'transitive verb' and 'proper noun.'

$$\begin{array}{rcl} \text{S} & \rightarrow & \text{NP} + \text{VP} \\ \text{VP} & \rightarrow & \text{TV} + \text{NP} \\ \text{NP} & \rightarrow & \text{PN} \end{array} \qquad \begin{array}{c} [\text{NP}] \sqsubseteq [\text{VP}] \\ [\text{TV}] : [\text{NP}] \\ [\text{PN}]. \end{array} \tag{1}$$

Production rules in the grammar are associated with a semantic function $[\cdot]$ in a compositional way as indicated in the right-hand side of (1). In other words, semantic trees are construed in parallel with syntactic derivation trees. The semantic trees are linked to extended relation algebras via a valuation that maps terminal symbols of G onto an element of the algebra, where nouns are mapped onto sets and transitive verbs onto binary relations, thus equipping our natural language fragment with a variable free semantics.

The use of relation algebra in proving properties of programs goes back at least to De Bakker and De Roever (1973). The calculus of *weakest prespecifications* of Hoare and He (1987) is used as a formal tool in program specification. In this calculus programs are binary relations that may be combined using relation algebraic connectives. A special class of relations is called *conditions*; they express conditional statements, and are defined as the right ideal elements, that is, elements R for which $R = R; \nabla$. As the right ideal elements form a Boolean algebra, the natural algebraic setting for the calculus of weakest prespecifications is a Peirce algebra with programs living in a relation algebra, conditions living in a separate Boolean algebra, and ^{c} and : being used to move across from one to the other, cf. Brink *et al.* (1994).

Arrow logic arises as an effort to do transition logic without the computational complexity that comes with transition logics based on the identification of transitions as ordered pairs. Instead, arrow logic as developed by Van Benthem (1991) takes transitions seriously as dynamic objects in their own right. The general program proposes a re-design of systems of transition logic to isolate the genuine computational aspects from the mathematical modeling aspects. Van Benthem (1994) contains samples of this program; in particular, it discusses a two-sorted arrow logic whose models have both states and arrows, and whose formulas are sorted accordingly. The models of this (decidable) arrow logic may be viewed as an 'arrow-ized' version of our Peirce algebras; the decidability result is obtained by abstracting away from any set-theoretical assumptions concerning objects and operations of Peirce algebras. For instance, a test $\phi?$ is successfully performed at an arrow x_a if there exists a state y_s that is 'test-related' to x_a and which satisfies ϕ: $x_a \models \phi?$ iff for some state y_s, $Tx_a y_s$ and $y_s \models \phi$.

In the setting of information processing Van Benthem (1994) and De Rijke (1992b) study a system of *dynamic modal logic* called *DML*. *DML* is similar to propositional dynamic logic (*PDL*) in that it has formulas and procedures. The formulas ϕ and programs α of *DML* are built up as follows

$$\phi ::= p \mid \bot \neg\phi \mid \phi \wedge \phi \mid \mathsf{do}(\alpha) \mid \mathsf{ra}(\alpha) \mid \mathsf{fix}(\alpha) \text{ and}$$

$$\alpha ::= \mathsf{exp}(\phi) \mid -\alpha \mid \alpha^{\smile} \mid \alpha \cap \alpha \mid \alpha;\alpha \mid \phi?.$$

Here $\mathsf{exp}(\phi)$ is the special relation of 'expanding one's information with ϕ', and $\mathsf{fix}(\alpha)$ is formula that is true precisely at fixed points for α. Like *PDL*, *DML* only allows equational reasoning with formulas – not with programs. The modal algebras for *DML* are Peirce algebras over a single relation, the information order underlying the exp construct. To obtain a proper match one has to allow multiple exp constructs, each with its own underlying information order. The corresponding structures give rise to full Peirce algebras, and conversely. Moreover, the (extended) DML-operators are definable in full Peirce algebras, and the operators of full Peirce algebras are definable on DML-models:

DML	$\mathsf{do}(\alpha)$	$\mathsf{exp}_i(\phi)$	$\phi?$
FPA	$(\nabla : \alpha)$	$(\sqsubseteq_i : \phi)$	$\phi^c \cap Id$

FPA	$\alpha : \phi$	ϕ^c
DML	$\mathsf{do}(\alpha; \phi?)$	$\phi?; (\delta \cup -(\delta))$.

This implies that the complete axiomatization of DML structures presented in De Rijke (1992b) also generates the 'set equations' valid in FPA.

3. Peirce algebras and relation algebras

Brink *et al.* (1994) show that the equivalence '$r : a = b$ iff $r; a^c = b^c$' holds for all Peirce algebras; they also show that the Boolean elements in a Peirce algebra are precisely the *right ideal elements*, that is the elements satisfying $r = r; 1$:

THEOREM 3.1 *(Brink et al. (1994))* Let $(\mathfrak{B}, \mathfrak{R}, :, ^c)$ be a Peirce algebra. Then $\mathfrak{B}$ and $(\{r \in R \mid r = r; 1\}, +, \cdot, -, 0, 1)$ are isomorphic.

This result has a number of consequences. On the face of it, it seems to suggest that Peirce algebras are not required to study interactions between sets and relations: relation algebras suffice. However, following Brink *et al.* (1994) we argue that for application purposes, Peirce algebras are the more natural framework for modeling such interactions. Consider terminological reasoning. Peirce algebras are not just a mathematical framework; they model the application domain with great clarity. Viewing terminological reasoning in terms of right ideal elements adds nothing comparable.

But one can also make a *mathematical* case for Peirce algebras. For example, as is well known, there is no mathematical need to work with Boolean algebras – one can work with Boolean rings instead. But no one would deny that thinking of Boolean algebras in non-ring-theoretic terms has been very useful. Moreover, Peirce algebras have a very natural representation theory: the usual representation techniques in relation algebra and arrow logic usually extract points from (a Cartesian product of) the diagonal to obtain a base set over which a full algebra can be built; Peirce algebras have all the required points available in their Boolean reduct, thus allowing for a very direct representation result – see De Rijke (1994) for details. Finally, the fact that one *can* reduce one theory to another does not prove that one *should* abandon the former theory in favour of the latter.

Theorem 3.1 certainly has interesting logical consequences: representable Peirce algebras cannot be finitely axiomatized. The idea is that any finite axiomatization of representable Peirce algebras would yield a finite axiomatization of representable relation algebras, and by Lyndon (1950) this is impossible. Assume that Σ is a finite axiomatization of RPA. Σ may contain set identities $a = b$; replace these by relation identities $a^c = b^c$ – by Brink *et al.* (1994) the first identity is valid (and hence derivable) iff the latter is. So we may assume that Σ contains only relation identities. These may still contain occurrences of the Peirce product or of the cylindrification operator; to obtain purely relational axioms we need to get rid of such occurrences. By Brink *et al.* (1994), cylindrification commutes with all operations; in particular, $(r : a)^c = r; a^c$. Further, as all identities in Σ are assumed to be relational, every occurrence of : is in the scope of a ^{c}. Using this we can remove all occurrences of :, and push all occurrences of ^{c} down to the atomic level.

Let $s = t$ be a relation identity thus transformed. Using the fact that the Boolean elements are precisely the right ideal elements (Theorem 3.1), we translate $s = t$ into a quasi-identity of the form

$$r_{a_1} = r_{a_1}; 1 \;\&\; \ldots \;\&\; r_{a_n} = r_{a_n}; 1 \rightarrow [r_{a_1}/{a_1}^c, \ldots, r_{a_n}/{a_n}^c](s = t),$$

where $a_1, \ldots, a_n$ are all the Boolean terms occurring in $s = t$, and $r_{a_1}, \ldots, r_{a_n}$ are fresh relation terms.

This results in a finite axiomatization of RRA by means of quasi-identities. Now RRA is a discriminator variety, and by general results from universal algebra, in discriminator varieties every quasi-identity is equivalent to an identity (Burris and Sankappanavar (1981)). Hence, we have obtained a finite axiomatization of RRA by means of identities – the desired contradiction. Thus, we conclude that RPA is not finitely axiomatizable.

4. Modalizing Peirce algebras

Our first goal in this section is to briefly review the equational theory of modal logic. We do this by introducing a modal language for Peirce algebras. On the face

TABLE I. A plethora of notations.

	Full version	Abstract version	Modal version
relations	R, S	r, s	α, β
top	∇	1	$\mathbf{1}$
bottom	$\emptyset$	0	$\mathbf{0}$
diagonal	Id	1'	δ
complement	$-$	$-$	$-$
converse	$^{-1}$	$\breve{}$	$\otimes$
union	$\cup$	$+$	$\cup$
implication	$\rightarrow$		$\rightarrow$
composition	$\mid$	;	$\circ$
sets	A, B	a, b	ϕ, ψ
top	$\top$	1	$\top$
bottom	$\bot$	0	$\bot$
complement	$\neg$	$-$	$\neg$
union	$\cup$	$+$	$\vee$
implication	$\rightarrow$	$\rightarrow$	$\rightarrow$
right cylindrification	^{c}	c_1	$\updownarrow\cdot$
Peirce product	:	:	$\langle\cdot\rangle\cdot$

of it this may seem like a detour; however, by the end of the section it will be clear that this is actually a very effective route towards our goal of determining the logic of Peirce algebras. Table I lists the notation we adopt.

DEFINITION 4.1. Let $\Phi = \{p_0, p_1, \ldots\}$ be a countable set of propositional variables. Let Ω be a countable set of atomic relation symbols. The formulas of the *two-sorted language* $\mathcal{ML}_2(\delta, \otimes, \circ, \langle\rangle, \updownarrow; \Phi; \Omega)$, or $\mathcal{ML}_2$ for short, are generated by the rules

$$\phi ::= \bot \mid \top \mid p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \langle\alpha\rangle\phi \text{ and}$$

$$\alpha ::= \mathbf{0} \mid \mathbf{1} \mid \delta \mid a \mid -\alpha \mid \alpha_1 \cap \alpha_2 \mid \otimes\alpha \mid \alpha_1 \circ \alpha_2 \mid \updownarrow\phi.$$

The first sort of formulas will be interpreted as sets and called *set formulas*; formulas of the second sort will be interpreted as relations and called *relation formulas*.

DEFINITION 4.2. A *two-sorted frame* is a tuple $\mathfrak{F} = (W_s, W_r, I, R, C, F, P)$, where $W_s \cap W_r = \emptyset$, $I \subseteq W_r$, $R \subseteq {}^2W_r$, $C \subseteq {}^3W_r$, $F \subseteq W_r \times W_s$, and $P \subseteq W_s \times W_r \times W_s$.

Given a set U, a two-sorted frame is called *the two-sorted Peirce frame over U* if, for some base set U, $W_s = U$ and $W_r = U \times U$, and

$$I = \{(u, v) \in U \times U \mid u = v\}$$

$$\begin{aligned}
R &= \{((u_1,v_1),(u_2,v_2)) \in {}^2(U\times U) \mid u_1 = v_2 \wedge u_2 = v_1\}\\
C &= \{((u_1,v_1),(u_2,v_2),(u_3,v_3))\\
&\quad \in {}^3(U\times U) \mid u_1 = u_2 \wedge v_1 = v_3 \wedge v_2 = u_3\}\\
F &= \{((u_1,v_1),u_2) \in (U\times U)\times U \mid u_1 = u_2\}\\
P &= \{(u_1,(u_2,v_2),u_3) \in U\times(U\times U)\times U \mid u_1 = u_2 \wedge v_2 = u_3\}.
\end{aligned}$$

The class of two-sorted Peirce frames is denoted by TPF.

A model for $\mathcal{ML}_2$ is a *model based on a two-sorted frame*, that is, a structure $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F}$ is a two-sorted frame, and V is a *two-sorted valuation*, a function assigning subsets of W_s to set variables, and subsets of W_r to relation variables. *Truth of a formula at a state* is defined inductively, with the interesting clauses being

$$\begin{aligned}
\mathfrak{M}, x_r \models \delta &\text{ iff } x_r \in I\\
\mathfrak{M}, x_r \models \otimes\alpha &\text{ iff } \exists y_r\,(Rx_ry_r \wedge y_r \models \alpha)\\
\mathfrak{M}, x_r \models \alpha\circ\beta &\text{ iff } \exists y_rz_r\,(Cx_ry_rz_r \wedge y_r \models \alpha \wedge z_r \models \beta)\\
\mathfrak{M}, x_s \models \langle\alpha\rangle\phi &\text{ iff } \exists y_rz_s\,(Px_sy_rz_s \wedge y_r \models \alpha \wedge z_s \models \phi)\\
\mathfrak{M}, x_r \models \updownarrow\phi &\text{ iff } \exists y_s\,(Fx_ry_s \wedge y_s \models \phi).
\end{aligned}$$

Here $x_s, y_s, \ldots$ are taken from W_s; $x_r, y_r, \ldots$ are taken from W_r.

In models based on Peirce frames all the modal connectives receive their intended interpretation. That is, one has $(u,v) \models \delta$ iff $u = v$; $(u,v) \models \otimes\alpha$ iff $(v,u) \models \alpha$; $(u,v) \models \alpha\circ\beta$ iff $\exists w\,((u,w) \models \alpha \wedge (w,v) \models \beta)$; $u \models \langle\alpha\rangle\phi$ iff $\exists v\,((u,v) \models \alpha \wedge v \models \phi)$; and $(u,v) \models \updownarrow\phi$ iff $u \models \phi$.

Remark 4.3. We adopt a *local* perspective on satisfiability and consequence. The two-sorted setting of this paper calls for some comments. To avoid messy complications we define consequence only for *one-sorted* sets of formulas Σ, and formulas ξ of the same sort (compare Section 6.1). For K a class of frames we put $\Sigma \models_{\mathsf{K}} \xi$ iff for all models $(\mathfrak{F}, V)$ with $\mathfrak{F} \in \mathsf{K}$, and for every element x in $\mathfrak{F}$ of the appropriate sort:

$$(\mathfrak{F}, V), x \models \Sigma \text{ implies } (\mathfrak{F}, V), x \models \xi.$$

For one-sorted sets of formulas, notions like satisfiability are defined as usual.

To be able to state the connection between two-sorted Peirce frames and Peirce algebras, we recall that the *complex algebra* $\mathfrak{Cm}\,\mathfrak{F}$ of a two-sorted frame $\mathfrak{F}$ is given as $\mathfrak{A} = ((2^{W_s}, -, \cap, \emptyset, W_s), (2^{W_r}, -, \cap, m_\delta, m_\otimes, m_\circ, \emptyset, W_r), m_{\langle\rangle}, m_\updownarrow)$, where, for # an n-ary modal operator, $m_\#$ is an n-ary operator on the power set(s) of the appropriate domain(s) of $\mathfrak{F}$. To be precise

$$m_\delta = \{x_r \mid x_r \in I\}$$

$$\begin{aligned}
m_{\otimes}(X) &= \{x_r \mid \exists y_r\,(Rx_ry_r \wedge y_r \in X)\}\\
m_{\circ}(X,Y) &= \{x_r \mid \exists y_rz_r\,(Cx_ry_rz_r \wedge y_r \in X \wedge z_r \in Y)\}\\
m_{\langle\rangle}(X,Y) &= \{x_s \mid \exists y_rz_s\,(Px_sy_rz_s \wedge y_r \in X \wedge z_s \in Y)\}\\
m_{\updownarrow}(X) &= \{x_r \mid \exists y_s\,(Fx_ry_s \wedge y_s \in X)\}.
\end{aligned}$$

For K a class of frames **Cm**(K) is the class of complex algebras of frames in K.

The following justifies our introduction of TPF and the modal language $\mathcal{ML}_2$ as tools for understanding full Peirce algebras: if $\mathfrak{F}$ is a two-sorted frame, then $\mathfrak{F}$ is a Peirce frame (or: in TPF) iff $\mathfrak{Cm}\,\mathfrak{F}$ is (isomorphic) to a full Peirce algebra. In other words: $\mathbf{Cm}(\mathsf{TPF}) = \mathsf{FPA}$. Thus, instead of studying full Peirce algebras by algebraic means we can as well study two-sorted Peirce frames by modal means.

However, Peirce frames can not be characterized in $\mathcal{ML}_2$; the reason is that $\mathsf{FPA} = \mathbf{Cm}\,(\mathsf{TPF})$ is not a variety as it is not closed under products or subalgebras. However, if we are wiling to extend the modal language, a characterization can be obtained.

More precisely, to characterize the Peirce frames we will use special modal operators called *difference operators*; their special feature is that they are interpreted using the diversity relation $\neq$, one for each domain in a two-sorted frame. We use D_s and D_r to denote them:

$$\begin{aligned}
&x_s \models D_s\phi \text{ iff for some } y_s \neq x_s, y_s \models \phi \text{ where } x_s, y_s \in W_s\\
&x_r \models D_r\alpha \text{ iff for some } y_r \neq x_r, y_r \models \alpha \text{ where } x_r, y_r \in W_r.
\end{aligned}$$

Using the difference operators we can define other useful operators such as E, where $E\xi := \xi \vee D\xi$ (there exists an object with ξ), and O, where $O\xi = E(\xi \wedge \neg D\xi)$ (there is only one object with ξ); these defined operators will be indexed with an s or an r. The reader is referred to De Rijke (1992a) for details about logics with difference operators.

Observe that on Peirce frames the difference operators can be defined as follows

$$D_s'\phi := \langle -\delta\rangle\phi \text{ and } D_r'\alpha := (-\delta \circ \alpha \circ 1) \cup (1 \circ \alpha \circ -\delta).$$

5. Characterizing Peirce frames

As a prelude to the completeness result for two-sorted Peirce frames, we briefly present a characterization of two-sorted Peirce frames; we refer the reader to De Rijke (1994) for proofs and details. We proceed in two steps.

In the first step we characterize a class of Peirce like frames. We need a number of axioms governing the structure of Peirce frames. We first list the modal axioms handling the relational component of two-sorted frames plus the conditions they impose on such frames; they are simply the modal counterparts of the earlier relation algebraic axioms (R1)–(R8), and the corresponding conditions have been calculated by Lyndon (1950) and Maddux (1982). We then list the modal counterparts of the Peirce axioms (P1)–(P8), and calculate the corresponding conditions on frames. (Recall that a first-order condition γ is said to *correspond* to a modal formula ξ if for all frames $\mathfrak{F}$, $\mathfrak{F} \models \gamma$ iff $\mathfrak{F} \models \xi$; if γ corresponds to ξ, we also say that ξ *defines* or *expresses* γ.)

The first axiom states that R, the interpretation of $\otimes$, is a function; this is proved by standard arguments.

(MR0) $\otimes a \leftrightarrow -\otimes -a$ (CR0) R is a function

So, in frames validating (MR0) we are justified in interpreting $\otimes$ using a unary function f, and evaluating formulas $\otimes\alpha$ as follows

$$\mathfrak{M}, x_r \models \otimes\alpha \text{ iff } \mathfrak{M}, f(x_r) \models \alpha.$$

A *two-sorted arrow frame* is simply a two-sorted frame $\mathfrak{F} = (W_s, W_r, I, f, C, F, P)$ in which the binary relation R used to interpret the operator $\otimes$ is a function from W_r to W_r, denoted by f. A *two-sorted arrow model* is a two-sorted model based on a two-sorted arrow frame, where $\otimes$ is interpreted using the function f as indicated above.

Here are the remaining axioms governing the behavior of δ, $\otimes$ and $\circ$, as well as the conditions expressed by these axioms.

(MR1)	$a \to \otimes\otimes a$	(CR1)	$f(f(x_r)) = x_r$
(MR2)	$a \circ (b \circ c) \to (a \circ b) \circ c$	(CR2)	$\forall y_r z_r u_r v_r\,(Cx_r y_r z_r \wedge Cz_r u_r v_r \to \exists w_r\,(Cx_r w_r v_r \wedge Cw_r y_r u_r))$
(MR3)	$(a \circ b) \circ c \to a \circ (b \circ c)$	(CR3)	$\forall y_r w_r u_r v_r\,(Cx_r w_r v_r \wedge Cw_r y_r u_r \to \exists z_r\,(Cx_r y_r z_r \wedge Cz_r u_r v_r))$
(MR4)	$a \to \delta \circ a,\ a \to a \circ \delta$	(CR4)	$\exists y_r\,(Iy_r \wedge Cx_r y_r x_r),\ \exists y_r\,(Cx_r x_r y_r \wedge Iy_r)$
(MR5)	$\delta \circ a \to a,\ a \circ \delta \to a$	(CR5)	$\forall y_r z_r\,(Cx_r y_r z_r \wedge Iy_r \to x_r = z_r)$, $\forall y_r z_r\,(Cx_r y_r z_r \wedge Iz_r \to x_r = y_r)$
(MR6)	$\otimes(a \circ b) \to (\otimes b \circ \otimes a)$	(CR6)	$\forall y_r z_r\,(Cf(x_r) y_r z_r \to Cx_r f(z_r) f(y_r))$
(MR7)	$(\otimes b \circ \otimes a) \to \otimes(a \circ b)$	(CR7)	$\forall y_r z_r\,(Cx_r f(z_r) f(y_r) \to Cf(x_r) y_r z_r)$
(MR8)	$\otimes a \circ -(a \circ b) \cap b \to \mathbf{0}$	(CR8)	$\forall y_r z_r\,(Cx_r f(y_r) z_r \to Cz_r y_r x_r)$.

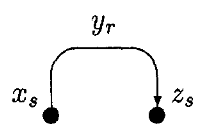
Next come the axioms governing the behavior of the Peirce product and cylindrification.

(MP1)	$\langle a\rangle\langle b\rangle p \to \langle a \circ b\rangle p$	(CP1)	$\forall y_r y_r' z_s z_s'\,(Px_s y_r z_s \wedge Pz_s y_r' z_s' \to \exists y_r''\,(Px_s y_r'' z_s' \wedge Cy_r'' y_r y_r'))$
(MP2)	$\langle a \circ b\rangle p \to \langle a\rangle(\langle b\rangle p)$	(CP2)	$\forall y_r y_r' y_r'' z_s\,(Px_s y_r z_s \wedge Cy_r y_r' y_r'' \to \exists z_s'\,(Px_s y_r' z_s' \wedge Pz_s' y_r'' z_s))$
(MP3)	$\langle\delta\rangle p \to p$	(CP3)	$\forall y_r z_s\,(Px_s y_r z_s \wedge Iy_r \to x_s = z_s)$
(MP4)	$p \to \langle\delta\rangle p$	(CP4)	$\exists y_r\,(Px_s y_r x_s \wedge Iy_r)$

(MP5)	$\langle\otimes a\rangle\neg\langle a\rangle p \wedge p \rightarrow \bot$	(CP5)	$\forall y_r z_s\,(Px_s y_r z_s \rightarrow Pz_s f(y_r) x_s)$
(MP6)	$\langle\updownarrow p\rangle\top \rightarrow p$	(CP6)	$\forall y_r z_s z'_s\,(Px_s y_r z_s \wedge Fy_r z'_s \rightarrow x_s = z'_s)$
(MP7)	$p \rightarrow \langle\updownarrow p\rangle\top$	(CP7)	$\exists y_r z_s\,(Px_s y_r z_s \wedge Fy_r x_s)$
(MP8)	$\updownarrow\langle a\rangle\top \rightarrow (a \circ \mathbf{1})$	(CP8)	$\forall y_s y'_r z_s\,(Fx_r y_s \wedge Py_s y'_r z_s \rightarrow \exists z'_r\,(Cx_r y'_r z'_r))$
(MP9)	$(a \circ \mathbf{1}) \rightarrow \updownarrow\langle a\rangle\top$	(CP9)	$\forall y_r z_r\,(Cx_r y_r z_r \rightarrow \exists y'_s z'_s\,(Fx_r y'_s \wedge Py'_s y_r z'_s))$.

It follows from the general results of De Rijke (1993) that the above axioms (MRi) and (MPi) correspond to the conditions (CMi) and (CPi). All axioms listed here are so-called Sahlqvist formulas, and for such formulas there is an explicit algorithm computing the corresponding relational condition. Here we compute one such correspondence result 'by hand.' We consider axiom (MP7) and condition (CP7), i.e., $p \rightarrow \langle\updownarrow p\rangle\top$ and $\exists y_r z_s\,(Px_s y_r z_s \wedge Fy_r x_s)$.

Assume first that x_s is a set element in some frame $\mathfrak{F}$ in which (MP7) is valid. This means that (MP7) is true at x_s under the special valuation that assigns p to x_s (and only to x_s). It follows that $x \models \langle\updownarrow p\rangle\top$, that is: there are y_r, z_s with $Px_s y_r z_s$ and $y_r \models \updownarrow p$. By the latter conjunct, there must be a x'_s with $Fy_r x'_s$ and $x'_s \models p$ – but as p is true at x_s only, we must have $x_s = x'_s$:



For the converse, assume that we are in a situation as depicted above, that is, $Px_s y_r z_s$, $Fy_r x_s$ and $x_s \models p$. We need that $x_s \models \langle\updownarrow p\rangle\top$. As $Fy_r x_s$ and $x_s \models p$, it follows that $y_r \models \updownarrow p$. From this and $Px_s y_r z_s$ we get $x_r \models \langle\updownarrow p\rangle\top$, as required.

DEFINITION 5.1. A two-sorted arrow frame is *Peirce like* if it satisfies conditions (CR1)–(CR8), as well as (CP1)–(CP9). The class of Peirce like frames is denoted by TPLF.

LEMMA 5.2. Let $\mathfrak{F}$ be a two-sorted arrow frame. Then $\mathfrak{F} \in \mathsf{TPLF}$ iff $\mathfrak{F} \models$ (MPi).

We now arrive at the second stage in our characterization result: we narrow down the two-sorted Peirce like frames to two-sorted Peirce frames. Briefly, what we need, to show that a two-sorted Peirce like frame is a two-sorted Peirce frame, is the following

- With every relational element we can associate a unique set element as its first coordinate and a unique set element as its second coordinate.
- With every two set elements we can associate a unique relational element having those set elements as first and second coordinate.

This boils down to having the following conditions satisfied by our Peirce like frames:

(CP10) $\forall x_r y_s y'_s\,(Fx_r y_s \wedge Fx_r y'_s \rightarrow y_s = y'_s)$
(CP11) $\forall x_r y_s y'_s\,(Ff(x_r)y_s \wedge Ff(x_r)y'_s \rightarrow y_s = y'_s)$
(CP12) $\forall x_r \exists y_s\,(Fx_r y_s)$
(CP13) $\forall x_r \exists y_s\,(Ff(x_r)y_s)$
(CP14) $\forall x_s y_s \exists z_r\,(Px_s z_r y_s)$
(CP15) $\forall x_s y_s z_r z'_r\,(Px_s z_r y_s \wedge Px_s z'_r y_s \rightarrow z_r = z'_r)$.

We leave it to the reader to check that Peirce-like frames validate each of (CP10)–(CP13), and that those conditions can be defined in $\mathcal{ML}_2$.

To pin down the Peirce frames we add two axioms, corresponding to (CP14) and (CP15); those axioms involve a difference operator. Recall from Section 4 that the operator E_s is short for $E_s p \equiv p \vee D_s p$ (there exists a state where p holds), and that the operator O_s is short for $O_s p \equiv (p \wedge \neg D_s p)$ (p is true only here).

We define the following two formulas:

(MP14) $E_s p \rightarrow \langle 1 \rangle p$
(MP15) $E_s O_s p \wedge \langle a \rangle p \wedge \langle b \rangle p \rightarrow \langle a \cap b \rangle p$.

It can then be shown that if $\mathfrak{F}$ is a two-sorted Peirce like frame, then $\mathfrak{F}$ satisfies (CP14) and (CP15) iff it validates (MP14) and (MP15), respectively.

Putting things together, De Rijke (1994) obtains a characterization of Peirce frames:

THEOREM 5.3. TPF $= \{\mathfrak{F} \mid \mathfrak{F} \models \bigwedge_{0\leq i\leq 8}(\text{MR}i) \wedge \bigwedge_{0\leq i\leq 9}(\text{MP}i) \wedge (\text{MP14}) \wedge (\text{MP15})\}$.

6. Completeness

We will now use our characterization of Peirce frames to obtain a complete axiomatization of Peirce frames. We will use a strategy due to Yde Venema (1991; 1993) to prove completeness of derivation systems involving difference operators. For this strategy to be applicable our logic should satisfy the following conditions:

- It needs to have difference operators; these difference operators should satisfy certain axioms and rules.
- Converses, and more generally, conjugates for all modal operators (see below), and axioms expressing the appropriate relationships between conjugates.
- Inclusion axioms for all modal operators, stating that for any of the accessibility relations in our Peirce frames, a move along one of these relations must either lead to the same point or to a point that can be reached by using a difference operator.

To actually verify that the above conditions are satisfied we have to work our way through a number of cumbersome technicalities. Below we start by axiomatizing the Peirce like frames as a first approximation. We then add all the required operators and axioms, and apply Venema's strategy to arrive at our completeness result for Peirce frames. And finally we briefly discuss a slight simplification of the logic.

6.1. A FIRST APPROXIMATION

As a first step we axiomatize the logic of Peirce like frames in the language $\mathcal{ML}_2$.

Let *MLPL* be the minimal normal modal axiom system in $\mathcal{ML}_2(\delta, \circ, \otimes, \langle\rangle, \updownarrow)$ that has (MR0)–(MR8) and (MP1)–(MP9) as axioms. So, besides (MR0)–(MR8) and (MP1)–(MP9), *MLPL* has the Boolean axioms for $\neg$, $\wedge$, $\bot$, $\top$; the Boolean axioms for $-$, $\cap$, $\mathbf{0}$, $\mathbf{1}$; and distribution axioms for the modal operators:

$$
\begin{array}{lll}
\otimes & : & \overline{\otimes}(a \to b) \to (\overline{\otimes}a \to \overline{\otimes}b), \text{ where } \overline{\otimes}\alpha \equiv -\otimes-\alpha \\
\circ & : & (a \to b)\,\bar{\circ}\,c \to ((a\,\bar{\circ}\,c) \to (b\,\bar{\circ}\,c)), \text{ where } \alpha\,\bar{\circ}\,\beta \equiv -(-\alpha \circ -\beta) \\
\circ & : & a\,\bar{\circ}\,(b \to c) \to ((a\,\bar{\circ}\,b) \to (a\,\bar{\circ}\,c)) \\
\langle\rangle & : & [\![a \to b]\!]p \to ([\![a]\!]p \to [\![b]\!]p), \text{ where } [\![\alpha]\!]\phi \equiv \neg\langle-\alpha\rangle\neg\phi \\
\langle\rangle & : & [\![a]\!](p \to q) \to ([\![a]\!]p \to [\![a]\!]q) \\
\updownarrow & : & \overline{\updownarrow}(p \to q) \to (\overline{\updownarrow}p \to \overline{\updownarrow}q), \text{ where } \overline{\updownarrow}\phi \equiv -\updownarrow\neg\phi.
\end{array}
$$

In addition, *MLPL* has the derivation rules modus ponens (MP), substitution (SUB), and necessitation (NEC), for all modal operators. The latter covers the following:

$(\mathrm{NEC}_{\otimes})$ $\alpha\,/\,\overline{\otimes}\alpha$	$(\mathrm{NEC}_{\langle\rangle})$ $\alpha\,/\,[\![\alpha]\!]\phi$	$(\mathrm{NEC}_{\circ})$ $\alpha\,/\,\alpha\,\bar{\circ}\,\beta$
$(\mathrm{NEC}_{\updownarrow})$ $\phi\,/\,\overline{\updownarrow}\phi$	$(\mathrm{NEC}_{\langle\rangle})$ $\phi\,/\,[\![\alpha]\!]\phi$	$(\mathrm{NEC}_{\circ})$ $\beta\,/\,\alpha\,\bar{\circ}\,\beta$.

For L a (two-sorted) modal logic we define an *L-derivation* to be a list of formulas from the language of L such that every formula is either a substitution instance of an axiom of L, or obtained from earlier formulas in the list by means of a derivation rule of L. An *L-theorem* is any formula that can occur as the last item in a derivation. We write $\vdash_{\mathbf{L}} \xi$ for ξ is an L-theorem, and $\Sigma \vdash_{\mathbf{L}} \xi$ for: there are $\sigma_1, \ldots, \sigma_n \in \Sigma$ such that $\vdash_{\mathbf{L}} (\sigma_1 \wedge \ldots \wedge \sigma_n) \to \xi$ (if ξ is a set formula), or $\vdash_{\mathbf{L}} (\sigma_1 \cap \ldots \cap \sigma_n) \to \xi$ (if ξ is a relation formula). (Compare Remark 4.3.)

THEOREM 6.1. *MLPL* is strongly sound and complete for TPLF.

Proof. To prove the theorem one may use the standard canonical model construction, or one may observe that all *MLPL*-axioms are Sahlqvist formulas, and derive immediately that *MLPL* is complete with respect Peirce like frames satisfying the conditions (CRi) and (CPi) (see De Rijke and Venema (1995)). □

6.2. A COMPLETE AXIOMATIZATION OF PEIRCE FRAMES

To be able to apply Venema's strategy we need to add (to the logic *MLPL* and its language) difference operators with their axioms and rules, conjugates with their axioms and rules, and so-called inclusion axioms. We now consider each of those components. Fortunately, it will turn out that nearly all of the required additions can already be defined or derived within *MLPL*.

Axioms for the difference operators. Instead of adding primitive difference operators, we consider the derived ones D'_s and D'_r:

$$D'_s\phi := \langle -\delta \rangle \phi \text{ and } D'_r\alpha := (-\delta \circ \alpha \circ 1) \cup (1 \circ \alpha \circ -\delta).$$

(Compare Section 4.) The difference operators are governed by the following axioms and rules:

(MD1) $\overline{D}(k \to l) \to (\overline{D}k \to \overline{D}l)$, where $\overline{D} \equiv \neg D \neg$,
(MD2) $DDk \to k \vee Dk$
(MD3) $k \to \overline{D}Dk$
(NEC_D) $\xi \,/\, \overline{D}\xi$
(IR_D) $k \wedge \neg Dk \to \xi / \xi$, provided k does not occur in ξ.

Axiom (MD1) is the usual distribution law for normal modal operators; (MD2) expresses that the diversity relation is pseudo-transitive (that is, it satisfies the condition $\forall xyz\,(Rxy \wedge Ryz \to x = z \vee Rxz)$), and (MD3) expresses that it is symmetric. To understand the *irreflexivity rule* (IR_D) we first reason semantically: assume that $\neg\xi$ is satisfiable at some state x and that k is an atomic symbol that does not occur in ξ; as the diversity relation $\neq$ is irreflexive we find that by making k true only at x we can satisfy the conjunction $k \wedge \neg Dk \wedge \neg\xi$. So, if $\neg\xi$ is satisfiable, so is $\neg\xi \wedge k \wedge \neg Dk$ for any k not occurring in ξ. Turning to validity: the rule 'if $\vdash k \wedge \neg Dk \to \xi$ then $\vdash \xi$ provided that k does not occur in ξ' is sound.

We will show below that except for the irreflexivity rules each of the axioms (MD1)–(MD3) and the rule (NEC_D) is derivable in *MLPL* for the defined difference operators.

Axioms for conjugated operators. Let R be an $(n+1)$-ary relation. A frame $\mathfrak{F} = (\ldots, R, \ldots)$ is called *versatile for* R if there are relations $R_1, \ldots, R_n$ such that for all $x_0, \ldots, x_n$ one has $(x_0, \ldots, x_n) \in R$ iff $(x_1, \ldots, x_n, x_0) \in R_1$ iff $\ldots$ iff $(x_n, x_0, \ldots, x_{n-1}) \in R_n$. Once we know that a frame is versatile for R, it suffices to mention just a single R, and suppress the other relations.

Let # be an n-ary modal operator whose semantics is based on an $(n+1)$-ary relation R; the *conjugates of* # are n operators $\#_1, \ldots, \#_n$ whose semantics are based on $(n+1)$-ary relations $R_1, \ldots, R_n$, respectively, such that $R, R_1, \ldots, R_n$ form a versatile system, and

$$x \models \#_i(\xi_1, \ldots, \xi_n) \text{ iff } \exists y_1 \ldots y_n\,(Rxy_1 \ldots y_n \wedge \textstyle\bigwedge_i y_i \models \xi_i).$$

Unary modal operators whose underlying relation is symmetric form their own conjugates; also, a frame is versatile for a binary relation B if it contains the converse B^{-1} of B.

We will now define conjugated operators for all modal operators in $\mathcal{ML}_2$. Note that the defined difference operators D_s', D_r', and the converse operator $\otimes$ are self-conjugated; so we do not need to add conjugates for them. For $\updownarrow$ we define a conjugate $\Updownarrow$ by putting $\Updownarrow\alpha := \langle\alpha\rangle\top$; so $\Updownarrow$ takes a relation formula and returns a set formula. For $\langle\cdot\rangle\cdot$ we define two conjugate operators $\langle\cdot\rangle_1$ and $\langle\cdot\rangle_2\cdot$ by putting $\langle\phi\rangle_1\psi := \otimes(\updownarrow\phi) \cap \updownarrow\psi$, and $\langle\phi\rangle_2\alpha := \langle\otimes\alpha\rangle\phi$. For $\circ$ we also add two operators, written $\circ_1$ and $\circ_2$, defined by $\alpha \circ_1 \beta := \beta \circ \otimes\alpha$ and $\alpha \circ_2 \beta := \otimes\beta \circ \alpha$.

All in all, we have have introduced the following abbreviations:

$$\begin{array}{rclrcl} \Updownarrow\alpha & := & \langle\alpha\rangle\top & & & \\ \langle\phi\rangle_1\psi & := & \otimes(\updownarrow\phi) \cap \updownarrow\psi & \alpha \circ_1 \beta & := & \beta \circ \otimes\alpha \\ \langle\phi\rangle_2\alpha & := & \langle\otimes\alpha\rangle\phi & \alpha \circ_2 \beta & := & \otimes\beta \circ \alpha. \end{array}$$

To motivate the above, observe that $y_r \models \langle p\rangle_1 q$ means that there exist x_s, z_s with $Pz_sy_rx_s$ and $x_s \models p$, $z_s \models q$; hence, $y_r \models (\otimes\updownarrow p) \cap (\updownarrow q)$ $(= \langle p\rangle_1 q)$. Second, $z_s \models \langle p\rangle_2 a$ iff there exist x_s, y_r with $Px_sy_rz_s$ and $x_s \models p$, $y_r \models a$. Hence $z_s \models \langle\otimes a\rangle p$ $(= \langle p\rangle_2 a)$. Similar remarks pertain to $\circ_1$, $\circ_2$.

We force the appropriate modal operators to be each others conjugates by imposing the axioms below; $\overline{\updownarrow}\phi$ abbreviates $-\updownarrow\neg\phi$, and $\overline{\Updownarrow}\alpha$ abbreviates $\neg\Updownarrow-\alpha$.

(MP16) $a \to \overline{\updownarrow}\Updownarrow a$
(MP17) $p \to \overline{\Updownarrow}\updownarrow p$
(MP18) $p \wedge \langle -\langle q\rangle_1 p\rangle q \to \bot$
(MP19) $a \cap \langle \neg\langle p\rangle_2 a\rangle_1 p \to \mathbf{0}$
(MP20) $p \wedge \langle \neg\langle a\rangle p\rangle_2 a \to \bot$
(MP21) $a \cap -(b \circ_1 a) \circ b \to \mathbf{0}$
(MP22) $a \cap -(b \circ_2 a) \circ_1 b \to \mathbf{0}$
(MP23) $a \cap -(b \circ a) \circ_2 b \to \mathbf{0}$

The first of the above two axioms are well-known from temporal logic; they simply express that $\updownarrow$ and $\Updownarrow$ are interpreted using converse relations; likewise, axioms (MP18)–(MP23) express that $\langle\cdot\rangle\cdot$, $\langle\cdot\rangle_1\cdot$, $\langle\cdot\rangle_2\cdot$ and $\circ$, $\circ_1$, $\circ_2$ form conjugate triples, cf. Venema (1991).

Inclusion axioms. One special feature of the difference operators is that if one moves along one of the accessibility relations using the modal operators $\circ$, $\updownarrow$, $\otimes$, or $\langle\ \rangle$, one either gets back to the starting point or to a point that one must be able to reach using one of the difference operators. This feature is implemented by the following so-called *inclusion axioms*:

(INC1) $\langle a\rangle p \to E_s'p$
(INC2) $\otimes a \to E_r'a$
(INC3) $a \circ b \to E_r'a \wedge E_r'b$
(INC4) $\langle D_r'(\updownarrow q)\rangle\top \to E_s'q$
(INC5) $\updownarrow(D_s'\langle a\rangle\top) \to E_r'a.$

The logic of Peirce frames. We are ready now to define the modal logic of two-sorted Peirce frames and to prove its completeness.

DEFINITION 6.2. We define one more axiom system: *MLP*. Its language is $\mathcal{ML}_2$. Its axioms are those of *MLPL* (Section 6.1), and its rules of inference are those of *MLPL* plus the following two irreflexivity rules:

(IR$_s$) $p \wedge \neg D'_s p \to \phi \,/\, \phi$, where p does not occur in ϕ
(IR$_r$) $a \cap -D'_r a \to \alpha \,/\, \alpha$, where a does not occur in α.

To prove *MLP* complete using Venema's strategy, we need to show that it derives the axioms and rules for the difference operators and conjugate operators, as well as the inclusion axioms. Showing this is in fact the heart of the completeness proof.

THEOREM 6.3. Let $\Delta \cup \{\xi\}$ be a set of $\mathcal{ML}_2$-formulas. Then $\Delta \vdash \xi$ in *MLP* iff $\Delta \models_{\mathsf{TPF}} \xi$.

Proof. Proving soundness is left to the reader. As to completeness, by Lemmas A.2 and A.3 *MLP* satisfies all the requirements needed for an application of Venema's strategy as described at the start of this section: both the axioms for the difference operators and the conjugated operators and the inclusion axioms are derivable in derivable in *MLP*, and the necessitation rules for the difference operators and the conjugated operators are all derived rules of *MLP*. So by Venema (1993, Theorem 7.7) *MLP* is strongly complete for the class of frames validating axioms (MR1)–(MR8) and (MP1)–(MP15). Hence, by our characterization result Theorem 5.3, *MLP* is strongly complete for Peirce frames. That is, $\Delta \models_{\mathsf{TPF}} \xi$ implies $\Delta \vdash \xi$. □

To conclude this section we briefly discuss the number of irreflexivity rules that we need. From our observations in Section 3 we know that we need at least some non-standard means to get a complete axiomatization for Peirce frames. To rephrase this somewhat inaccurately, we need at least one irreflexivity rule. Further, by Theorem 6.3. we know that we need at most two. However, we can get by with just one, namely the irreflexivity rule for D'_r. The one for D'_s can be replaced by the derived rule

$$\langle a \cap -D'_r a \cap \delta\rangle\top \wedge \neg D'_s\langle a \cap -D'_r a \cap \delta\rangle\top \to \phi \,/\, \phi, \quad \text{provided } a \text{ does not occur in } \phi. \tag{2}$$

As with the irreflexivity rule for D'_s, the intuition is that if ϕ is consistent then it is consistent to have ϕ together with a 'unique name'. With the rule (2) a unique name for a set element (namely $\langle a \cap -D'_r \cap \delta\rangle\top \wedge \neg D'_s\langle a \cap -D'_r \cap \delta\rangle\top$) is borrowed from a unique name for a diagonal element – and being relation elements such elements will get unique names by the irreflexivity rule for D'_r.

It can be shown that (2) is a derived rule in *MLP* minus the irreflexivity rule for D'_s, but the proof is too cumbersome (and uninformative) to be included here.

TABLE II. The first-order translation for $\mathcal{ML}_2$.

$ST_i(\top)$	=	$(x_i = x_i)$
$ST_i(p)$	=	$P(x_i)$
$ST_i(\neg\phi)$	=	$\neg ST_i(\phi)$
$ST_i(\phi \wedge \psi)$	=	$ST_i(\phi) \wedge ST_i(\psi)$
$ST_i(\langle\alpha\rangle\phi)$	=	$\exists x_j\,(ST_{ij}(\alpha) \wedge ST_j(\phi))$
$ST_{ij}(\mathbf{1})$	=	$(x_i = x_i) \wedge (x_j = x_j)$
$ST_{ij}(\delta)$	=	$(x_i = x_j)$
$ST_{ij}(a)$	=	$A(x_i, x_j)$
$ST_{ij}(-\alpha)$	=	$\neg ST_{ij}(\alpha)$
$ST_{ij}(\alpha \cap \beta)$	=	$ST_{ij}(\alpha) \wedge ST_{ij}(\beta)$
$ST_{ij}(\otimes\alpha)$	=	$ST_{ji}(\alpha)$
$ST_{ij}(\alpha \circ \beta)$	=	$\exists x_k\,(ST_{ik}(\alpha) \wedge ST_{kj}(\beta))$
$ST_{ij}(\updownarrow\phi)$	=	$ST_i(\phi) \wedge (x_j = x_j)$.

7. Expressive power

Consider the triangle 'algebraic logic – modal logic – first-order logic' depicted in the introduction again. In sections 4–6 we concentrated on the 'algebraic logic – modal logic' side of the triangle to arrive at a complete axiomatization. In the present section we will put together some results from the modal literature that bear on the 'modal logic – first-order logic' side. Concretely, we will characterize the fragment of first-order logic that corresponds to the modal language $\mathcal{ML}_2$ using an appropriate notion of bisimulation; along the way we obtain a definability result.

When interpreted on Peirce models (that is, on models based on Peirce frames), $\mathcal{ML}_2$-formulas become equivalent to first-order formulas of the following kind. Let τ be the (first-order) vocabulary $\{P_1, P_2, \ldots, A_1, A_2, \ldots\}$, where the P_i's are unary relation symbols corresponding to the atomic set variables p_i in our language, and the A_i's are binary relation symbols corresponding to the atomic relation variables. Let $\mathcal{L}(\tau)$ be the set of all first-order formulas over τ (with identity).

We now define a translation ST taking $\mathcal{ML}_2$-formulas to formulas in $\mathcal{L}(\tau)$. Fix three distinct individual variables x_1, x_2, and x_3, and let i, j, k denote distinct objects in $\{1, 2, 3\}$. Consider Table II. For ϕ a set formula in $\mathcal{ML}_2$, we define the *standard translation* $ST(\phi)$ of ϕ by $ST(\phi) := ST_1(\phi)$; and if α is a relation formula in $\mathcal{ML}_2$, we define its *standard translation* by $ST(\alpha) := ST_{12}(\alpha)$.

Peirce models may be viewed as models for $\mathcal{L}(\tau)$: to interpret the predicate symbols in $\mathcal{L}(\tau)$ we simply use the the values that the valuation assigns to the corresponding modal symbols.

PROPOSITION 7.1. Let ϕ be a set formula in $\mathcal{ML}_2$, α a relation formula in $\mathcal{ML}_2$, and $\mathfrak{M}$ a Peirce model. For any x in $\mathfrak{M}$, $\mathfrak{M}, x \models \phi$ iff $\mathfrak{M} \models ST(\phi)[x]$. For any x, y in $\mathfrak{M}$, $\mathfrak{M}, (x, y) \models \alpha$ iff $\mathfrak{M} \models ST(\alpha)[xy]$.

So, on models every $\mathcal{ML}_2$-formula is equivalent to a first-order formula, or more precisely, to a first-order formula containing at most three variables. A natural question at this point is to ask for special *semantic* features that isolate the fragment of $\mathcal{L}(\tau)$ that corresponds to $\mathcal{ML}_2$. Similar questions have been raised and answered before in the literature on modal logic; we refer the reader to Van Benthem (1976; 1991) and De Rijke (1992a; 1995) for examples. Bisimulations have turned out to be an important tool in answering such questions – and in our case too we will use an appropriate notion of bisimulation.

DEFINITION 7.2. A *bisimulation for* $\mathcal{ML}_2$ between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is a non-empty relation $Z \subseteq (W_1 \times W_2) \cup (W_1^2 \times W_2^2)$ such that

1. $Z\mathbf{xy}$ implies $\mathrm{lh}(\mathbf{x}) = \mathrm{lh}(\mathbf{y})$, where $\mathrm{lh}(\mathbf{x})$ is the length of $\mathbf{x}$,
2. if $Z(x_1x_2)(y_1y_2)$ then Zx_1y_1, Zx_2y_2, and $Z(x_2x_1)(y_2y_1)$,
3. if Zx_1y_1 then x_1 and y_1 agree on all set variables p,
4. if $Z(x_1x_2)(y_1y_2)$ then (x_1, x_2) and (y_1, y_2) agree on all relation variables a and on δ,
5. if Zx_1y_1 and $x_2 \in \mathfrak{M}_1$, then there exists y_2 in $\mathfrak{M}_2$ with $Z(x_1x_2)(y_1y_2)$, and similarly in the opposite direction, if $Z(x_1x_2)(y_1y_2)$ and $x_3 \in \mathfrak{M}_1$, then there exists y_3 in $\mathfrak{M}_2$ with $Z(x_1x_3)(y_1y_3)$ and $Z(x_3x_2)(y_3y_2)$, and similarly in the opposite direction.

(As an aside, it is clear from the standard translation that $\mathcal{ML}_2$ contains the equivalent of the full 2-variable fragment of $\mathcal{L}(\tau)$. Hence, as the latter is characterized by its invariance under 2-partial isomorphisms (see Van Benthem (1991, Chapter 15)), any relation between models that is to preserve truth of $\mathcal{ML}_2$-formulas should at least act like a family of 2-partial isomorphisms. This is indeed the case.)

PROPOSITION 7.3. Bisimilar states satisfy the same $\mathcal{ML}_2$-formulas. More precisely, let $\mathfrak{M}_1, \mathfrak{M}_2$ be Peirce models, and let Z be a bisimulation for $\mathcal{ML}_2$ between $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

For any x_1 in $\mathfrak{M}_1$, $y_1 \in \mathfrak{M}_2$, and for any set formula ϕ, if we have Zx_1y_1, then $\mathfrak{M}_1, x_1 \models \phi$ iff $\mathfrak{M}_2, y_1 \models \phi$.

Likewise, for any x_1, x_2 in $\mathfrak{M}_1$, $y_1, y_2 \in \mathfrak{M}_2$, and for any relation formula ϕ, if we have $Z(x_1x_2)(y_1y_2)$, then $\mathfrak{M}_1, (x_1, x_2) \models \alpha$ iff $\mathfrak{M}_2, (y_1, y_2) \models \alpha$.

In fact, the above preservation result Proposition 7.3 is characteristic for $\mathcal{ML}_2$-formulas; below we will briefly sketch a proof of this claim. We first observe that the converse of Proposition 7.3 does not hold: $\mathcal{ML}_2$-equivalent models need not be bisimilar (see Goldblatt (1994), Hollenberg (1994), De Rijke (1995) for a counterexample). Following Goldblatt (1994) and Hollenberg (1994), we call a class K of $\mathcal{ML}_2$-models a *Hennessy–Milner class* if every pair of models in K is $\mathcal{ML}_2$-equivalent iff it is bisimilar. As an example, the class of finite $\mathcal{ML}_2$-models is a Hennessy–Milner class, as is the class of ω-saturated models (in the sense of standard first-order model theory); we refer to Goldblatt (1994), Hollenberg (1994), De Rijke (1995) for further details.

Call a first-order formula $\lambda \in \mathcal{L}(\tau)$ *invariant for bisimulations* if for all $\mathcal{L}(\tau)$-models $\mathfrak{M}$, $\mathfrak{M}'$ and all tuples $\mathbf{x}$, $\mathbf{x}'$ in $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, and all binary relations Z between $\mathfrak{M}$ and $\mathfrak{M}'$, we have that if Z is bisimulation linking $\mathbf{x}$ and $\mathbf{z}'$, then $\mathfrak{M} \models \lambda[\mathbf{x}]$ iff $\mathfrak{M}' \models \lambda[\mathbf{x}']$.

THEOREM 7.4. Let $\lambda(\mathbf{x}) \in \mathcal{L}(\tau)$ be a first-order formula in one or two free variables. Then $\lambda(\mathbf{x})$ is invariant for bisimulations iff it is equivalent to (the translation of) an $\mathcal{ML}_2$-formula.

Proof. The direction from right to left is Proposition 7.3. Proving the converse requires more work. We will sketch the proof. Assume $\lambda(\mathbf{x}) = \lambda(x)$ has just a single free variable, and assume $\lambda(x)$ is invariant for bisimulations. Consider the set of modal consequences of λ in a single free variable:

$$\text{MOD-CON}(\lambda) = \{ST(\phi) \mid \lambda \models ST(\phi)(x),\ \ \phi \text{ is a set formula in } \mathcal{ML}_2\}.$$

By compactness it suffices to show that MOD-CON$(\lambda) \models \lambda$. For then there exists a finite $\Gamma \subseteq$ MOD-CON(λ) such that $\Gamma \models \lambda$ (and conversely) and $\bigwedge \Gamma$ is an $\mathcal{ML}_2$-formula.

Assume $\mathfrak{M} \models$ MOD-CON$(\lambda)[w]$. We have to show that $\mathfrak{M} \models \lambda[w]$. Our first observation is that by a simple compactness argument the set $X := \{\lambda\} \cup \{ST(\psi) \mid \mathfrak{M}, w \models \psi\}$ is consistent. Let $\mathfrak{N}$ be a model with $\mathfrak{N} \models X[v]$, for some v. Note that w in $\mathfrak{M}$ and v in $\mathfrak{N}$ satisfy the same $\mathcal{ML}_2$-formulas.

If $\mathfrak{M}$ and $\mathfrak{N}$ both lived in a Hennessy–Milner class, then the fact that w and v satisfy the same $\mathcal{ML}_2$-formulas would imply that there exists a bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ that links w and v, and from this we would be able to infer $\mathfrak{M} \models \lambda[w]$, which would complete the proof. We can get away with slightly less: it is enough to make a detour through a Hennessy–Milner class, as follows. By general model-theoretic considerations from first-order logic, both $\mathfrak{M}$ and $\mathfrak{N}$ have ω-saturated elementary extensions $\mathfrak{M}^*$ and $\mathfrak{N}^*$; it follows that w in $\mathfrak{M}^*$ and v in $\mathfrak{N}^*$ satisfy the same $\mathcal{ML}_2$-formulas, and that $\mathfrak{N}^* \models \lambda[w]$. The class of ω-saturated models is a Hennessy–Milner class, hence there exists a bisimulation between $\mathfrak{M}^*$ and $\mathfrak{N}^*$ that links w and v. By invariance under bisimulations we get $\mathfrak{M}^* \models \lambda[w]$. As $\mathfrak{M}^*$ is an elementary extension of $\mathfrak{M}$ we infer that $\mathfrak{M} \models \lambda[w]$ – and we are done. □

COROLLARY 7.5. Let K be a class of $\mathcal{ML}_2$-models that is defined by a set of first-order formulas. Then K is modally definable iff it is closed under bisimulations.

COROLLARY 7.6. A class of $\mathcal{ML}_2$-models is modally definable iff it is closed under bisimulations and ultraproducts, and its complement is closed under ultrapowers.

Further definability results along these lines may be obtained using general techniques from modal model theory; see De Rijke (1995) for details.

8. Concluding remarks

In this paper we studied Peirce algebras via modal logic. Known techniques from modal completeness theory supplied us with a finite axiomatization of Peirce frames, and thereby of the equational theory of full Peirce algebras, and general results from modal model theory helped us to analyze the expressive power of Peirce algebras.

A lot remains to be done. Some questions were already mentioned in the main body of the paper. To conclude, here are further questions.

In Section 2 we briefly mentioned a connection between a system of Arrow Logic and Peirce algebras. There is a whole hierarchy of calculi in between this Arrow Logic and *MLP*, the logic of Peirce frames, just like there is a hierarchy of subsystems of relation algebra. About the former hierarchy one can ask the same kind of questions as for the latter. For example, where does undecidability strike? Is there an arrow version of Peirce algebras which is sufficiently expressive for applications (say, in terminological logic), but still decidable? Recent work by Marx (1994) presents partial answers, but more work needs to be done.

Another point in connection with the use of Peirce algebras in terminological logic is this. In terminological reasoning one often needs to be able to *count* the number of objects related to a given object; this is done using so-called *number restrictions* (see Brink *et al.* (1994)). The modal logic of such counting expressions is analyzed by Van der Hoek and De Rijke (1992). One direction for further work is to combine the results of the latter with the results of the present paper.

Finally, a more general point. We have used unorthodox derivation rules like the irreflexivity rule to arrive at our completeness result. To which extent do such rules capture our operators? We know from De Rijke (1992a) that the irreflexivity rule goes a long way towards determining the difference operator. But what about the other operators, like $\circ$, $\updownarrow$, :? Which aspects of their behavior are determined by our unorthodox derivation rules?

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Appendix

A. Two lemmas

We still need two lemmas to round off the completeness theorem for *MLP*: one saying that the axioms for the difference operators and the conjugated operators, as well as the inclusion axioms are derivable in *MLP*, and another lemma showing that the necessitation rules for the difference operators and the conjugated operators are derived rules of *MLP*.

We need the following proposition which is proved in De Rijke (1994).

PROPOSITION A.1. Let $\mathfrak{F}$ be a two-sorted Peirce like frame. Then

1. $\mathfrak{F} \models \forall x_s y_s z_r\,(Px_s z_r y_s \rightarrow Fz_r x_s \wedge Ff(z_r)y_s)$, and
2. $\mathfrak{F} \models \forall x_s y_s z_r\,(Fz_r x_s \wedge Ff(z_r)y_s \rightarrow Px_s z_r y_s)$.

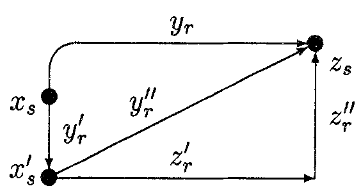
LEMMA A.2. *MLP* proves axioms (MD1)–(MD3) for the difference operators D'_s and D'_r. It also proves the distribution axioms for the defined operators $\langle\rangle_1$, $\langle\rangle_2$, $\circ_1$, and $\circ_2$, as well as (MP14), (MP15), (MP16)–(MP23) and (INC1)–(INC5).

Proof. We use the completeness of *MLPL* established in Theorem 6.1 to argue *semantically* that *MLPL*, and hence *MLP*, proves the axioms mentioned.

Showing that *MLPL* proves the distribution axioms is left to the reader. (MD$_s$1), (MD$_s$2) are easy consequences of (CP1) and (CP3); (MD$_r$1)–(MD$_r$3) are proved by Venema (1991, Proposition 3.3.38); (INC1) is easy and (INC2), (INC3) are dealt with by Venema (1991, Proposition 3.3.38); (INC5) is an easy consequence of (CP1) and (CP8); (MP14), (MP16)–(MP20) are easy, and (M21)–(MP23) are in Venema (1991, Proposition 3.3.38). This only leaves (MD$_s$3), (INC4) and (MP15).

(MD$_s$3) $p \rightarrow \neg\langle -\delta\rangle\neg\langle -\delta\rangle p$. Assume that $x_s \models p$, $\langle -\delta\rangle\neg\langle -\delta\rangle p$. We will derive a contradiction. As $x_s \models \langle -\delta\rangle\neg\langle -\delta\rangle p$ there are y_r, z_s with $Px_s y_r z_s$, $y_r \notin I$ and $z_s \models \neg\langle -\delta\rangle p$. By (CP5) this implies $Pz_s f(y_r)x_s$. Now, if $f(y_r) \notin I$ then $z_s \models \langle -\delta\rangle p$, and we have arrived at the desired contradiction. So it suffices to show $f(y_r) \notin I$. Assume $f(y_r) \in I$, then, as $Px_s y_r z_s$ and $Pz_s f(y_r)x_s$ there exists y''_r with $Cy''_r y_r f(y_r)$, by (CP1). Hence, by (CR5) and $f(y_r) \in I$ $Cy_r y_r f(y_r)$. By (CR1) and (CR7) this implies $Cf(y_r)y_r f(y_r)$, and by (CR5) this in turns yields $f(y_r) = y_r$. Therefore $y_r \in I$ – a contradiction.

(INC4) $\langle(-\delta \circ \updownarrow q \circ 1) \cup (1 \circ \updownarrow q \circ -\delta)\rangle\top \rightarrow q \vee \langle -\delta\rangle q$. Assume x_s satisfies the antecedent of the axiom, say $x_s \models \langle -\delta \circ \updownarrow q \circ 1\rangle\top$. Then there are y_r, z_s with $Px_s y_r z_s$ and $y_r \models -\delta \circ \updownarrow q \circ 1$. This means that there are y'_r, y''_r, z'_r, z''_r such that $Cy_r y'_r y''_r$, $Cy''_r z'_r z''_r$ and $y'_r \models -\delta$, and $z'_r \models \updownarrow q$. The latter implies that there is an x'_s with $Fz'_r x'_s$ and $x'_s \models q$.



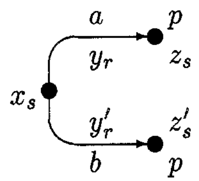
It suffices to show that $Px_s y'_r x'_s$, for then $x_s \models \langle -\delta\rangle q$. Now, to see that $Px_s y'_r x'_s$, observe that

$$Px_s y_r z_s \wedge Cy_r y'_r y''_r \quad\Rightarrow\quad Px_s y'_r z'_s \wedge Pz'_s y''_r z_s, \tag{3}$$

for some z'_s by (CP2). Furthermore, $Pz'_s y''_r z_s$ and $Cy''_r z'_r z''_r$ imply that for some z''_s, $Pz'_s z'_r z''_s$, by (CP2). Next, $Pz'_s z'_r z''_s$ implies $Fz'_r z'_s$ by Proposition A.1. On the other hand, we already have that $Fz'_r x'_s$, so by (CP10) it follows that $z'_s = x'_s$. But, then, by (3) we must have $Px_s y'_r x'_s$, as required.

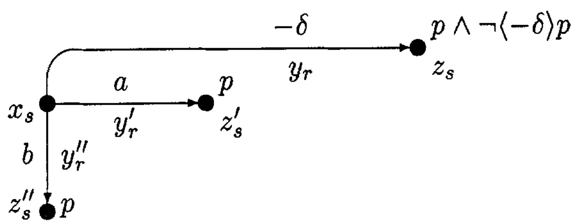
The case that $x_s \models \langle 1 \circ \updownarrow q \circ -\delta \rangle \top$ is proved entirely analogously.

(MP15) $((p \wedge \neg\langle -\delta \rangle p) \vee \langle -\delta \rangle(p \wedge \neg\langle -\delta \rangle p)) \wedge \langle a \rangle p \wedge \langle b \rangle p \rightarrow \langle a \cap b \rangle p$. Assume first that $x_s \models p \wedge \neg\langle -\delta \rangle p \wedge \langle a \rangle p \wedge \langle b \rangle p$. Then, for some y_r, y'_r, z_s, z'_s we have $Px_s y_r z_s$, $Px_s y'_r z'_s$ and $y_r \models a$, $y'_r \models b$, and $z_s, z'_s \models p$.



Observe that $x_s \models \neg\langle -\delta \rangle p$ implies $y_r, y'_r \in I$. Hence, by (CP3), $x_s = z_s = z'_s$. Furthermore, $Px_s y_r z_s$ implies $Pz_s f(y_r) x_s$; together with $Px_s y'_r z'_s$ and (CP1) this yields a y''_r such that $Cy''_r f(y_r) y'_r$, and so by (CR8) such that $Cy'_r y_r y''_r$. By (CR5) we find $y'_r = y''_r$ and $y''_r = f(y_r)$. So we have $Cf(y_r)f(y_r)f(y_r)$, and by (CR8) $Cf(y_r)y_r f(y_r)$. As $y_r \in I$ implies $f(y_r) \in I$, (CR5) now gives $y_r = f(y_r)$. All in all we find $y_r = f(y_r) = y''_r = y'_r$. Hence $x_s \models \langle a \cap b \rangle p$.

Assume next that $x_s \models \langle -\delta \rangle(p \wedge \neg\langle -\delta \rangle p) \wedge \langle a \rangle p \wedge \langle b \rangle p$. Then there are y_r, y'_r, y''_r and z_s, z'_s, z''_s with $Px_s y_r z_s$, $Px_s y'_r z'_s$, $Px_s y''_r z''_s$, and $y_r \notin I$, $y'_r \models a$, $y''_r \models b$, and $z_s, z'_s, z''_s \models p$, and $z_s \not\models \langle -\delta \rangle p$.



By (CP1) there is a y'''_r with $Pz_s y'''_r z'_s$ and $Cy'''_r f(y_r) y'_r$. If $y'''_r \notin I$, then $z'_s \not\models p$ – a contradiction. Hence $y'''_r \in I$. Likewise we find a $y''''_r \in I$ with $Cy''''_r f(y_r) y''_r$. By (CR8) and (CR5) we have $y'_r = y_r$ and $y''_r = y_r$. Hence $y_r \models a \cap b$, and $x_s \models \langle a \cap b \rangle p$. □

LEMMA A.3. The necessitation rules for the defined operators $\Updownarrow$, $\langle\cdot\rangle_1\cdot$, $\langle\cdot\rangle_2\cdot$, $\circ_1$, $\circ_2$ and D'_s, D'_r are derived rules in *MLP*.

Proof. By way of example we show that the two necessitation rules for $\langle\rangle_2$ are derived rules. Assume $\vdash_{MLP} \alpha$; we need $\vdash_{MLP} [\![\phi]\!]_2\alpha$. Now, by (NEC$_\otimes$) and (MR0), $\vdash_{MLP} \alpha$ implies $\vdash_{MLP} \otimes\alpha$. Hence, by (NEC$_{\langle\rangle}$) we have $\vdash_{MLP} [\![\otimes\alpha]\!]\phi$, and by definition of $\langle\rangle_2$ this means $\vdash_{MLP} [\![\phi]\!]_2\alpha$. Next, assume $\vdash_{MLP} \phi$; we need $\vdash_{MLP} [\![\phi]\!]_2\alpha$, but this is immediate from (NEC$_\otimes$). □