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A Modal Characterization of Peirce Algebras

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Abstract

Peirce algebras combine sets, relations and various operations linking the two in a unifying setting. This note offers a modal perspective on Peirce algebras. It uses modal logic to characterize the full Peirce algebras.

1 Introduction

The work of Helena Rasiowa that I am most familiar with is her work on algebraizations of non-classical logic, and especially with algebraizations of modal logics [17]; in this note I will be concerned with modalizing an algebraic logic. Of course, Rasiowa is also well-known for her work on the interface of logic and computer science [18]; in this note, we look at the so-called Peirce algebras that have arisen in the computational field of knowledge engineering, and our aim will be to settle a purely logical (or algebraic) question, namely, to characterize the full Peirce algebras. Thus, like much of Rasiowa's work, the topic of this note is part of an enterprise to relate modal languages, algebraic languages, and fragments of first-order logic.

Peirce algebras (Brink et al [5]) have emerged as the common mathematical structures underlying many phenomena being studied in program semantics, AI and natural language analysis. Peirce algebras are two-sorted algebras in which sets and relations co-exist together with operations between them that model their interaction. The most important such operations considered here are the *Peirce product* : that takes a relation and a set, and returns a set

$$R : A = \{x \mid \exists y\, ((x,y) \in R \wedge y \in A)\},$$

and *right cylindrification* ^{c} which takes a set and returns a relation

$$A^c = \{(x,y) \mid x \in A\}.$$

We will show how Peirce algebras arise as algebraic counterparts of a two-sorted modal language $\mathcal{ML}_2$; this language extends the modal formalism $CC\delta$

that was designed by Venema [26] to reason about binary relations. The main contribution of this note is a characterization of the 'intended' or 'concrete' frames underlying $\mathcal{ML}_2$ and thereby of the full Peirce algebras. Due to space limitations important consequences of this result (such as a completeness result for these frames) are discussed elsewhere (see De Rijke [20]).

The next section quickly reviews relevant algebraic definitions. §3 relates Peirce algebras to other structures found in the literature. §4 introduces modal languages for describing relational counterparts of Peirce algebras. §5 presents the main result, a characterization of the modal frames corresponding to full Peirce algebras. §6 discusses further results on Peirce algebras, and §7 concludes with some questions.

2 Definitions

This section introduces the main definitions; we refer to Henkin et al. [6] or Rasiowa [17] for further details on algebraic logic.

Let U be a set; $Re(U)$ is $\{R \mid R \subseteq U \times U\}$. R, S typically denote elements of $Re(U)$, while A, B typically denote elements of 2^U, the power set of U.

Recall the following operations on elements of $Re(U)$.

top	∇	$\{(r,s) \in (U \times U) \mid r, s \in U\}$
complement	$-R$	$\{(r,s) \in (U \times U) \mid (r,s) \notin R\}$
converse	R^{-1}	$\{(r,s) \in (U \times U) \mid (s,r) \in R\}$
diagonal	Id	$\{(r,s) \in (U \times U) \mid r = s\}$
composition	$R \mid S$	$\{(r,s) \in (U \times U) \mid \exists u\,((r,u) \in R \wedge (u,s) \in S)\}$

We also consider the following operations from $Re(U)$ and $Re(U) \times 2^U$ to 2^U

domain	$\mathsf{do}(R)$	$\{x \in U \mid \exists y \in U\,((x,y) \in R)\}$
range	$\mathsf{ra}(R)$	$\{x \in U \mid \exists y \in U\,((y,x) \in R)\}$
Peirce product	$R : A$	$\{x \in U \mid \exists y \in U\,((x,y) \in R \wedge y \in A)\}$,

as well as the following operations going from 2^U to $Re(U)$

tests	$A?$	$\{(x,y) \in (U \times U) \mid x = y \wedge x \in A)\}$
right cylindrification	A^c	$\{(x,y) \in (U \times U) \mid x \in A\}$.

A *relation type algebra* is a Boolean algebra with a binary operation ;, a unary operation $\breve{}$, and a constant 1'. The class FRA of *full relation algebras* consists of all relation type algebras isomorphic to an algebra of the form $\mathfrak{R}(U) = (Re(U), \cup, -, |, ^{-1}, Id)$. RRA is the class of *representable relation algebras*, that is, RRA = $\mathbf{SP}$(FRA) (= $\mathbf{HSP}$(FRA) by a result due to Birkhoff). RA is the class of *relation algebras*, that is, of relation type algebras $\mathfrak{A} = (A, +, -, ;, \breve{}, 1')$ satisfying the axioms

(R0)	$(A, +, -, \emptyset)$ is a Boolean algebra	(R5)	$x \,;\, 1' = x = 1' \,;\, x$
(R1)	$(x + y) \,;\, z = x \,;\, z + y \,;\, z$	(R6)	$(x^{\smile})^{\smile} = x$
(R2)	$(x + y)^{\smile} = x^{\smile} + y^{\smile}$	(R7)	$(x \,;\, y)^{\smile} = y^{\smile} \,;\, x^{\smile}$
(R4)	$(x \,;\, y) \,;\, z = x \,;\, (y \,;\, z)$	(R8)	$x^{\smile} \,;\, -(x \,;\, y) \leq -y.$

We refer the reader to Jónsson [7, 8] for the essentials on relation algebra.

A *Peirce type algebra* is a two-sorted algebra $(\mathfrak{B}, \mathfrak{R}, :, ^c)$, where $\mathfrak{B}$ is a Boolean algebra, $\mathfrak{R}$ is a relation type algebra, $:$ is a function from $\mathfrak{R} \times \mathfrak{B}$ to $\mathfrak{B}$, and $^c : \mathfrak{B} \to \mathfrak{R}$. The class FPA of *full Peirce algebras* consists of all Peirce type algebras isomorphic to an algebra of the form

$$\mathfrak{P}(U) = \big((2^U, \cup, -, \emptyset), (Re(U), \cup, -, ^{-1}, |, Id), :, ^c\big).$$

The class RPA of *representable Peirce algebras* is defined as RPA $=$ $\mathbf{HSP}$(FPA), the variety generated by FPA. PA is the class of *Peirce algebras*, that is of all Peirce type algebras $\mathfrak{A} = (\mathfrak{B}, \mathfrak{R}, :, ^c)$ where $\mathfrak{B}$ is a Boolean algebra, $\mathfrak{R}$ is a relation algebra, $:$ is a mapping $\mathfrak{R} \times \mathfrak{B} \to \mathfrak{B}$ such that

(P1)	$r : (a + b) = (r : a) + (r : b)$	(P4)	$1' : a = a$
(P2)	$(r + s) : a = (r : a) + (s : a)$	(P5)	$0 : a = 0$
(P3)	$r : (s : a) = (r \,;\, s) : a$	(P6)	$r^{\smile} : -(r : a) \leq -a,$

while ^{c} is a mapping $\mathfrak{B} \to \mathfrak{R}$ such that

(P7)	$a^c : 1 = a$	(P8)	$(r : 1)^c = r \,;\, 1.$

Reducts of the form $(\mathfrak{B}, \mathfrak{R}, :)$ were introduced by Brink [3] as *Boolean modules*; see also Henkin et al. [6]. Sources for Peirce algebras are Brink et al. [5] and Schmidt [23].

Unlike the one-sorted language of relation algebras, the algebraic language of Peirce algebras has two sorts of terms: one interpreted in $\mathfrak{B}$, the other in $\mathfrak{R}$. Terms of the first sort are called *set terms*, terms of the second sort *relation terms*. Identities between set terms are called *set identities*; identities between relation terms are *relation identities*.

3 Peirce algebras and transition logics

By a modal transition logic I mean a modal-like logic whose intended semantics uses a collection of transitions to interpret (some of) the expressions of the logic. Over the past two decades a rich landscape of such transition logics has arisen. They may be classified according to their repertoire of operators, and according to the status that give to states and transitions. On one end of the spectrum one finds *arrow logic* in the sense of Venema [27], that is, logics whose algebras are relation type algebras. Formulas of arrow logic are interpreted on transitions (or arrows) only, and the modal operators of arrow logic correspond to the operations familiar from relation algebra. *Standard* *modal logic* and *propositional dynamic logic* (PDL, Pratt [16]) are extremes on the other end of the scale: their formulas are evaluated at states only, although in the case of propositional dynamic logic, the programs need transitions for their interpretations.

Thirdly, there are hybrid systems whose languages come with two sorts of formulas, one referring to states, the other referring to transitions. Van Benthem [2] presents an abstract approach based on two-sorted structures, with states and arrows. Marx [15] has results on concrete one- and two-dimensional interpretations of sorted transition calculi. Peirce algebras may be viewed as the full square case of these calculi. In addition to the computational origins of Peirce algebras mentioned earlier, they also arise as the modal algebras of a system of dynamic modal logic (DML, Van Benthem [1], De Rijke [21]).

DML is similar to propositional dynamic logic (PDL) in that it has formulas and programs. The formulas ϕ and programs α of DML are built up as follows

$$\begin{array}{rcl}\phi & ::= & p \mid - \mid \neg\phi \mid \phi \wedge \phi \mid \mathsf{do}(\alpha) \mid \mathsf{ra}(\alpha) \mid \mathsf{fix}(\alpha) \\ \alpha & ::= & \mathsf{exp}(\phi) \mid -\alpha \mid \alpha^{\smile} \mid \alpha \cap \alpha \mid \alpha \,;\, \alpha \mid \phi?.\end{array}$$

Here, $\mathsf{exp}(\phi)$ is the special relation of 'expanding one's information with ϕ', and $\mathsf{fix}(\alpha)$ is a formula that is true precisely at the fixed points for α. Like PDL, DML only allows equational reasoning with formulas, not with programs.

The modal algebras for DML are Peirce algebras over a single relation, the information order underlying the exp construct, and to obtain a proper match between DML and Peirce algebras one has to allow multiple exp operators with accompanying information orders. The corresponding structures give rise to full Peirce algebras, and conversely. Moreover, the (extended) DML-operators are definable in full Peirce algebras, and the operators of full Peirce algebras are definable on DML-models. As a result, the complete axiomatization of DML structures presented in [21] also generates the 'set identities' valid in FPA.

To conclude we should mention the dynamic algebras of Kozen [10]. Like Peirce algebras, dynamic algebras are two-sorted algebras of sets and relations. But their relations are organized in a Kleene algebra, not in a relation algebra, and their sets are only assumed to form a semi-lattice; any join complete Peirce algebra gives rise to a dynamic algebra. Another class of algebras closely related to Peirce algebras are the extended relation algebras of Suppes [25]. Roughly, an extended relation algebra is term-definably equivalent with a Peirce algebra in which the sortal distinctions are left out.

4 A modal language for Peirce algebras

In this section we introduce a modal language for Peirce algebras. The attractive feature of using modal languages is that they allow us to reason with simple pictures; additional motivation for the general program of relating algebraic logic, modal logic and first-order logic can be found in Rasiowa [17] or Venema [26].

4.1 Basic definitions

To start, Table 1 lists the notation we adopt.

	Full version	Abstract version	Modal version
relations	R, S	x, y	α, β
top	∇	1	$\mathbf{1}$
bottom	$\emptyset$	0	$\mathbf{0}$
diagonal	Id	1'	δ
complement	$-$	$-$	$-$
converse	$^{-1}$	$\breve{}$	$\otimes$
union	$\cup$	$+$	$\cup$
implication	$\rightarrow$		$\rightarrow$
composition	$\mid$	;	$\circ$
sets	A, B	x, y	ϕ, ψ
top	$\top$	1	$\top$
bottom	$-$	0	$-$
complement	$\neg$	$-$	$\neg$
union	$\cup$	$+$	$\vee$
implication	$\rightarrow$	$\rightarrow$	$\rightarrow$
right cylindrification	$(\cdot)^c$	c_1	$\updownarrow\cdot$
Peirce product	:	:	$\langle\cdot\rangle\cdot$

Table 1: A plethora of notations.

Definition 4.1 Let $\Phi = \{p_0, p_1, \ldots\}$ be a countable set of propositional variables. Let Ω be a countable set of atomic relation symbols. The formulas of the *two-sorted language* $\mathcal{ML}_2(\delta, \otimes, \circ, \langle\rangle, \updownarrow; \Phi; \Omega)$, or $\mathcal{ML}_2$ for short, are generated by the rules

$$\begin{aligned} \phi &::= \; - \mid \top \mid p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \langle\alpha\rangle\phi, \\ \alpha &::= \; \mathbf{0} \mid \mathbf{1} \mid \delta \mid a \mid -\alpha \mid \alpha_1 \cap \alpha_2 \mid \otimes\alpha \mid \alpha_1 \circ \alpha_2 \mid \updownarrow\phi. \end{aligned}$$

The first sort of formulas will be interpreted as sets and called *set formulas*; formulas of the second sort will be interpreted as relations and called *relation formulas*.

Definition 4.2 A *two-sorted frame* is a tuple $\mathfrak{F} = (W_s, W_r, I, R, C, F, P)$, where $W_s \cap W_r = \emptyset$, $I \subseteq W_r$, $R \subseteq W_r^2$, $C \subseteq W_r^3$, $F \subseteq W_r \times W_s$, and $P \subseteq W_s \times W_r \times W_s$.

Given a set U, a two-sorted frame is called *the two-sorted Peirce frame over* U if, for some base set U, $W_s = U$ and $W_r = U \times U$, and

$$I = \{(u, v) \in U \times U \mid u = v\}$$

$$
\begin{aligned}
R &= \{((u_1,v_1),(u_2,v_2)) \in (U\times U)^2 \mid u_1 = v_2 \wedge u_2 = v_1\}\\
C &= \{((u_1,v_1),(u_2,v_2),(u_3,v_3)) \in (U\times U)^3 \mid u_1 = u_2 \wedge v_1 = v_3 \wedge v_2 = u_3\}\\
F &= \{((u_1,v_1),u_2) \in (U\times U)\times U \mid u_1 = u_2\}\\
P &= \{(u_1,(u_2,v_2),u_3) \in U\times (U\times U)\times U \mid u_1 = u_2 \wedge v_2 = u_3\}.
\end{aligned}
$$

The class of two-sorted Peirce frames is denoted by TPF.

A model for $\mathcal{ML}_2$ is a *model based on a two-sorted frame*, that is, a structure $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F}$ is a two-sorted frame, and V is a *two-sorted valuation*, a function assigning subsets of W_s to set variables, and subsets of W_r to relation variables. Truth of a formula at a state is defined inductively, with the interesting clauses being

$$
\begin{aligned}
\mathfrak{M}, x_r \models \delta &\quad\text{iff}\quad x_r \in I\\
\mathfrak{M}, x_r \models \otimes\alpha &\quad\text{iff}\quad \exists y_r\,(Rx_ry_r \wedge y_r \models \alpha)\\
\mathfrak{M}, x_r \models \alpha\circ\beta &\quad\text{iff}\quad \exists y_rz_r\,(Cx_ry_rz_r \wedge y_r \models \alpha \wedge z_r \models \beta)\\
\mathfrak{M}, x_s \models \langle\alpha\rangle\phi &\quad\text{iff}\quad \exists y_rz_s\,(Px_sy_rz_s \wedge y_r \models \alpha \wedge z_s \models \phi)\\
\mathfrak{M}, x_r \models \updownarrow\phi &\quad\text{iff}\quad \exists y_s\,(Fx_ry_s \wedge y_s \models \phi).
\end{aligned}
$$

Here x_s, y_s, ... are taken from W_s; x_r, y_r, ... are taken from W_r; see Figure 1 for a picture.

A formula ξ is *valid* on a two-sorted frame $\mathfrak{F}$ (notation: $\mathfrak{F} \models \xi$) if for all valuations V and for all states x of the appropriate sort, $(\mathfrak{F}, V), x \models \xi$.

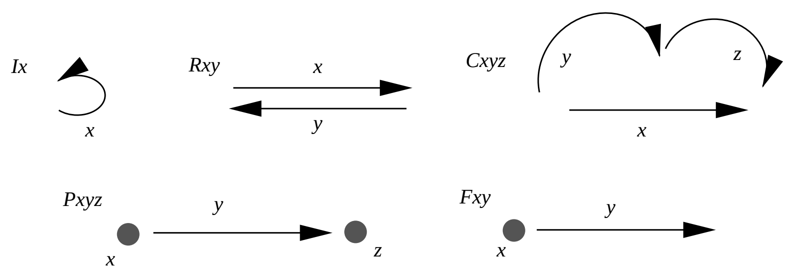

Figure 1: Relations in a two-sorted frame.

In models based on Peirce frames all modal connectives receive their intended interpretation. That is, one has $(u,v) \models \delta$ iff $u = v$; $(u,v) \models \otimes\alpha$ iff $(v,u) \models \alpha$; $(u,v) \models \alpha\circ\beta$ iff $\exists w\,((u,w) \models \alpha \wedge (w,v) \models \beta)$; $u \models \langle\alpha\rangle\phi$ iff $\exists v\,((u,v) \models \alpha \wedge v \models \phi)$; and $(u,v) \models \updownarrow\phi$ iff $u \models \phi$.

To be able to state the connection between two-sorted Peirce frames and Peirce algebras, we recall that the *complex algebra* (Jónsson and Tarski [9]) or *power structure* (Brink [4]) $\mathfrak{Em}\,\mathfrak{F}$ of a two-sorted frame $\mathfrak{F}$ is given as $\mathfrak{A} = ((2^{W_s}, -, \cap, \emptyset, W_s), (2^{W_r}, -, \cap, m_\delta, m_\otimes, m_\circ, \emptyset, W_r), m_{\langle\rangle}, m_\updownarrow)$, where, for $\#$ an n-ary modal operator, $m_\#$ is an n-ary operator on the power set(s) of the appropriate domain(s) of $\mathfrak{F}$. To be precise

$$\begin{aligned} m_\delta &= \{ x_r \mid x_r \in I \} \\ m_\otimes(X) &= \{ x_r \mid \exists y_r \, (Rx_r y_r \wedge y_r \in X \} \\ m_\circ(X,Y) &= \{ x_r \mid \exists y_r z_r \, (Cx_r y_r z_r \wedge y_r \in X \wedge z_r \in Y) \} \\ m_{\langle\rangle}(X,Y) &= \{ x_s \mid \exists y_r z_s \, (Px_s y_r z_s \wedge y_r \in X \wedge z_s \in Y) \} \\ m_\updownarrow(X) &= \{ x_r \mid \exists y_s \, (Fx_r y_s \wedge y_s \in X) \}. \end{aligned}$$

For K a class of frames $\mathbf{Cm}(\mathsf{K})$ is the class of complex algebras of frames in K.

Proposition 4.3 *Let $\mathfrak{F}$ be a two-sorted frame. Then $\mathfrak{F}$ is a Peirce frame (or: in* TPF*) iff $\mathfrak{Cm}\,\mathfrak{F}$ is (isomorphic) to a full Peirce algebra. In other words:* $\mathbf{Cm}(\mathsf{TPF}) = \mathsf{FPA}$.

4.2 Adding a difference operator

Peirce frames cannot be characterized in $\mathcal{ML}_2$; the reason is that $\mathsf{FPA} = \mathbf{Cm}\,(\mathsf{TPF})$ is not a variety as it is not closed under products or subalgebras. However, if we are wiling to extend the modal language, a characterization can be obtained.

More precisely, to characterize the Peirce frames we will use a special modal operator D_s called the *difference operator*; its special feature is that it is interpreted using the diversity relation $\neq$ on set elements:

$$x_s \models D_s\phi \text{ iff for some } y_s \neq x_s,\ y_s \models \phi \text{ where } x_s,\ y_s \in W_s.$$

Observe that on Peirce frames the difference operator can be defined as follows

$$D'_s\phi := \langle -\delta \rangle \phi.$$

Using the difference operator we can define other useful operators such as E_s, where $E_s\phi := \phi \vee D_s\phi$ (there exists an object with ϕ), and O_s, where $O_s\phi = E_s(\phi \wedge \neg D_s\phi)$ (there is only one object with ϕ). The reader is referred to De Rijke [19] for details about logics with difference operators.

5 Characterizing Peirce algebras

In this section we characterize Peirce frames. We do this in two steps. We first define a class of Peirce like frames and characterize those in the language $\mathcal{ML}_2$. We then extend the language with the difference operator D_s and characterize the Peirce frames.

5.1 A first approximation

To characterize the Peirce frames among the two-sorted frames, we need a number of axioms. We first list the modal axioms handling the relational component of two-sorted frames plus the conditions they impose on such frames; they are simply the modal counterparts of the earlier relation algebraic axioms (R1)–(R8), and the corresponding conditions have been calculated by Lyndon [12] and Maddux [13]. We then list the modal counterparts of the Peirce axioms (P1)–(P8), and calculate the corresponding conditions on frames. (Recall that a first-order condition γ is said to *correspond* to a modal formula ξ if for all frames $\mathfrak{F}$, $\mathfrak{F} \models \xi$ iff $\mathfrak{F} \models \gamma$.) The reader is strongly advised to draw pictures while checking the correspondence results.

The first axiom states that R, the interpretation of $\otimes$, is a function; this is proved by standard arguments.

(MR0)	$\otimes a \leftrightarrow -\otimes -a$
(CR0)	R is a function

So, in frames validating (MR0) we are justified in interpreting $\otimes$ using a unary function f, and evaluating formulas $\otimes\alpha$ as follows

$$\mathfrak{M}, x_r \models \otimes\alpha \text{ iff } \mathfrak{M}, f(x_r) \models \alpha.$$

Definition 5.1 A *two-sorted arrow frame* is simply a two-sorted frame $\mathfrak{F} = (W_s, W_r, I, f, C, F, P)$ in which the binary relation R used to interpret the operator $\otimes$ is a function from W_r to W_r, denoted by f. A *two-sorted arrow model* is a two-sorted model based on a two-sorted arrow frame, where $\otimes$ is interpreted using the function f as indicated above.

Here are the remaining axioms governing the behaviour of δ, $\otimes$ and $\circ$, as well as the conditions expressed by these axioms.

(MR1)	$a \to \otimes\otimes a$
(CR1)	$f(f(x_r)) = x_r$
(MR2)	$a \circ (b \circ c) \to (a \circ b) \circ c$
(CR2)	$\forall y_r z_r u_r v_r\, (Cx_r y_r z_r \wedge Cz_r u_r v_r \to \exists w_r\, (Cx_r w_r v_r \wedge Cw_r y_r u_r))$
(MR3)	$(a \circ b) \circ c \to a \circ (b \circ c)$
(CR3)	$\forall y_r w_r u_r v_r\, (Cx_r w_r v_r \wedge Cw_r y_r u_r \to \exists z_r\, (Cx_r y_r z_r \wedge Cz_r u_r v_r))$
(MR4)	$a \to \delta \circ a$, $a \to a \circ \delta$
(CR4)	$\exists y_r\, (Iy_r \wedge Cx_r y_r x_r)$, $\exists y_r\, (Cx_r x_r y_r \wedge Iy_r)$
(MR5)	$\delta \circ a \to a$, $a \circ \delta \to a$
(CR5)	$\forall y_r z_r\, (Cx_r y_r z_r \wedge Iy_r \to x_r = z_r)$, $\forall y_r z_r\, (Cx_r y_r z_r \wedge Iz_r \to x_r = y_r)$
(MR6)	$\otimes(a \circ b) \to (\otimes b \circ \otimes a)$
(CR6)	$\forall y_r z_r\, (Cf(x_r) y_r z_r \to Cx_r f(z_r) f(y_r))$
(MR7)	$(\otimes b \circ \otimes a) \to \otimes(a \circ b)$
(CR7)	$\forall y_r z_r\, (Cx_r f(z_r) f(y_r) \to Cf(x_r) y_r z_r)$

(MR8)	$\otimes a \circ -(a \circ b) \cap b \to \mathbf{0}$
(CR8)	$\forall y_r z_r\,(Cx_r f(y_r) z_r \to Cz_r y_r x_r)$.

Next come the axioms governing the behaviour of the Peirce product and cylindrification.

(MP1)	$\langle a\rangle\langle b\rangle p \to \langle a \circ b\rangle p$
(CP1)	$\forall y_r y'_r z_s z'_s\,(Px_s y_r z_s \wedge Pz_s y'_r z'_s \to \exists y''_r\,(Px_s y''_r z'_s \wedge Cy''_r y_r y'_r))$
(MP2)	$\langle a \circ b\rangle p \to \langle a\rangle\langle b\rangle p$
(CP2)	$\forall y_r y'_r y''_r z_s\,(Px_s y_r z_s \wedge Cy_r y'_r y''_r \to \exists z'_s\,(Px_s y'_r z'_s \wedge Pz'_s y''_r z_s))$
(MP3)	$\langle\delta\rangle p \to p$
(CP3)	$\forall y_r z_s\,(Px_s y_r z_s \wedge Iy_r \to x_s = z_s)$
(MP4)	$p \to \langle\delta\rangle p$
(CP4)	$\exists y_r\,(Px_s y_r x_s \wedge Iy_r)$
(MP5)	$\langle\otimes a\rangle\neg\langle a\rangle p \wedge p \to -$
(CP5)	$\forall y_r z_s\,(Px_s y_r z_s \to Pz_s f(y_r) x_s)$
(MP6)	$\langle\updownarrow p\rangle\top \to p$
(CP6)	$\forall y_r z_s z'_s\,(Px_s y_r z_s \wedge Fy_r z'_s \to x_s = z'_s)$
(MP7)	$p \to \langle\updownarrow p\rangle\top$
(CP7)	$\exists y_r z_s\,(Px_s y_r z_s \wedge Fy_r x_s)$
(MP8)	$\updownarrow\langle a\rangle\top \to (a \circ \mathbf{1})$
(CP8)	$\forall y_s y'_r z_s\,(Fx_r y_s \wedge Py_s y'_r z_s \to \exists z'_r\,(Cx_r y'_r z'_r))$
(MP9)	$(a \circ \mathbf{1}) \to \updownarrow\langle a\rangle\top$
(CP9)	$\forall y_r z_r\,(Cx_r y_r z_r \to \exists y'_s z'_s\,(Fx_r y'_s \wedge Py'_s y_r z'_s))$.

Lemma 5.2 *Let $\mathfrak{F}$ be a two-sorted arrow frame. Then $\mathfrak{F} \models$ (MRi) iff $\mathfrak{F} \models$ (CRi), for $1 \leq i \leq 8$, and $\mathfrak{F} \models$ (MPi) iff $\mathfrak{F} \models$ (CPi), for $1 \leq i \leq 9$.*

Proof. As pointed out before, the proof that the above axioms (MRi) correspond to the conditions (CRi) is due to Lyndon and Maddux. For the Peirce axioms (CPi) ($1 \leq i \leq 9$) the correspondence result follows from the general results of De Rijke [22]: all axioms listed here are so-called Sahlqvist formulas, and for such formulas there is an explicit algorithm computing the corresponding relational condition. Consider, for example, axiom (MP5). For any two-sorted arrow frame $\mathfrak{F}$ and x_s in $\mathfrak{F}$ we have

$$\begin{aligned}
&\mathfrak{F}, x_s \models \text{(MP5)}\\
&\quad\text{iff}\quad \mathfrak{F}, x_s \models \forall a \forall p \Big(\exists y_r z_s\,(Px_s y_r z_s \wedge a(f(y_r)) \wedge\\
&\qquad\qquad \neg\exists y'_r z'_s\,(Pz_s y'_r z'_s \wedge a(y'_r) \wedge p(z'_s)) \wedge p(x_s)) \to -(x_s)\Big)\\
&\quad\text{iff}\quad \mathfrak{F}, x_s \models \forall a \forall p \forall y_r z_s \Big(Px_s y_r z_s \wedge a(f(y_r)) \wedge p(x_s) \to\\
&\qquad\qquad \exists y'_r z'_s\,(Pz_s y'_r z'_s \wedge a(y'_r) \wedge p(z'_s))\Big).
\end{aligned}$$

To turn the latter formula into an equivalent first-order formula we will find special instantiations for the universally quantified variables a and p in such a way that substituting these instantiations produces a formula that is equivalent to the above one; these instantiations are special because they are the minimal ones needed to verify the antecedent of the above formula. Given that the formula is the second-order transcription of a Sahlqvist formula, the required 'minimal' instantiations can be read off from the antecedent: take $\lambda u.\, u = f(y_r)$ for a, and $\lambda u.\, u = x_s$ for p. Substituting these predicates for a and p, respectively, yields the equivalent formula

$$\forall y_r z_s \left(P x_s y_r z_s \to \exists y'_r z'_s \, (P z_s y'_r z'_s \wedge y'_r = f(y_r) \wedge z'_s = x_s) \right).$$

And this, in turn, is equivalent to $\forall y_r z_s \, (P x_s y_r z_s \to P z_s f(y_r) x_s)$. ⊣

Definition 5.3 A two-sorted arrow frame is *Peirce like* if it satisfies conditions (CR1)–(CR8), as well as (CP1)–(CP9).

Lemma 5.4 *Let $\mathfrak{F}$ be a two-sorted arrow frame. Then $\mathfrak{F}$ is Peirce like iff* $\mathfrak{F} \models \bigwedge_{1 \le i \le 8} \mathrm{MR}_i \wedge \bigwedge_{1 \le i \le 9} \mathrm{MP}_i$.

5.2 Characterizing two-sorted Peirce frames

We now narrow down the two-sorted Peirce like frames to Peirce frames. Briefly, what we need to show that a two-sorted Peirce like frame is a Peirce frame, is the following

- With every relational element we can associate a unique set element as its first coordinate and a unique set element as its second coordinate.
- With every two set elements we can associate a unique relational element having those set elements as first and second coordinate.

This boils down to having the following conditions satisfied by our Peirce like frames:

(CP10) $\forall x_r y_s y'_s \, (F x_r y_s \wedge F x_r y'_s \to y_s = y'_s)$
(CP11) $\forall x_r y_s y'_s \, (F f(x_r) y_s \wedge F f(x_r) y'_s \to y_s = y'_s)$
(CP12) $\forall x_r \exists y_s \, (F x_r y_s)$
(CP13) $\forall x_r \exists y_s \, (F f(x_r) y_s)$
(CP14) $\forall x_s y_s \exists z_r \, (P x_s z_r y_s)$
(CP15) $\forall x_s y_s z_r z'_r \, (P x_s z_r y_s \wedge P x_s z'_r y_s \to z_r = z'_r)$.

Lemma 5.5 *Assume that $\mathfrak{F}$ be a two-sorted Peirce like frame. Then* $\mathfrak{F} \models$ (CP10)–(CP13).

Observe that conditions (CP10)–(CP13) are expressed by the following four modal formulas, respectively:

(MP10)	$\updownarrow p \cap \updownarrow q \to \updownarrow(p \wedge q)$
(MP11)	$\otimes(\updownarrow p) \wedge \otimes(\updownarrow q) \to \otimes\updownarrow(p \wedge q)$
(MP12)	$\updownarrow\top$
(MP13)	$\otimes(\updownarrow\top)$.

The proof of this claim is left to the reader.

We will now give a representation result for full Peirce algebras. I like to think that the representation below is more elegant than the usual representations in relation algebra and arrow logic; the latter usually extract points from (a Cartesian product of) the diagonal to obtain a base set over which a full algebra can be built. In the case of Peirce algebras we already have our points available in the domain of set points; we will be able to simply map every relation point z_r in a Peirce frame onto a pair of set points x_s, y_s already present.

We need the following lemma.

Lemma 5.6 *Let $\mathfrak{F}$ be a two-sorted Peirce like frame. Then*

(1) $\mathfrak{F} \models \forall x_s y_s z_r\,(Px_s z_r y_s \to Fz_r x_s \wedge Ff(z_r)y_s)$, *and*

(2) $\mathfrak{F} \models \forall x_s y_s z_r\,(Fz_r x_s \wedge Ff(z_r)y_s \to Px_s z_r y_s)$.

Proof. To prove (1) assume $Px_s z_r y_s$. By (CP12) $Fz_r x'_s$, for some x'_s. By (CP6) $x_s = x'_s$, hence $Fz_r x_s$. Likewise, by (CP13), (CP5) and (CP6) we have $Ff(z_r)y_s$. For (2), assume that $Fz_r x_s$, $Ff(z_r)y_s$. By (CR4) there exists y_r with $Cz_r z_r y_r$. By (CP9) this implies there exist y'_s, z'_s with $Py'_s z_r z'_s$. By (i) $Fz_r y'_s$ and $Ff(z_r)z'_s$. (CP10) and (CP11) then yield $x_s = y'_s$ and $y_s = z'_s$. Hence $Px_s z_r y_s$. $\dashv$

Theorem 5.7 *Let $\mathfrak{F} = (W_s, W_r, I, f, C, F, P)$ be a two-sorted Peirce like frame. If $\mathfrak{F} \models$ (CP14), (CP15), then $\mathfrak{F}$ is isomorphic to the two-sorted Peirce frame over W_s.*

Proof. If $\mathfrak{F}$ is a Peirce like frame satisfying (CP14) and (CP15), then, with every $z_r \in W_r$ we can associate a *unique* x and y such that $Fz_r x$ and $Ff(z_r)y$. Define a mapping $g : W_r \to W_s \times W_s$ by $g(z) = (z_0, z_1)$, where z_0, z_1 are the unique x and y with $Fz_r x$ and $Ff(z_r)y$. We prove that g is an isomorphism.
g is surjective. Let x_s, $y_s \in W_s$. By (CP14) $Px_s z_r y_s$, for some z_r. By Lemma 5.6 $Fz_r x_s$ and $Ff(z_r)y_s$. Hence $g(z) = (x_s, y_s)$.
g is injective. Let z_r, $z'_r \in W_r$, and assume $g(z_r) = g(z'_r)$. Then, for some x_s, y_s we have $Fz_r x_s$, $Ff(z_r)y_s$, and $Fz'_r x_s$, $Ff(z'_r)y_s$. By Lemma 5.6 this implies $Px_s z_r y_s$ and $Px_s z'_r y_s$. Hence, by (CP15) $z_r = z'_r$.
g is a homomorphism. To establish this claim we need to consider 5 cases: I, f, C, P, F. Here we go.
I: let $z_r \in I$; we need to show that $g(z_r) = (x_s, x_s)$ for some x_s. Choose x_s, y_s such that $g(z_r) = (x_s, y_s)$. By definition $Fz_r x_s$, $Ff(z_r)y_s$ and so $Px_s z_r y_s$ by Lemma 5.6. By (CP3) this gives $x_s = y_s$.

f: we need to show that $f(g(z_r)) = g(f(z_r))$. If $g(z_r) = (x_s, y_s)$, then $Px_sz_ry_s$, and, by (CP5), $Pf(z_r)y_sx_s$. Hence, $g(f(z_r)) = (y_s, x_s) = f(g(z))$.
C: we need to show that $Cx_ry_rz_r$ implies that $g(x_r)$ is the composition of $g(y_r)$ and $g(z_r)$. That is: if $g(x_r) = (x_0, x_1)$, $g(y_r) = (y_0, y_1)$, $g(z_r) = (z_0, z_1)$, then $x_0 = y_0$, $y_1 = z_0$, $z_1 = x_1$. Observe that by (CP2) we have Px_0y_rz', $Pz'z_rx_1$, for some z'. By Lemma 5.6, (CP5), (CP10) and (CP11) this implies the three identities.
F: here we need to show that Fz_rx_s implies that if $g(z) = (z_0, z_1)$ then $z_0 = x_s$. But this is immediate from the definition of g and (CP10).
P: we need to show that $Px_sz_ry_s$ implies $g(z) = (x_s, y_s)$; this is immediate by Lemma 5.6.
g^{-1} *is a homomorphism.* Again, this requires us to consider 5 cases.
I: we need to show that whenever $g(z_r) = (x_s, x_s)$, then $z_r \in I$. If $g(z_r) = (x_s, x_s)$, then $Px_sz_rx_s$. By (CP4) there is a y_r such that $Px_sy_rx_s$ and Iy_r. By (CP15) this implies $y_r = z_r$; hence Iz_r.
f: this has already been proved above.
C: assume $g(x_r)$ is the composition of $g(y_r)$ and $g(z_r)$, that is, assume $g(x_r) = (x_0, x_1)$, $g(y_r) = (y_0, y_1)$, $g(z_r) = (z_0, z_1)$; We need to show that $Cx_ry_rz_r$. By definition $x_0 = y_0$, $y_1 = z_0$, $z_1 = x_1$; so $Px_0y_rz_0$ and $Pz_0z_rx_1$. By (CM1) the latter implies that for some u_r, $Px_0u_rx_1$ and $Cu_ry_rz_r$. By (CP15) $u_r = x_r$, hence $Cx_ry_rz_r$.
F: assume $g(z_r) = (x_s, y_s)$; we need to show that Fz_rx_s; but this is immediate from the definitions.
P: assume that $g(z_r) = (x_s, y_s)$; we have to show that $Px_sz_ry_s$. But $g(z_r) = (x_s, y_s)$ implies Fz_rx_s and $Ff(z_r)y_s$; now apply Lemma 5.6. ⊣

Corollary 5.8 *Let $\mathfrak{F}$ be a two-sorted arrow frame. Then*

$$\mathfrak{F} \in \mathsf{TPF} \quad \textit{iff} \quad \mathfrak{F} \models \text{(CR1)–(CR8), (CP1)–(CP9), (CP14), (CP15)}.$$

Recall from §4 that the operator E_s is short for $E_sp \equiv p \vee D_sp$ (there exists a state where p holds), and that the operator O_s is short for $O_sp \equiv E_s(p \wedge \neg D_sp)$ (there exists only one state with p).

Definition 5.9 We define the following two formulas:

(MP14) $E_sp \to \langle \mathbf{1} \rangle p$
(MP15) $E_sO_sp \wedge \langle a \rangle p \wedge \langle b \rangle p \to \langle a \cap b \rangle p.$

Lemma 5.10 *Let $\mathfrak{F}$ be a two-sorted Peirce like frame. Then $\mathfrak{F}$ satisfies* (CP14) *iff it validates* (MP14)*; it satisfies* (CP15) *iff it validates* (MP15).

Proof. We first prove that (CP14) is defined by (MP14). Assume $\mathfrak{F} \not\models$ (CP14). Then there exist x_s, y_s such that $x_sz_ry_s$ holds for no z_r. Defining a valuation V such that $V(p) = \{y_s\}$ refutes (MP14) at x_s.

For the converse, fif $\mathfrak{F} \not\models$ (MP14), then for some valuation V and state x_s in $\mathfrak{F}$ we have $x_s \models E_s p$ and $x_s \not\models \langle \mathbf{1} \rangle p$. Hence there exists y_s with $y_s \models p$. As $x_s \not\models \langle \mathbf{1} \rangle p$, we can't have $P x_s z_r y_s$ for any z_r. Therefore $\mathfrak{F} \not\models$ (CP14).

Next we prove that (CP15) is defined by (MP15). Assume $\mathfrak{F} \not\models$ (CP15). Then there are z_r, z'_r, x_s, y_s such that $P x_s z_r y_s$ and $P x_s z'_r y_s$, but $z_r \neq z'_r$. Defining a valuation V such that $V(p) = \{y_s\}$, $V(a) = \{z_r\}$, $V(b) = \{z'_r\}$ refutes (MP15) at x_s.

For the converse, if $\mathfrak{F} \not\models$ (MP15), then for some valuation V and x_s in $\mathfrak{F}$ we have $x_s \models E_s O_s p \wedge \langle a \rangle p \wedge \langle b \rangle p$ and $x_s \not\models \langle a \cap b \rangle p$. Hence, there exists a unique y_s in $\mathfrak{F}$ with $y_s \models p$, and there exist z_r, z'_r with $P x_s z_r y_s$, $P x_s z'_r y_s$ and $z_r \models a$, $z'_r \models b$. As $x_s \not\models \langle a \cap b \rangle p$, we must have $z_r \neq z'_r$. So $\mathfrak{F} \not\models$ (CP15). $\dashv$

Theorem 5.11 $\mathsf{TPF} = \{\mathfrak{F} \mid \mathfrak{F} \models \bigwedge_{0 \leq i \leq 8} (\mathrm{MR}i) \wedge \bigwedge_{0 \leq i \leq 9} (\mathrm{MP}i) \wedge (\mathrm{MP14}) \wedge (\mathrm{MP15})\}$.

Proof. This follows from 5.4, 5.8 and 5.10. $\dashv$

As pointed out in §4, the difference operator is definable on Peirce frames (cf. the operator D'_s). If we take versions of axioms (MP14), (MP15) in which D_s is replaced by D'_s, don't we get a characterization of Peirce frames in the original modal language $\mathcal{ML}_2$ from Theorem 5.11 after all? The answer is 'no.' And the reason is that the semantics of the difference operator as a primitive operator is based on the diversity relation $\neq$; for the defined difference operator this does not hold for all two-sorted frames for the language $\mathcal{ML}_2$.

6 Further results

Building on the characterization result Theorem 5.11 one can give a complete axiomatization of Peirce frames (or equivalently, of the full Peirce algebras). The result is that the axioms (MR0)–(MR8), (MP1)–(MP9) extended with the derivation rules of basic modal logic as well as a so-called *irreflexivity rule* for the defined difference operator D'_s, are complete for Peirce frames; see [20].

Building on work of Wadge [28], Maddux [14] develops a sequent system for relation algebras. This work has recently been extended to Peirce algebras by Stebletsova [24].

In §2 we briefly mentioned a connection between a system of Arrow Logic and Peirce algebras. There is a whole hierarchy of calculi in between this Arrow Logic and the logic of full Peirce algebras, just like there is a hierarchy of subsystems of relation algebra. About the former hierarchy one can ask the same kind of questions as for the latter. For example, where does undecidability strike? For arrow logic this question was answered in Kurucz et al. [11]. In recent work Marx [15] presents a list of answers for the case of hybrid calculi in which sets and relations coexist.

In [20] techniques from modal logic such as bisimulations and the so-called standard translation are used are used to describe the expressive power of Peirce algebra.

7 Conclusion

In this note we studied Peirce algebras via modal logic. By extending the modal language for the frames corresponding to Peirce algebras, we were able to characterize the 'intended' frames, i.e., the frames corresponding to full Peirce algebras.

We already mentioned further work on Peirce algebras that has been reported elsewhere; to conclude this note we mention an unresolved issue related to the use of Peirce algebras in knowledge engineering. In terminological reasoning one often needs to be able to *count* the number of objects related to a given object; this is done using so-called *number restrictions* as in KL-ONE (see [5]). One direction for further work is to try to characterize and axiomatize Peirce algebras with counting.

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