Modal Logics and Description Logics

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1 Introduction

In their early days, description logics did not appear to be much more than a convenient notation for talking about structured knowledge. But once they were equipped with a proper syntax and semantics, model and proof theory — in short: once they grew up to be *logics*, it became possible to relate description logics to other areas of logic. In particular, the connection between description logics on the one hand and modal logics on the other hand has received considerable attention.

Schild [8] was the first to make the connection between description logic and modal logic explicit. He developed the correspondence between description logics and propositional dynamic logics, which are logics designed for reasoning about programs. The links provided a valuable a tool for devising decision procedures for very expressive description logics. Later, Schild [9] and De Giacomo and Lenzerini [3] identified a correspondence between description logics and the modal mu-calculus; again this link was exploited to transfer decidability and complexity results from modal to description logics. Van der Hoek and De Rijke [5] considered connections between description logics with number restrictions and modal and other logics with counting expressions.

In this talk I will take a more general look at the connection between description logics, modal logics, and various fragments of first-order logic. I will start by looking at a particular description logic and its modal counterpart, and will gradually adopt a more general perspective, viewing description and modal logics as restricted formalisms for talking about graph-like structures. While we will see that semantic characterizations can be given for particular description logics, we also consider the question which syntactic restrictions on first-order formulas produce 'good' description logics.

2 From Modal Logic to Description Logic, and Back

To make matters concrete, I will first look at the description logic $\mathcal{ALC}$ and its modal counterpart. Let us assume that $\mathcal{ALC}$ has a collection of atomic concepts $\mathcal{C}$ and a collection of atomic roles $\mathcal{R}$.

The corresponding (multi-) modal logic has formulas are produced by the following rule:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \langle a\rangle\phi \mid [a]\phi.$$

Here, p is a proposition letter taken from some collection $\mathcal{P}$, and a is an index also taken from some collection $\mathcal{A}$. (The semantics for this multi-modal language is based on tuples $(W, \{R_a \mid a \in \mathcal{A}\}, V)$, where W is a non-empty domain, the R_a's are binary relations on W — each modal operator $\langle a\rangle$ is associated with its own binary relation R_a —, and V is a valuation assigning subsets of W to the proposition letters in $\mathcal{P}$; observe that we can view these multi-modal models as interpretations for $\mathcal{ALC}$.)

Now, to connect $\mathcal{ALC}$ to multi-modal logic, all we really do is the following. Semantically, when we go from $\mathcal{ALC}$ to multi-modal logic, we simply view interpretations of atomic concepts as values of proposition letters, and atomic roles are binary relations to be used as interpretations for the modal operators. Syntactically, consider the following mapping τ from concepts to formulas:

$$\begin{aligned}
\tau(A) &= p_A,\ A \text{ atomic}\\
\tau(\neg C) &= \neg\tau(C)\\
\tau(C \sqcap D) &= \tau(C) \wedge \tau(D)\\
\tau(\exists R.C) &= \langle a_R\rangle\tau(C)\\
\tau(\forall R.C) &= [a_R]\tau(C)
\end{aligned}$$

Then, for every concept C, we have that C is equivalent to its translation $\tau(C)$ in the following sense:

$$(*) \qquad w \in C^{\mathcal{I}} \text{ iff } \mathcal{I}, w \models \tau(C).$$

(Here $\mathcal{I}$ is an interpretation for $\mathcal{ALC}$, and on the right-hand side I view it as a multi-modal model; the notation on the right-hand side means: $\tau(C)$ is true of w in $\mathcal{I}$.)

As pointed out in the introduction, this correspondence between description and modal logic may be extended in a variety of ways. In the talk several further examples will be considered.

3 A More General Perspective

What makes the correspondence recorded in (∗) work? It's the fact that $\mathcal{ALC}$ and multi-modal logic really talk about the same thing. In a sense to be made precise below, they are both restricted formalisms for talking about graph-like structures: collections of objects (or worlds, or states, or points in time,...) equipped with one or more binary relations. As such there is nothing unique about description logics and modal logics — there are plenty of ways of talking about graph-like structures, and first-order logic is probably the best known of these.

So what distinguishes description logics from first-order logic as a means for talking about graph-like structures? First of all, description logics — usually — very small fragments of first-order languages. Let us make this precise. Let $\mathcal{FO}$ be a first-order language that has unary predicate symbols corresponding to atomic concepts, and binary relation symbols corresponding to atomic roles (below, I won't distinguish between concepts and unary predicate symbols, and similarly for roles and binary predicate symbols).

Now, $\mathcal{ALC}$ can be translated into $\mathcal{FO}$ in the following way. Let x and y be two individual variables. Define two functions taking concepts to first-order formulas:

$$
\begin{aligned}
st_x(A) &= Ax \\
st_y(A) &= Ay \\
st_x(\neg C) &= \neg st_x(C) \\
st_y(\neg C) &= \neg st_y(C) \\
st_x(C \sqcap D) &= st_x(C) \land st_x(D) \\
st_y(C \sqcap D) &= st_y(C) \land st_y(D) \\
st_x(\exists R.C) &= \exists y\,(Rxy \land st_y(C)) \\
st_y(\exists R.C) &= \exists x\,(Ryx \land st_x(C)) \\
st_x(\forall R.C) &= \forall y\,(Rxy \to st_y(C)) \\
st_y(\forall R.C) &= \forall x\,(Ryx \to st_x(C)).
\end{aligned}
$$

Then, viewing interpretations for $\mathcal{ALC}$ as first-order models, one can formulate the following equivalence. For all interpretations $\mathcal{I}$ and all objects w and all $\mathcal{ALC}$-concepts C:

$$w \in C^{\mathcal{I}} \text{ iff } \mathcal{I} \models st_x(C)[w].$$

The notation on the right-hand side means that w is assigned to the free variable x.

So, this identifies $\mathcal{ALC}$ (and hence multi-modal logic) as a fragment of first-order logic. Actually, it identifies $\mathcal{ALC}$ as a part of the *2-variable fragment* of first-order logic — we only need 2 variables in the st-translation! Which fragment of $\mathcal{FO}$ is $\mathcal{ALC}$? What's special about it? This is where bisimulations come in. Call a unary first-order formula $\alpha(x)$ *invariant for bisimulations* if it cannot distinguish between bisimilar objects. Then, a first-order formula is equivalent to (the translation of) an $\mathcal{ALC}$-concept if, and only if, it is invariant under bisimulations. This result is basically an old result for modal logic proved by Van Benthem in his thesis [2]; the terminology was quite different, and only uni-modal languages were considered but the definitions and proofs easily extend to the multi-modal case. In [6, 7] this characterization result has been extended and adapted to many description logics other than $\mathcal{ALC}$. The main use of these characterizations is in understanding the expressive power of description logics.

4 The Right Fragment?

So, fragments of first-order logic that correspond to description logics can be characterized in terms of preservation under suitable notions of simulation relations between models. This is an interesting and useful but fairly abstract semantic description. But what are these fragments like *syntactically*? For over a decade finite variable fragments were thought to be the natural counterpart of modal logics within first-order logic, and given the correspondence (∗) noted above, this view carries over to description logics as well. And indeed, following Gabbay [4], a correspondence can indeed be set up between finite variable fragments and modal logics (this give rise to the issue of expressive completeness of a modal or description logic w.r.t. a finite variable fragment).

But, in general, finite variable fragments lack the 'good' computational and logical properties that usually come with description and modal logics (decidability, interpolation, ...). So, finite variable fragments don't seem to be the appropriate syntactic counterpart of description logics. The *guarded fragment* has recently been proposed by Andréka, Van Benthem and Németi [1] as the appropriate generalization of 'good fragments' of first-order logic.

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