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A Note on the Interpretability Logic of Finitely Axiomatized Theories[1]

Abstract. In [6] Albert Visser shows that *ILP* completely axiomatizes all schemata about provability and relative interpretability that are provable in finitely axiomatized theories. In this paper we introduce a system called ILP^{ω} that completely axiomatizes the arithmetically valid principles of provability in and interpretability over such theories. To prove the arithmetical completeness of ILP^{ω} we use a suitable kind of tail models; as a byproduct we obtain a somewhat modified proof of Visser's completeness result.

1. Introduction

In [5] Albert Visser introduces a logic *ILP* in a modal language $\mathcal{L}(\Box, \rhd)$ with a unary operator $\Box$, to be interpreted arithmetically as provability, and a binary operator $\rhd$, to be interpreted arithmetically as relative interpretability over some fixed theory U. In [6] he shows that *ILP* completely axiomatizes all schemata about provability and relative interpretability that are provable in Σ^0_1-sound finitely axiomatized sequential theories U that extend $\mathrm{I}\Delta_0 + \mathrm{SupExp}$. In this paper we present a complete axiomatization, called ILP^{ω}, of all *true* such schemata; on the way we obtain a somewhat modified proof of Visser's completeness result.

The main difference between Visser's proof of the arithmetical completeness of *ILP* and ours, is that we use infinite Kripke-like models, instead of finite ones, to find arithmetical interpretations for unprovable modal formulas. The models we use are variations on the *tail models* for provability logic as developed by Albert Visser (cf. [4]). We think that the use of tail models in this setting is rather natural. The advantage of using these models is two-fold. First of all, it allows us to set up things in such a way, that we can prove the arithmetical completeness of *ILP* and ILP^{ω} (almost) in one go.

To understand the second advantage, recall that the arithmetical sentences needed to prove the arithmetical completeness of some given logic Λ are usually found by embedding models of Λ into arithmetic. If these models are finite, the embedding will only be partial, in the following sense.

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Consider a formula $A(\vec{p})$ as a polynomial in the truth values of the *p*s, and suppose that $[B]$ is a representation in arithmetic of the extension of B in a given model. To justify the use of the phrase 'embedding into arithmetic' we want the equivalence $A([\vec{p}]) \leftrightarrow [A(\vec{p})]$ to be provable in our arithmetical theory, for all formulas A. But, assuming that our arithmetical theory is Σ^0_1-sound, this is not possible when we are working with finite models: for if $\mathcal{M}$ is such a model then for some n, $\mathcal{M} \models \Box^n\bot \leftrightarrow \Box^{n+1}\bot$. By using infinite models we will be able to obtain complete embeddings.

The rest of this paper is organized as follows: in §2 the systems ILP and ILP^ω are introduced; in §3 we review the arithmetical notions we need and assumptions we make for our completeness results. Then, in §4, we state and prove the arithmetical completeness of ILP and ILP^ω.

Two last remarks: we assume that the reader is familiar with the discussion of systems and arithmetization in [7]; he or she is also advised to keep a copy of Visser's [6] at hand.

2. The systems ILP and ILP^ω

The provability logic L is propositional logic plus all instance of the schemas $\Box(A \to B) \to (\Box A \to \Box B)$, $\Box A \to \Box\Box A$ and $\Box(\Box A \to A) \to \Box A$; its rules of inference are A, $A \to B/B$ (Modus Ponens), and $A/\Box A$ (Necessitation). Let $\mathcal{L}(\Box, \rhd)$ denote the language of interpretability logic. The interpretability logic ILP extends L with all instances of the following schemas:

$(J1)$	$\Box(A \to B) \to A \rhd B$;	$(J4)$	$A \rhd B \to (\Diamond A \to \Diamond B)$;
$(J2)$	$(A \rhd B) \wedge (B \rhd C) \to A \rhd C$;	$(J5)$	$A \rhd \Diamond A$;
$(J3)$	$(A \rhd C) \wedge (B \rhd C) \to (A \rhd C)$;	(P)	$A \rhd B \to \Box(A \rhd B)$.

ILP^ω has as axioms all theorems of ILP plus all instances of the schema of *reflection*: $\Box A \to A$; its sole rule of inference is Modus Ponens. Since $ILP \vdash \Box A \leftrightarrow \neg A \rhd \bot$, we may consider '$\Box$' to be defined in terms of '$\rhd$'.

ILP has been shown to be modally complete with respect to two kinds of (finite) models, notably with respect to Veltman models for ILP in [1, Theorem 5.2], and with respect to Friedman models for ILP in [6, Theorem 8.1].

DEFINITION 2.1. A *Friedman tail model* is a tuple $\mathcal{M} = \langle \omega, 0, Q, P, \Vdash \rangle$ with

1. $Q \subseteq \omega^2$ is transitive, irreflexive and tree-like;
2. $P \subseteq Q$ is given by a set $X \subseteq \omega$ such that $0 \in X$, and $xPy \Leftrightarrow xQy$ and $y \in X$, and such that $y \in X$, yPz implies $yQz'Pz$, for some z';

3. if $xQyPz$ then xPz;
4. if $n \neq 0$ then $0Qn$, and if $0 \neq nQm$ then $n > m$;
5. there is an $N \in \omega$ such that
   (a) for every $n, m \geq N$, if $m < n$ then nQm;
   (b) for every $n \geq N$, if for some k, $n = 2k + N$ then mPn for all $m > n$;
   (c) for every $n \geq N$, $N \Vdash p$ iff $n \Vdash p$ iff $0 \Vdash p$.

An N which satisfies 5 is called a *tail element.* We define $R := Q \circ P$, i.e., xRy iff $\exists z\, xQzPy$. $\Vdash$ satisfies the usual clauses, with R as the accessibility relation for '$\Box$', and

$$x \Vdash A \rhd B \quad \Longleftrightarrow \quad \forall u\,(xQu \Rightarrow (\exists y\,(uPy \wedge y \Vdash A) \Rightarrow \exists z\,(uPz \wedge z \Vdash B))).$$

Finally, if $\mathcal{M}$ is a Friedman tail model, and A a formula. Then $[\![A]\!]_{\mathcal{M}} := \{x \in \mathcal{M} : x \Vdash A\}$.

DEFINITION 2.2. We introduce two operators Δ_p, Δ_q with forcing conditions $x \Vdash \Delta_p A$ iff for all y with xPy, $y \Vdash A$, and $x \Vdash \Delta_q A$ iff for all y with xQy, $y \Vdash A$. We write ∇_p, ∇_q for $\neg \Delta_p \neg$, $\neg \Delta_q \neg$ respectively. $\mathcal{L}(\Delta_p, \Delta_q)$ denotes the language with the two new operators.

Define a translation $(\cdot)^\tau : \mathcal{L}(\Box, \rhd) \longrightarrow \mathcal{L}(\Delta_p, \Delta_q)$ as follows: $(\cdot)^\tau$ is the identity on proposition letters and the constants $\top$, $\bot$, and it commutes with the Boolean connectives; furthermore $(A \rhd B)^\tau := \Delta_q\,(\nabla_p A^\tau \to \nabla_p B^\tau)$.

We write $\tau\mathcal{L}(\Box, \rhd)$ for the image of $\mathcal{L}(\Box, \rhd)$ under τ, and define $\tau\mathcal{L}(\Box, \rhd)^*$ to be the sublanguage of $\mathcal{L}(\Delta_p, \Delta_q)$ in which Δ_q occurs only in front of implications of the form $\nabla_p C \to \nabla_p D$. Clearly, then, $\tau\mathcal{L}(\Box, \rhd) \subseteq \tau\mathcal{L}(\Box, \rhd)^*$.

REMARK 2.3. $\mathcal{L}(\Delta_p, \Delta_q)$ is in fact the language of the bi-modal provability logic PRL_1 discussed in [3] (with the modal operators interpreted as *tableaux provability* instead of ordinary provability). Using $(\cdot)^\tau$ and 2.7 one easily verifies that PRL_1 is a conservative extension of ILP.

PROPOSITION 2.4. *Let $\mathcal{M}$ be a Friedman tail model, and let $A \in \mathcal{L}(\Box, \rhd)$. Then for all $n \in \mathcal{M}$, $n \Vdash A \leftrightarrow A^\tau$.*

PROPOSITION 2.5. *Let $\mathcal{M}$ be a Friedman tail model in which P is given by some set X. Let $\Delta_q B \in \tau\mathcal{L}(\Box, \rhd)^*$. If $n \in X$ and $n \Vdash \Delta_q B$ then $n \Vdash B$.*

PROOF. If $\Delta_q B \in \tau\mathcal{L}(\Box, \rhd)^*$ then B has the form $\nabla_p C \to \nabla_p D$. Moreover, if $n \in X$ and nPm then nRm. These observations yield the result. $\Box$

PROPOSITION 2.6. *1. Let $A \in \mathcal{L}(\Box, \rhd) \cup \mathcal{L}(\Delta_p, \Delta_q)$. Then $[\![A]\!]_{\mathcal{M}}$ is either finite or cofinite.*

2. $0 \Vdash A$ iff for some N and all $n \geq N$, $n \Vdash A$;

3. $0 \nVdash A$ iff for some N and all $n \geq N$, $n \nVdash A$.

THEOREM 2.7. *Let $A \in \mathcal{L}(\Box, \rhd)$. Then*

1. *$ILP \vdash A$ iff for every Friedman tail model $\mathcal{M}$, and all $n \in \mathcal{M}$, $n \Vdash A$;*
2. *$ILP^\omega \vdash A$ iff for every Friedman tail model $\mathcal{M}$, $0 \Vdash A$.*

PROOF. By [6, Theorem 8.1] ILP is modally complete with respect to Friedman models; since such models are in fact Friedman tail models without tail, our first claim is immediate. To prove the second one, note first that 0 forces the theorems of ILP in any Friedman tail model. Closure under Modus Ponens is trivial. Assume that $0 \Vdash \Box A$, then for all n with $0Rn$, $n \Vdash A$. So $[\![A]\!]_{\mathcal{M}}$ is infinite, and hence, by 2.6, cofinite. Thus $0 \Vdash A$, again by 2.6.

Next assume that $ILP^\omega \nvdash A$, then, obviously, $ILP \nvdash \mathrm{T}(A) \to A$, where

$$\mathrm{T}(A) := \bigwedge_{\Box B \in \mathrm{S}(A)} (\Box B \to B) \wedge \bigwedge_{C \rhd D \in \mathrm{S}(A)} (C \to \Diamond C),$$

and $\mathrm{S}(A)$ is the set of subformulas of A. So by 1 there is a tail model $\mathcal{M}$ such that for some tail element N in $\mathcal{M}$, $N \Vdash \mathrm{T}(A) \wedge \neg A$. An easy induction now establishes that for $C \in \mathrm{S}(A)$ the following facts hold: (1) if $N \Vdash C$ then for all m with mRN, $m \Vdash C$; and (2) if $N \nVdash C$ then for all m with mRN, $m \nVdash C$. So $0 \nVdash A$. $\Box$

3. Arithmetical completeness: preliminaries

To prove the arithmetical completeness of ILP^ω we want to use several results from [6]. To be able to do so, we only consider arithmetical theories that satisfy a number of conditions to be given now. (Details about the notions used below may be looked up in [2], [5], [6] and [7].)

Officially we will be working in a relational version of the language of arithmetic, in which successor, addition and multiplication are (2-, 3- and 3-place) relation symbols. We will, however, *pretend* that we are working with function symbols. We assume that the theories T we consider are given by an R_1^+-formula $\alpha_T(x)$ having just x free plus the relevant information on what the set of natural numbers of T is; α_T gives the set of codes of non-logical axioms of the theory (cf. [7]). We also assume that the numbers of T satisfy $\mathrm{I}\Delta_0 + \Omega_1$, and that T is finitely axiomatized and sequential.

Wilkie and Paris [7] show that $I\Delta_0 + \Omega_1$ is a completely adequate theory for arithmetizing syntax. E.g., if T is a theory satisfying the assumptions made above, we can formalize in $I\Delta_0 + \Omega_1$ (as an R_1^+-formula) $\text{Proof}_T(x, y)$, which represents the relation 'x is a proof of the formula y from T'. We further define $\text{Prov}_T(y) := \exists x\ \text{Proof}_T(x, y)$.

One of the key results needed to prove our arithmetical completeness results, is a result by Friedman, extended by Visser, that gives a characterization of interpretability in terms of consistency. To state it we need a notion of cut free proof. We follow [6] in choosing tableaux provability. We write $\text{TabProof}_T(x, y)$ for (a formalization of) the relation 'x is a tableau proof of the formula y from T'. Furthermore, $\text{TabProv}_T(y) := \exists x\, \text{TabProof}_T(x, y)$, and $\text{TabCon}_T(\ulcorner\varphi\urcorner) := \neg\text{TabProv}_T(\ulcorner\neg\varphi\urcorner)$. Using this notation we can state the Friedman-Visser characterization as follows: let U be finitely axiomatized and sequential, and let Interp_U denote (a formalization of) relative interpretability over U, then $I\Delta_0 + \text{Exp}$ proves

$$\text{Interp}_U(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \leftrightarrow \text{TabProv}_{Exp}(\ulcorner\text{TabCon}_U(\ulcorner\varphi\urcorner) \to \text{TabCon}_U(\ulcorner\psi\urcorner)\urcorner).$$

A proof of this result may be found in [6, Section 7.4].

4. Arithmetical completeness: the main result

Before setting off, let us briefly outline the arithmetical completeness proofs we are about to give. Starting with a formula A that is non-provable in ILP or ILP^ω, we find a Friedman tail model refuting A; using a Solovay-like function defined on this model we define an arithmetical interpretation of the modal language in arithmetic. Up to this point we will follow Visser's original completeness proof for ILP (modulo some changes necessitated by the fact that we are working with infinite models instead of finite ones). The way we subsequently prove that the arithmetical interpretation thus defined really is an embedding of the model refuting A into arithmetic, differs from Visser's set up. Finally, we derive the arithmetical completeness of both ILP and ILP^ω.

For the remainder of this paper, let U be a Σ_1^0-sound extension of $I\Delta_0 +$ SupExp that satisfies all the requirements from §3.

Our first aim is to embed Friedman tail models into U. To do so we fix $\mathcal{M} = \langle \omega, 0, Q, P, \Vdash \rangle$ to be a tail model; we assume that P is given by a set X as in item 2 of the definition of a Friedman tail model. Define as formulas in the language of U:

$$(x \in [\![A]\!]_{\mathcal{M}}) := \begin{cases} \bigvee_i \{ (x = \underline{i}) : i \Vdash A \}, & \text{if } [\![A]\!]_{\mathcal{M}} \text{ is finite} \\ \bigwedge_i \{ (x \neq \underline{i}) : i \nVdash A \}, & \text{if } [\![A]\!]_{\mathcal{M}} \text{ is cofinite.} \end{cases}$$

It is easily verified that $\mathrm{I}\Delta_0 + \mathrm{Exp}$ proves

- $(x \in [\![A]\!]_{\mathcal{M}}) \wedge (x \in [\![B]\!]_{\mathcal{M}}) \leftrightarrow (x \in [\![A \wedge B]\!]_{\mathcal{M}})$;
- $(x \in [\![A]\!]_{\mathcal{M}}) \vee (x \in [\![B]\!]_{\mathcal{M}}) \leftrightarrow (x \in [\![A \vee B]\!]_{\mathcal{M}})$;
- $x \notin [\![A]\!]_{\mathcal{M}} \leftrightarrow x \in [\![\neg A]\!]_{\mathcal{M}}$.

Using the Recursion Theorem we define a Solovay-like function H as follows:

$$\begin{array}{rcl} H(0) & = & 0 \\ H(x+1) & = & \begin{cases} y, & \text{if } H(x)Py \text{ and } \mathrm{TabProof}_U(x+1, \ulcorner L \neq y \urcorner) \\ y, & \text{if } H(x)Qy \text{ and } \mathrm{TabProof}_{Exp}(x+1, \ulcorner L \neq y \urcorner) \\ H(x), & \text{otherwise} \end{cases} \\ L & = & \text{the limit of } H. \end{array}$$

We leave it to the reader to check that the formula '$H(x) = u$' is $\Delta_0(2^x)$, and that for any x, y

1. $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash xQy \to (H(x) = H(y) \vee H(x)QH(y))$;
2. $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash$ 'L exists';
3. $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash L = x \leftrightarrow \exists y\,(H(y) = x) \wedge \forall uv\,(H(u) = x \wedge v > u \to H(v) = x)$;
4. $L = 0$.

DEFINITION 4.1. We define the representation $[A]_{\mathcal{M}}$ of $[\![A]\!]_{\mathcal{M}}$ in the language of U by $[A]_{\mathcal{M}} := (L \in [\![A]\!]_{\mathcal{M}})$.

Let g be any function that takes the proposition letters from $\mathcal{L}(\Box, \rhd)$ (or $\mathcal{L}(\triangle_p, \triangle_q)$) to sentences in the language of arithmetic. Then the *arithmetical interpretation* $\langle\cdot\rangle_g$ of $\mathcal{L}(\Box, \rhd) \cup \mathcal{L}(\triangle_p, \triangle_q)$ into the language of arithmetic is defined by

$$\begin{array}{rclrcl} \langle p \rangle_g & := & [p]_g & \langle \Box A \rangle_g & := & \mathrm{Prov}_U(\ulcorner \langle A \rangle_g \urcorner) \\ \langle \bot \rangle_g & := & `0 = 1' & \langle A \rhd B \rangle_g & := & \mathrm{Interp}_U(\ulcorner \langle A \rangle_g \urcorner, \ulcorner \langle B \rangle_g \urcorner) \\ \langle \neg A \rangle_g & := & \neg \langle A \rangle_g & \langle \triangle_p\, A \rangle_g & := & \mathrm{TabProv}_U(\ulcorner \langle A \rangle_g \urcorner) \\ \langle A \wedge B \rangle_g & := & \langle A \rangle_g \wedge \langle B \rangle_g & \langle \triangle_q\, A \rangle_g & := & \mathrm{TabProv}_{Exp}(\ulcorner \langle A \rangle_g \urcorner). \end{array}$$

In case $g(p) = [p]_{\mathcal{M}}$ for some model $\mathcal{M}$, we write $\langle\cdot\rangle_{\mathcal{M}}$ for $\langle\cdot\rangle_g$.

PROPOSITION 4.2. *Let* $\psi \in \Pi_2^0$. *Then* $I\Delta_0 + \mathrm{Exp} \vdash \mathrm{TabProv}_{Exp}(\ulcorner\psi\urcorner) \rightarrow \mathrm{TabProv}_U(\ulcorner\psi\urcorner)$.

PROOF. Cf. [6, Lemma 8.2]. □

PROPOSITION 4.3. $U \vdash L \in X$.

PROOF. Reason in U: by our earlier remarks the limit L exists. So assume $L = \underline{i} \notin X$. Then, by the definition of H, $i > 0$ and $\mathrm{TabProv}_{Exp}(\ulcorner L \neq \underline{i}\urcorner)$. Recall that U extends $I\Delta_0 + \mathrm{SupExp}$. By [6, Consequence 7.3.7], $I\Delta_0 + \mathrm{SupExp}$ proves Π_2^0-reflection for $I\Delta_0 + \mathrm{Exp}$. (This is in fact the only place where we really need U to be an extension of $I\Delta_0 + \mathrm{SupExp}$.) Therefore, in U we have $L \neq \underline{i}$—a contradiction. □

LEMMA 4.4. *Let* $A \in \tau\mathcal{L}(\Box, \rhd)^*$. *Then* $I\Delta_0 + \mathrm{Exp} \vdash [A]_{\mathcal{M}} \leftrightarrow \langle A\rangle_{\mathcal{M}}$.

PROOF. Induction on A. The propositional case and the Boolean cases are immediate from the fact that the limit provably exists and the induction hypothesis, respectively.

Suppose $A \equiv \Delta_p\, B$. First we assume that $[\Delta_p\, B]\!]_{\mathcal{M}}$ is cofinite. Then $[\![\Delta_p\, B]\!]_{\mathcal{M}} = \omega$. So $[\Delta_p\, B]_{\mathcal{M}} \equiv \bigwedge_i\{(L \neq \underline{i}) : i \Vdash \bot\} \equiv \top$. So $I\Delta_0 + \mathrm{Exp} \vdash [\Delta_p\, B]_{\mathcal{M}}$, and hence $I\Delta_0 + \mathrm{Exp} \vdash \langle\Delta_p\, B\rangle_{\mathcal{M}} \rightarrow [\Delta_p\, B]_{\mathcal{M}}$. To prove the other direction it suffices to show that $I\Delta_0 + \mathrm{Exp} \vdash \mathrm{TabProv}_U(\ulcorner[B]_{\mathcal{M}}\urcorner)$. Clearly, $[\![B]\!]_{\mathcal{M}}$ is cofinite and $X \subseteq [\![B]\!]_{\mathcal{M}}$; therefore $[B]_{\mathcal{M}} \equiv \bigwedge_i\{(L \neq \underline{i}) : i \nVdash B\}$. Reason in $I\Delta_0 + \mathrm{Exp}$: if $i \nVdash B$, then $\mathrm{TabProv}_U(\ulcorner L \neq \underline{i}\urcorner)$, because $U \vdash L \in X$. Therefore $\mathrm{TabProv}_U(\ulcorner[B]_{\mathcal{M}}\urcorner)$.

Next we assume that $[\Delta_p\, B]\!]_{\mathcal{M}}$ is finite. Let $\{j_0, \ldots, j_s\}$ be all j with $j \Vdash \Delta_p\, B, \neg B$. Then, if $i \nVdash \Delta_p\, B$, there is a $j \in \{j_0, \ldots, j_s\}$ with iPj. By the induction hypothesis it suffices to show that $I\Delta_0 + \mathrm{Exp} \vdash [\Delta_p\, B]_{\mathcal{M}} \leftrightarrow \mathrm{TabProv}_U(\ulcorner[B]_{\mathcal{M}}\urcorner)$. Reason in $I\Delta_0 + \mathrm{Exp}$:

'$\leftarrow$': Assume $\mathrm{TabProv}_U(\ulcorner[B]_{\mathcal{M}}\urcorner)$. Let $j \in \{j_0, \ldots, j_s\}$. Then $\mathrm{TabProv}_U(\ulcorner L \neq \underline{j}\urcorner)$. So assume that $\mathrm{TabProof}_U(p+1, \ulcorner L \neq \underline{j}\urcorner)$. If LPj then $H(p)Pj$—so $H(p+1) = j$, which is a contradiction. Therefore, $\neg LPj$, so $\bigvee_i\{(L = \underline{i}) : i \Vdash \Delta_p\, B\}$.

'$\rightarrow$': Assume $L = \underline{i}$, $i \Vdash \Delta_p\, B$. Then $i \neq 0$. So $\mathrm{TabProv}_{Exp}(\ulcorner L \neq \underline{i}\urcorner)$ or $\mathrm{TabProv}_U(\ulcorner L \neq \underline{i}\urcorner)$, so by 4.2 $\mathrm{TabProv}_U(\ulcorner L \neq \underline{i}\urcorner)$. We also have for some x, $H(x) = i$, and hence $\mathrm{TabProv}_U(\ulcorner\exists x\, H(x) = \underline{i}\urcorner)$. This implies $\mathrm{TabProv}_U(\ulcorner iQL\urcorner)$. Given that $U \vdash L \in X$, this entails $\mathrm{TabProv}_U(\ulcorner iPL\urcorner)$. Finally, iPj implies $j \Vdash B$. Therefore $\mathrm{TabProv}_U(\ulcorner\bigvee_j\{(L = \underline{j}) : j \Vdash B\}\urcorner)$.

Assume next that $A \equiv \Delta_q\, B$, and that $[\![\Delta_q\, B]\!]_{\mathcal{M}}$ is cofinite. Then $[\![\Delta_q\, B]\!]_{\mathcal{M}} = [\![B]\!]_{\mathcal{M}} = \omega$. So by the induction hypothesis $I\Delta_0 + \mathrm{Exp} \vdash \langle B\rangle_{\mathcal{M}}$, and hence

$$\begin{aligned} \mathrm{I}\Delta_0 + \mathrm{Exp} &\vdash \mathrm{TabProv}_{Exp}(\ulcorner\langle B\rangle_{\mathcal{M}}\urcorner) \\ &\vdash \langle\triangle_q\, B\rangle_{\mathcal{M}} \leftrightarrow \top \\ &\vdash \langle\triangle_q\, B\rangle_{\mathcal{M}} \leftrightarrow [\triangle_q\, B]_{\mathcal{M}}. \end{aligned}$$

Next we assume that $[\triangle_q\, B]\!]_{\mathcal{M}}$ is finite. As in the case of $\triangle_p\, B$, let $\{j_0,\ldots,j_s\}$ be all j with $j \Vdash \triangle_q\, B, \neg B$. Then, if $i \not\Vdash \triangle_q\, B$, there is a $j \in \{j_0,\ldots,j_s\}$ with iQj. By the induction hypothesis it suffices to show that $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash [\triangle_q\, B]_{\mathcal{M}} \leftrightarrow \mathrm{TabProv}_{Exp}(\ulcorner[B]_{\mathcal{M}}\urcorner)$. Reason in $\mathrm{I}\Delta_0 + \mathrm{Exp}$:

'$\leftarrow$': This direction is analogous to the corresponding direction in the case of $\triangle_p\, B$.

'$\rightarrow$': Assume $L = \underline{i}$, $i \Vdash \triangle_q\, B$. Then there exists an x with $H(x) = i$. In other words, $\mathrm{TabProv}_{Exp}(\ulcorner\exists x\, H(x) = i\urcorner)$, and hence $\mathrm{TabProv}_{Exp}(\ulcorner\bigvee_k\{(L = \underline{k}) : i = k \vee iQk\}\urcorner)$. Now, if $i \notin X$, then $\mathrm{TabProv}_{Exp}(\ulcorner L \neq \underline{i}\urcorner)$ by the definition of H. Therefore $\mathrm{TabProv}_{Exp}(\ulcorner\bigvee_k\{(L = \underline{k}) : iQk\}\urcorner)$. Thus $\mathrm{TabProv}_{Exp}(\ulcorner\bigvee_k\{(L = \underline{k}) : k \Vdash B\}\urcorner)$. If, on the other hand, $i \in X$, then $i \Vdash \triangle_q\, B$ implies by 2.5 that $i \Vdash B$. But then we have $\mathrm{TabProv}_{Exp}(\ulcorner\bigvee_k\{(L = \underline{k}) : k \Vdash B\}\urcorner)$. □

LEMMA 4.5. *Let* $A \in \mathcal{L}(\Box, \rhd)$. *Then* $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash [A]_{\mathcal{M}} \leftrightarrow \langle A\rangle_{\mathcal{M}}$.

PROOF. Since, by 2.4, for all $A \in \mathcal{L}(\Box, \rhd)$, and all i, $i \Vdash A \leftrightarrow A^\tau$, we trivially have $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash [A]_{\mathcal{M}} \leftrightarrow [A^\tau]_{\mathcal{M}}$. Since $A^\tau \in \tau\mathcal{L}(\Box, \rhd)^*$, we can apply Lemma 4.4 to conclude that $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash [A]_{\mathcal{M}} \leftrightarrow \langle A^\tau\rangle_{\mathcal{M}}$ $(\star)$.

Recall that by the Friedman-Visser characterization of relative interpretability over finitely axiomatized sequential theories, $\mathrm{I}\Delta_0 + \mathrm{Exp}$ proves

$$\mathrm{Interp}_U(\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) \leftrightarrow \mathrm{TabProv}_{Exp}(\ulcorner\mathrm{TabCon}_U(\ulcorner\varphi\urcorner) \rightarrow \mathrm{TabCon}_U(\ulcorner\psi\urcorner)\urcorner).$$

This characterization allows us to show by induction on A that $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash \langle A^\tau\rangle_{\mathcal{M}} \leftrightarrow \langle A\rangle_{\mathcal{M}}$. Together with $(\star)$ this yields the Lemma. □

We need one more definition and a proposition before we can prove the arithmetical completeness of ILP and ILP^ω. From now on $\mathcal{M}$ is no longer a fixed Friedman tail model.

DEFINITION 4.6. Let $\mathcal{M}$ be a Friedman tail model. Define $\mathrm{d}_{\mathcal{M}}(k) := \sup\{\mathrm{d}_{\mathcal{M}}(l) + 1 : kRl\}$, and

$$\mathrm{d}(A) := \begin{cases} \mu n.\, \exists\mathcal{M}\exists m\, (\mathrm{d}_{\mathcal{M}}(m) = n \wedge m \not\Vdash A), & \text{if such an } n \text{ exists} \\ \omega, & \text{otherwise.} \end{cases}$$

PROPOSITION 4.7. *Let* $A \in \mathcal{L}(\Box, \rhd)$. *Then there is a function* g *taking proposition letters to sentences in the language of* U *such that* $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash \langle A \wedge \Box A\rangle_g \leftrightarrow \mathrm{Prov}_U^{\mathrm{d}(A)}(\ulcorner 0 = 1\urcorner)$.

PROOF. If $\mathrm{d}(A) = \omega$ then $ILP \vdash A$, so any g does the trick. If $\mathrm{d}(A) < \omega$, then there is a tail model $\mathcal{M}$ with tail element N such that $\mathrm{d}_{\mathcal{M}}(N) = n$, and $N \not\Vdash A$. Define $g(p) := [p]_{\mathcal{M}}$. Then for every k with NRk, $k \Vdash A \wedge \Box A$. Moreover, if $k = N$ or kRN, then $k \not\Vdash A \wedge \Box A$. Therefore

$$\begin{aligned} \mathrm{I}\Delta_0 + \mathrm{Exp} \vdash \langle A \wedge \Box A\rangle_g &\leftrightarrow [A \wedge \Box A]_{\mathcal{M}}, \text{ by 4.5} \\ &\leftrightarrow [\Box^{\mathrm{d}(A)} \bot] \\ &\leftrightarrow \mathrm{Prov}_U^{\mathrm{d}(A)}(\ulcorner 0 = 1\urcorner). \ \Box \end{aligned}$$

THEOREM 4.8. *Let* $A \in \mathcal{L}(\Box, \rhd)$. *Then* $ILP \vdash A$ *iff for every interpretation* $\langle\cdot\rangle_g$, $U \vdash \langle A\rangle_g$.

PROOF. The direction from left to right is left to the reader. To prove the other one, assume that $ILP \nvdash A$. Then there is a tail model $\mathcal{M}$ and a tail element N such that $\mathrm{d}_{\mathcal{M}}(N) = \mathrm{d}(A) < \omega$, $N \not\Vdash A$ and $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash \langle A \wedge \Box A\rangle_{\mathcal{M}} \leftrightarrow \mathrm{Prov}_U^{\mathrm{d}(A)}(\ulcorner 0 = 1\urcorner)$. If $U \vdash \langle A\rangle_{\mathcal{M}}$ then $U \vdash \langle A \wedge \Box A\rangle_{\mathcal{M}}$, and hence $U \vdash \mathrm{Prov}_U^{\mathrm{d}(A)}(\ulcorner 0 = 1\urcorner)$—contradicting our assumption that U is Σ_1^0-sound. Conclude that $U \nvdash \langle A\rangle_{\mathcal{M}}$. $\Box$

THEOREM 4.9. *Let* $A \in \mathcal{L}(\Box, \rhd)$. *Then* $ILP^\omega \vdash A$ *iff for every interpretation* $\langle\cdot\rangle_g$, $\mathbb{N} \models \langle A\rangle_g$.

PROOF. Again, the direction from left to right is left to the reader. Assume, to prove the converse, that $ILP^\omega \nvdash A$. Then there is a Friedman tail model $\mathcal{M}$ with $0 \not\Vdash A$. By 4.5 $\mathbb{N} \models \langle A\rangle_{\mathcal{M}} \leftrightarrow [A]_{\mathcal{M}}$. Moreover, $\mathbb{N} \models L = 0$. It follows that $\mathbb{N} \models \neg\langle A\rangle_{\mathcal{M}}$. $\Box$

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