Journal of Logic, Language, and Information **6:** 403–425, 1997.

Bisimulations for Temporal Logic

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(Received 20 January 1997)

Abstract. We define bisimulations for temporal logic with Since and Until. This new notion is compared to existing notions of bisimulations, and then used to develop the basic model theory of temporal logic with Since and Until. Our results concern both invariance and definability. We conclude with a brief discussion of the wider applicability of our ideas.

Key words: modal and temporal logic, expressive power, model theory, definability

1. Introduction

Labeled transition systems are probably the simplest structures used to model dynamic phenomena: they are simply structures equipped with a collection of states and one or more transition relations that indicate how one state can evolve into another. Numerous languages have been proposed as suitable description tools for talking about transition systems. *Process algebraic* languages take an external view on transition systems in that each process algebraic term denotes an entire transition system. *Modal* and *temporal* languages, on the other hand, offer an internal perspective on transition systems, as they describe (local) properties of states and transitions between them.

This paper deals with the model theory of one particular 'internal' description language for transition systems: the temporal language with Since and Until. This language, and languages closely related to it, have been proposed by a number of authors as suitable for describing dynamic phenomena. For example, van Benthem (1991) suggests that we use Since and Until to describe operations of theory change. Also, information change often involves an 'economy principle' saying that one should change as little information as possible when accommodating new data; languages with Since and Until (or Since and Until-like operators) are the obvious candidates if one wants to express this idea of minimal change, and, indeed, in most of the more powerful dynamic languages one can define them (see, for example, van Benthem et al., 1994; van Eijck et al., 1996; de Rijke, 1992).

In a properly developed theory of dynamics the relation between the models of dynamic phenomena on the one hand, and the description language used to specify such models is a central issue. In this paper we analyze the model theory of the temporal language with Since and Until; the main tool in our analysis is a special kind of bisimulations.

The relevance of bisimulations to dynamics lies in the answer one can give to the following question: when do two transition systems represent the same process? Obviously, it depends on the character of the states and transitions, and on the features of transition systems that one finds important. If we are modeling dialogues one can think of the information that a participant in a conversation has as a state, and the transitions are changes to his information induced as the conversation progresses. Here, a criterion for identifying two systems could be that a given statement should produce equivalent outputs on equivalent inputs. As a second example, in reasoning about theory change, states represent databases and the actions or transitions represent insertions and deletions of information. Here, a criterion for calling two states equivalent could be that they have the same logical consequences or that an insertion or deletion in the one state can always be mimicked by insertions or deletions in the other state to yield (logically) equivalent results. And, of course, in concurrency theory states represent the state of a machine, and transitions represent executions of atomic programs. Here a minimal requirement for states to be identified is that they have the same choices of atomic programs enabled. If we use Since and Until to describe our systems we need to require more than this if we insist that states to be identified are logically indistinguishable. The details will emerge in Section 3 below, but just to give an idea, one thing we shall need is that if an action is enabled in a state s, then we should not only find the same action enabled in any state t that we want to identify with s, but we should also ensure that the 'interval' or 'period' leading from s to the result of the action can be matched by a similar interval starting from t.

In addition there are also more technical reasons to work with bisimulations in trying to understand the model theory of Since and Until. Recent work in the model theory of modal languages is characterized by a pervasive use of bisimulations. Van Benthem (1991) first observed the close resemblance of bisimulations to partial isomorphism. This observation has inspired a systematic investigation of the model theory of basic poly-modal logic along the lines of first-order model theory in de Rijke (1995b), whose results take the following 'heuristic equation' as their starting point:

$$\frac{\text{partial isomorphims}}{\text{first-order logic}} = \frac{\text{bisimulations}}{\text{modal logic}}\ .$$

Andréka et al. (1995) further explore the links between modal logic and first-order logic using bisimulations as a central tool, and the investigations of van Benthem et al. (1994), van Benthem and Bergstra (1994), and de Rijke (1995a) also revolve around the use of bisimulations in the model theory of modal logic.

Most of the results in the papers cited above concern only basic modal diamonds $\langle\alpha\rangle$ and boxes $[\alpha]$ with their familiar truth definitions, or simple variations thereof. The model theory of modal and temporal languages with more complex operators is not as well developed. In particular, in the case of the temporal language with Since and Until, there is no proper notion of bisimulation that allows for the development of its model theory in analogy with basic poly-modal logic; this has been observed by a number of authors (see van Benthem and Bergstra, 1994; van Benthem et al., 1994; de Rijke, 1995b). In this paper we address this issue by introducing a notion of bisimulation that 'works' for the temporal language with Since and Until. That is, we define a notion of bisimulation that can serve as a central tool in the model theory of temporal logic by allowing us to prove basic preservation and definability results.

The structure of the paper is as follows. In Section 2 we recall some basic concepts; in Section 3 we introduce a notion of bisimulations for Since and Until, and compare it to related equivalence relations on models. Section 4 considers the question when temporal equivalence implies bisimilarity, and Section 5 then uses bisimulations to establish basic model-theoretic results on preservation and definability for the temporal language with Since and Until. We conclude with some questions and suggestions for future work.

2. Definitions

This section introduces the concepts we need. First, SU*-formulas* are built up using propositional variables p, q, ..., the constants $\top$ and $\bot$, boolean connectives $\neg$, $\wedge$, and the binary temporal operators S (Since) and U (Until). We use $\mathcal{L}_{SU}$ to denote this language. We use the usual abbreviations: $F\phi \equiv U(\phi, \top)$, $G\phi \equiv \neg F \neg\phi$, $P\phi \equiv S(\phi, \top)$, $H\phi \equiv \neg P \neg \phi$.

A *flow of time*, *temporal order* or *frame* is a pair $F = (W, <)$, where W is a non-empty set of *time points* or *states*, and $<$ is a binary relation on W. A *valuation* is a function assigning a subset of W to every proposition letter. A *model* is a pair $M = (F, V)$ where F is a frame and V a valuation.

The *satisfaction relation* is defined in the familiar way for the atomic and boolean cases, while for the temporal connectives we put

$M, t \models S(\phi, \psi)$ iff there exists $v < t$ such that $M, v \models \phi$, and for all u with $v < u < t$: $M, u \models \psi$;

$M, t \models U(\phi, \psi)$ iff there exists $v > t$ such that $M, v \models \phi$, and for all u with $v > u > t$: $M, u \models \psi$.

To talk about the points involved in interpreting temporal formulas, the notion of an interval proves useful. Let $M = (W, <, V)$ be a model. An *interval* in M is simply a pair of points $w, v \in W$. An interval wv is called a *pseudo-interval* if there is no $u \in W$ such that $w < u$ and $u < v$. If wv is an interval, and ϕ a temporal formula, then define *truth* of ϕ in wv by putting

$$wv \models \phi \text{ iff for all } u \text{ with } w < u < v \text{ we have } u \models \phi.$$

Using our notion of intervals we can rewrite the truth condition for S as $w \models S(\phi, \psi)$ iff there exists $v < w$ with $v \models \phi$ and $vw \models \psi$.

The *temporal theory* of a point w is the set *tp*$(w) = \{\phi \in \mathcal{L}_{SU} \mid w \models \phi\}$, and the *temporal theory* of an interval wv is the set *tp*$(wv) = \{\phi \in \mathcal{L}_{SU} \mid wv \models \phi\}$. If we want to emphasize the model M in which w (or wv) lives, we write $tp_M(w)$ (or $tp_M(wv)$). Observe that if wv is a pseudo-interval, then its temporal theory is simply the set of all temporal formulas. Two points w, v are *temporally equivalent* if *tp*$(w) =$ *tp*(v) (notation $w \equiv v$); temporal equivalence for intervals is defined analogously.

Let $\mathcal{L}_1$ be the first-order language with unary predicate symbols corresponding to the proposition letters in $\mathcal{L}_{SU}$, and with one binary relation symbol $<$. $\mathcal{L}_1$ is called the *correspondence language* for $\mathcal{L}_{SU}$. $\mathcal{L}_1(x)$ denotes the set of all $\mathcal{L}_1$-formulas having one free variable x.

Models can be viewed as $\mathcal{L}_1$-structures in the usual first-order sense. The *standard translation* takes temporal formulas ϕ into equivalent formulas $ST(\phi)$ in the correspondence language. It maps proposition letters p onto unary predicate symbols Px, it commutes with the booleans, and the temporal case is

$$ST(S(\phi, \psi)) = \exists y\,(y < x \wedge ST(\phi)(y) \wedge \forall z\,(y < z < x \to ST(\psi)(z))),$$
$$ST(U(\phi, \psi)) = \exists y\,(x < y \wedge ST(\phi)(y) \wedge \forall z\,(x < z < y \to ST(\psi)(z))).$$

For all models M and points t we have $M, t \models \phi$ iff $M \models ST(\phi)[t]$, where the latter denotes first-order satisfaction of $ST(\phi)$ under the assignment of t to the free variable of $ST(\phi)$.

3. Bisimulations for S and U

Several notions of bisimulation that preserve temporal formulas have already been proposed in the literature. But none of these provides an exact characterization of the expressive power of the language with Since and Until. To fill this gap, we introduce a notion of bisimulation for Since and Until in this section, and compare it to related equivalence relations on models; our findings are summarized in a diagram at the end of the section (Figure 5).

To define bisimulations that work for temporal logic, we will use relations that link points to points and intervals to intervals.

DEFINITION 3.1 (Bisimulations). Let $M_1 = (W_1, <_1, V_1)$ and $M_2 = (W_2, <_2, V_2)$ be two models. A *bisimulation between* M_1 *and* M_2 is a triple $Z = (Z_0, Z_1, Z_2)$, where $Z_0 \subseteq |M_1| \times |M_2|$, $Z_1 \subseteq |M_1|^2 \times |M_2|^2$, and $Z_2 \subseteq |M_2|^2 \times |M_1|^2$ such that

$Z_0 \neq \emptyset$ and the following clauses hold:

1. If $x_1 Z_0 x_2$ then x_1 and x_2 satisfy the same proposition letters.
2. If $x_1 Z_0 x_2$ and $x_1 <_1 y_1$, then there exists y_2 in M_2 with $x_2 <_2 y_2$ such that $y_1 Z_0 y_2$ and $x_1 y_1 Z_1 x_2 y_2$.
3. If $x_1 y_1 Z_1 x_2 y_2$ and there exists z_2 with $y_2 <_2 z_2 <_2 x_2$, then there exists z_1 with $x_1 <_1 z_1 <_1 y_1$ and $z_1 Z_0 z_2$.
4. If $x_1 Z_0 x_2$ and $x_2 <_2 y_2$, then there exists y_1 in M_1 with $x_1 <_1 y_1$ such that $y_1 Z_0 y_2$ and $x_2 y_2 Z_2 x_1 y_1$.
5. If $x_2 y_2 Z_2 x_1 y_1$ and there exists z_1 with $y_1 <_1 z_1 <_1 x_1$, then there exists z_2 with $x_2 <_2 z_2 <_2 y_2$ and $z_1 Z_0 z_2$.
6. Clauses 2–5 with $>_1$ ($>_2$) instead of $<_1$ ($<_2$).

If there is a bisimulation $Z = (Z_0, Z_1, Z_2)$ with $x_1 Z_0 x_2$, then we say that x_1 and x_2 are *bisimilar* (notation $x_1 \leftrightarroweq x_2$, or $Z : x_1 \leftrightarroweq x_2$), and similarly for intervals $x_1 y_1$ and $x_2 y_2$. If necessary, the models in which x_1 and x_2 live will also be included in the notation: $M_1, x_1 \leftrightarroweq M_2, x_2$.

A few remarks are in order. First, in the semantics of dynamic formalisms both states and transitions play an important role; the semantics of Since and Until may seem to suggest that the transitions only have a secondary role to play in determining the truth value of a formula involving Since and Until. Our notion of bisimulation, however, clearly shows that both properties of states and of intervals are important: points are related to points, and intervals to intervals.

Second, observe that we have back and forth conditions for the first component, Z_0, of a bisimulation Z: a move from a point in the first model should be matched with a move to a Z_0-related state in the second model, and, vice versa, a move in the second model is matched with a move in the first one to a Z_0-related point. For the second and third component (Z_1 and Z_2) we only have one direction: intervals in the first model are Z_1-related to intervals in the second model, but to relate intervals in the second model to intervals in the first one we use a separate relation Z_2. The reason for the use of two relations in linking intervals is the following. The back-and-forth character of Z_0 ensures that negated formulas are preserved; but the way we have set up things, we do not have proper boolean negations of formulas interpreted on intervals, and thus a relation connecting intervals in a back-and-forth manner would be too strong for our purposes. See Kurtonina and de Rijke (1996) for further details on (bi-)simulations for negation free languages.

Finally, it is easily verified that arbitrary (component-wise) unions of bisimulation relations are again bisimulations, and that $\leftrightarroweq$ is the maximal bisimulation and an equivalence relation.

In Section 5 we show that a first-order formula in the correspondence language $\mathcal{L}_1$ is equivalent to a temporal formula with Since and Until iff it is invariant for the notion of bisimulation defined in Definition 3.1. In the remainder of the present section we compare our notion of bisimulation to closely related equivalence relations on models. Such comparisons can take place at two levels: one can compare particular instances of bisimulation relations, but at a more abstract level one can also compare the equivalence classes of models modulo the various notions of bisimilarity.

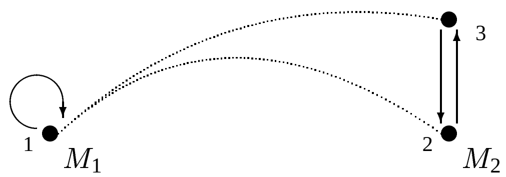
Our goal in comparing these equivalence relations is to locate our notion in the wider landscape of such relations, and to show that our notion of bisimulation is the weakest one that allows for a direct development of the model theory of Since and Until without a detour through richer languages.

3.1. Modal Bisimulations

We start with bisimulations for standard modal languages, often called *strong* bisimulations in the computational literature (see Hennessy and Milner, 1985). These are defined by clause 1 of Definition 3.1 together with clauses 2 and with their last conjuncts (‘and $x_1y_1Z_1x_2y_2$’ or ‘$x_2y_2Z_2x_1y_1$’) left out. Strong bisimulations are much weaker than our bisimulations: they do not take the ‘past’ of nodes into account. An obvious way of taking the past into account is by extending the language so as to include the familiar forward looking modality F and backward looking modality P. The corresponding notion of bisimulation is defined as follows. Let M_1, M_2 be two models; a non-empty relation $Z \subseteq W_1 \times W_2$ is a relation of F, P*-bisimulation* if it satisfies condition 1 of Definition 3.1 and a trimmed down version of its condition 2 in which references to intervals have been deleted:

2$'$. If x_1Zx_2 and $x_1 <_1 y_1$, then there exists y_2 in M_2 with $x_2 <_2 y_2$ and y_1Zy_2,

and similar conditions with $>_1$ instead of $<_1$, and going from M_2 to M_1. We write $x_1 \underline{\leftrightarrow}_{F,P}\ x_2$ to denote that there exists a F, P-bisimulation between x_1 and x_2. Clearly, $x_1 \underline{\leftrightarrow}\ x_2$ implies $x_1 \underline{\leftrightarrow}_{F,P}\ x_2$, but the converse need not hold, as is witnessed by the following example.



Here we have $M_1 \underline{\leftrightarrow}_{F,P}\ M_2$ via the relation indicated with dotted lines; but $M_1 \not\underline{\leftrightarrow} M_2$, because any candidate bisimulation Z should link 1 to both 2 and 3; so it would follow that $11Z_123$, and by the definition of bisimulations, there would be a state z between 2 and 3 – a contradiction.

All in all, then, we have the following.

PROPOSITION 3.2.
1. $M_1, w_1 \leftrightarroweq M_2, w_2$ implies $M_1, w_1 \leftrightarroweq_{F,P} M_2, w_2$.
2. $M_1, w_1 \leftrightarroweq_{F,P} M_2, w_2$ does not imply $M_1, w_1 \leftrightarroweq M_2, w_2$.

3.2. $\mathcal{U}$-BISIMULATIONS

Next we consider so-called $\mathcal{U}$-bisimulations. These were defined by van Benthem et al. (1994: definition 4.2) as candidate bisimulations for temporal logic. A non-empty relation $Z \subseteq W_1 \times W_2$ is a *$\mathcal{U}$-bisimulation* if it satisfies clause 1 of Definition 3.1, clause $2'$ above, and

$3'$. if $x_1 Z x_2$, $x_1 <_1 y_1$, $x_2 <_2 y_2$, $y_1 Z y_2$, and $x_1 <_1 z_1 <_1 y_1$, then there exists a z_2 in W_2 such that $x_2 <_2 z_2 <_2 y_2$ and $z_1 Z z_2$,

as well as similar conditions with $>_1$ ($>_2$) instead of $<_1$ ($<_2$), and going from M_2 to M_1. We use $x_1 \leftrightarroweq_{\mathcal{U}} x_2$ to denote that there exists a $\mathcal{U}$-bisimulation between x_1 and x_2.

It is easily verified that $M_1, w \leftrightarroweq_{\mathcal{U}} M_2, v$ implies $M_1, w \leftrightarroweq M_2, v$: any $\mathcal{U}$-bisimulation can be extended to a bisimulation in our sense. Let Z be a $\mathcal{U}$-bisimulation, and define Z' by

- $Z'_0 := Z$;
- $x_1 y_1 Z'_1 x_2 y_2$ iff $x_1 <_1 y_1$, $x_2 <_2 y_2$, $x_1 Z x_2$ and $y_1 Z y_2$; and
- $x_2 y_2 Z'_2 x_1 y_1$ iff $x_1 y_1 Z'_1 x_2 y_2$.

By way of example let us check clauses 2 and 3 of Definition 3.1. Assume $x_1 Z'_0 x_2$ and $x_1 <_1 y_1$. By $\mathcal{U}$-bisimilarity there exists y_2 with $x_2 <_2 y_2$ and $y_1 Z y_2$; putting these things together yields $x_1 y_1 Z'_1 x_2 y_2$, as required. To check clause 3, assume $x_1 y_1 Z'_1 x_2 y_2$ and $y_2 <_2 z_2 <_2 x_2$; we need to find a z_1 with $x_1 <_1 z_1 <_1 y_1$. Now, $x_1 y_1 Z'_1 x_2 y_2$ implies $x_1 <_1 y_1$, $x_2 <_2 y_2$, $x_1 Z x_2$ and $y_1 Z y_2$, so by the third clause in the definition of $\mathcal{U}$-bisimulation there exists a z_1 as required.

The upshot of the above is that any $\mathcal{U}$-bisimulation induces a bisimulation in a straightforward way. What about the converse? If Z is a bisimulation in our sense, is its first component Z_0 a $\mathcal{U}$-bisimulation? As the following example shows, the answer is 'no.' Consider Figure 1. The dotted curves depict the first component of a bisimulation in our sense that is not a $\mathcal{U}$-bisimulation. To be precise, let $M_1 = (\mathbb{Z}, <, V)$, where V is constant, and $<$ is the usual less-than relation; $M_2 = (\mathbb{Z}, <, V)$, where V and $<$ are as in M_1.

Define relations $Z_0 \subseteq \mathbb{Z} \times \mathbb{Z}$, Z_1, $Z_2 \subseteq (\mathbb{Z}^2 \times \mathbb{Z}^2)$ as follows:

$$\begin{aligned} Z_0 \; := \; & \{(n, n') \mid n \in \mathbb{Z}\} \cup \\ & \{(n, (n+1)') \mid n \in \mathbb{Z}\} \end{aligned}$$

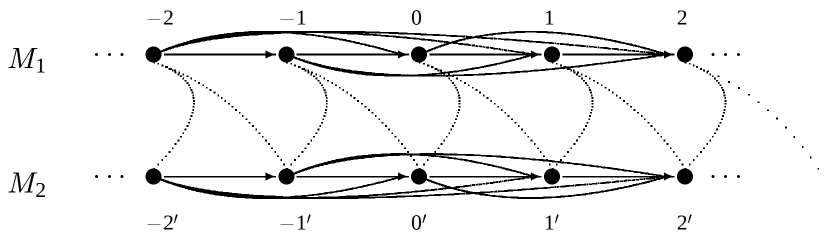


Figure 1. A bisimulation whose first component is not a $\mathcal{U}$-bisimulation.

$$
\begin{aligned}
Z_1 &:= \{(n\,m, n'\,m') \mid n < m\} \cup \\
&\quad\ \{(n\,m, (n+1)'\,(m+1)') \mid n < m\} \\
Z_2 &:= Z_1^{\smile}, \text{ the converse of } Z_1.
\end{aligned}
$$

We leave it to the reader to check that $Z : M_1, 0 \leftrightarroweq M_2, 0'$. However, this is not enough to make Z_0 into a $\mathcal{U}$-bisimulation. To see that $Z_0 : M_1, 0 \not\leftrightarroweq_{\mathcal{U}} M_2, 0'$, observe first that $0 < 2$, $0 < 1 < 2$, $1' < 2'$, $0Z_01'$, and $2Z_02'$. Hence, by clause $3'$, for Z_0 to be a $\mathcal{U}$-bisimulation we should be able to find a z with $1' < z < 2'$ and $1Z_0z$ – but there is no such point.

PROPOSITION 3.3.

1. $M_1, w \leftrightarroweq_{\mathcal{U}} M_2, v$ implies $M_1, w \leftrightarroweq M_2, v$.
2. $Z : M_1, w \leftrightarroweq M_2, v$ does not imply $Z_0 : M_1, w \leftrightarroweq_{\mathcal{U}} M_2, v$; and, more generally, $M_1, w \leftrightarroweq M_2, v$ does not imply $M_1, w \leftrightarroweq_{\mathcal{U}} M_2, v$ (cf. Proposition 3.4 below).

3.3. $\mathcal{B}$-BISIMULATIONS

Van Benthem et al. (1994) also consider an alternative notion, called $\mathcal{B}$-bisimulation, which relates points to points and pairs of points to pairs of points, much like our notion of bisimulation; the notion of $\mathcal{B}$-bisimulation is used to analyze a two-dimensional counterpart of the language of temporal logic with S and U. To be precise, a relation $Z \subseteq (W_1 \times W_2) \cup (W_1^2 \times W_2^2)$ with $Z \cap (W_1 \times W_2) \neq \emptyset$ is a *$\mathcal{B}$-bisimulation* if it satisfies clause 1 of Definition 3.1 and

$2''$. if $x_1 Z x_2$ and $x_1 <_1 y_1$, then there exists y_2 with $x_2 <_2 y_2$ and $x_1 y_1 Z x_2 y_2$

$3''$. if $x_1 y_1 Z x_2 y_2$, then $x_1 Z x_2$ and $y_1 Z y_2$

$4''$. if $x_1 y_1 Z x_2 y_2$ and $x_1 <_1 z_1 <_1 y_1$, then there exists z_2 with $x_2 <_2 z_2 <_2 y_2$ and both $x_1 z_1 Z x_2 z_2$ and $z_1 y_1 Z z_2 y_2$,

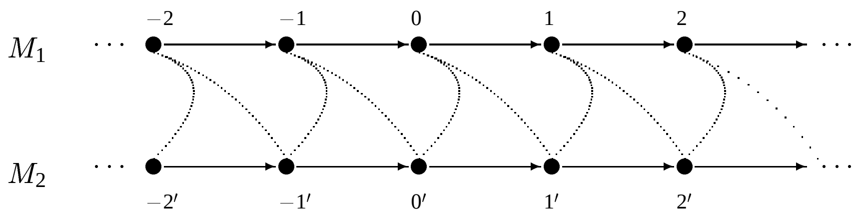


Figure 2. A bisimulation which is not a $\mathcal{B}$-bisimulation.

and similar conditions with $>_1$ ($>_2$) instead of $<_1$ ($<_2$), and going from M_2 to M_1.* We use $x_1 \;\underline{\leftrightarrow}_{\mathcal{B}}\; x_2$ to denote that there exists a $\mathcal{B}$-bisimulation between x_1 and x_2. Van Benthem et al. (1994: proposition 4.8) show that $x_1 \;\underline{\leftrightarrow}_{\mathcal{U}}\; x_2$ implies $x_1 \;\underline{\leftrightarrow}_{\mathcal{B}}\; x_2$: any $\mathcal{U}$-bisimulation can be extended to a $\mathcal{B}$-bisimulation. What about the relation between $\underline{\leftrightarrow}$ and $\underline{\leftrightarrow}_{\mathcal{B}}$? It is clear that any $\mathcal{B}$-bisimulation induces a bisimulation in our sense: if Z is a $\mathcal{B}$-bisimulation between M_1 and M_2, simply define Z' by putting $Z'_0 = Z \restriction (|M_1| \times |M_2|)$; $Z'_1 = Z \restriction (|M_1|^2 \times |M_2|^2)$, and $Z'_2 = Z'^{\smile}_1$.

The converse does not hold: a bisimulation Z need not induce a $\mathcal{B}$-bisimulation simply by taking the union of the components of Z (even when $Z_2 = Z_1^{\smile}$). To see this, look at Figure 1 again, but redefine the relations in the models to arrive at the picture in Figure 2. That is, define $M_1 = (\mathbb{Z}, R_1, V)$, where V is constant, and R_1nm iff $m = n + 1$; and $M_2 = (\mathbb{Z}, R_2, V)$, where V and R_2 are as in M_1.

Define relations $Z_0 \subseteq \mathbb{Z} \times \mathbb{Z}$, and Z_1, $Z_2 \subseteq (\mathbb{Z}^2 \times \mathbb{Z}^2)$ by putting

$$Z_0 := \{(n, n') \mid n \in \mathbb{Z}\} \cup \{(n, (n+1)') \mid n \in \mathbb{Z}\}$$

$$Z_1, Z_2 := \{(n\,(n+1), m'\,(m+1)') \mid n, m \in \mathbb{Z}\}.$$

We leave it to the reader to check that $Z : M_1, 0 \;\underline{\leftrightarrow}\; M_2, 0'$. Now, defining $Z' = Z_0 \cup Z_1$ does not produce a $\mathcal{B}$-bisimulation. In particular, $Z' : M_1, 0 \;\not\underline{\leftrightarrow}_{\mathcal{B}}\; M_2, 0'$, because if $Z' : 0\,1 \;\underline{\leftrightarrow}_{\mathcal{B}}\; 2'\,3'$ were to hold, we would also have $Z' : 0 \;\underline{\leftrightarrow}_{\mathcal{B}}\; 2'$, which is not the case.

The above observations can be strengthened: there are models that are bisimilar in our sense, but not $\mathcal{B}$-bisimilar (and hence, not $\mathcal{U}$-bisimilar either). Here is an example that is originally due to Holger Sturm. Consider Figure 3. The two models M_1 and M_2 depicted there are clearly not $\mathcal{B}$-bisimilar, but they are bisimilar in our sense. Define the following relations between M_1 and M_2:

$$Z_0 := \{(u_i, u_j), (v_i, v_j), (w_i, w_j) \mid i \leq 1,\ j \geq 2\}$$

* As one of the referees pointed out, actually van Benthem et al. (1994) only use $\mathcal{B}$-bisimulations to describe the forward looking fragment of their language (that is: only for the fragment with temporal operators exploring $<$, discarding $>$), and for this fragment it is definitively too strong. But for their full language (with forward and backward looking features) it is appropriate.

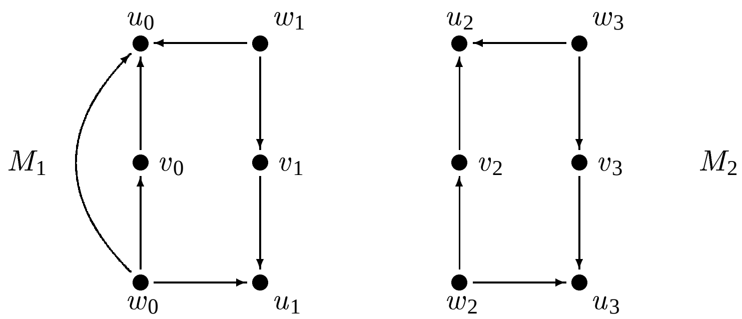


Figure 3. Bisimilar but not $\mathcal{B}$-bisimilar models.

$$\begin{aligned} Z_1 \ := \ & \{(w_0\,u_0, w_2\,u_3), (w_0\,u_0, w_3\,u_2)\} \cup \\ & \{(w_i\,v_i, w_j\,v_j), (v_i\,u_i, v_j\,u_j) \mid i \leq 1,\ j \geq 2\} \cup \\ & \{(w_i\,u_{i'}, w_j\,u_{j'}) \mid i \neq i' \leq 1,\ j \neq j' \geq 2\} \\ Z_2 \ := \ & \{(w_j\,v_j, w_i\,v_i), (v_j\,u_j, v_i\,u_i) \mid i \leq 1,\ j \geq 2\} \cup \\ & \{(w_j\,u_{j'}, w_i\,u_{i'}) \mid i \neq i' \leq 1,\ j \neq j' \geq 2\}. \end{aligned}$$

We leave it to the reader to check that $Z = (Z_0, Z_1, Z_2)$ is indeed a bisimulation.

PROPOSITION 3.4.
1. $M_1, w \;\underline{\leftrightarrow}_{\mathcal{B}}\; M_2, v$ implies $M_1, w \;\underline{\leftrightarrow}\; M_2, v$.
2. $M_1, w \;\underline{\leftrightarrow}\; M_2, v$ does not imply $M_1, w \;\underline{\leftrightarrow}_{\mathcal{B}}\; M_2, v$, and hence it does not imply $M_1, w \;\underline{\leftrightarrow}_{\mathcal{U}}\; M_2, v$ either.

3.4. S-SIMULATIONS

Sturm (1997) defines a notion of bisimulation, called S-simulation, for the forward looking fragment of our temporal language as follows. Let M_1, M_2 be two models; a non-empty relation $Z \subseteq W_1 \times W_2$ is a relation of *S-simulation* if it satisfies condition 1 of Definition 3.1 as well as

– If $x_1 Z x_2$ and $x_1 <_1 y_1$, then there exists y_2 in M_2 with $y_1 Z y_2$ and $x_2 <_2 y_2$ such that for every z_2 in M_2 with $x_2 <_2 z_2 <_2 y_2$ there exists z_1 in M_1 with $x_1 <_1 z_1 <_1 y_1$ and $z_1 Z z_2$.
– A similar clause going from M_2 to M_1.

Observe that S-simulations only 'look forward'; they do not take the converse $>_1$ of $<_1$ into account. Sturm (1997: lemma 2.11.6) shows that all forward looking temporal formulas (that is: formulas without occurrences of Since) are preserved under S-similarity.

For a proper comparison between S-simulations and our bisimulations we extend the above definition with backward looking clauses in the obvious way:

– If $x_1 Z x_2$ and $x_1 >_1 y_1$, then there exists y_2 in M_2 with $y_1 Z y_2$ and $x_2 >_2 y_2$ such that for every z_2 in M_2 with $x_2 >_2 z_2 >_2 y_2$ there exists z_1 in M_1 with $x_1 >_1 z_1 >_1 y_1$ and $z_1 Z z_2$.
– A similar clause going from M_2 to M_1.

It turns out that bisimilarity in our sense and S-similarity are equivalent notions, and therefore they preserve the same formulas. Clearly bisimilarity implies S-similarity (simply take the first component of a bisimulation). To see that the converse holds as well, let Z be an S-simulation, and define $Z' = (Z'_0, Z'_1, Z'_2)$ as follows:

$$\begin{aligned} Z'_0 &:= Z \\ Z'_1 &:= \{(x_1\, y_1, x_2\, y_2) \mid \forall z_2\, (x_2 <_2 z_2 <_2 y_2 \to \\ &\qquad \exists z_1\, (x_1 <_1 z_1 <_1 y_1 \wedge z_1 Z z_2))\} \\ Z'_2 &:= \{(x_2\, y_2, x_1\, y_1) \mid \forall z_1\, (x_1 <_1 z_1 <_1 y_1 \to \\ &\qquad \exists z_2\, (x_2 <_2 z_2 <_2 y_2 \wedge z_1 Z z_2))\} \end{aligned}$$

Then Z' is a bisimulation.

PROPOSITION 3.5. $M_1, w \leftrightarrows M_2, v$ is equivalent to $M_1, w \leftrightarrows_S M_2, v$.

To conclude our discussion of S-similarity we want to emphasize the following. We have seen that S-similarity (extended with backward looking clauses) coincides with our notion of bisimulation. This may seem to be a reason to prefer S-similarity over our notion of bisimilarity, especially since S-simulations are relations between points only, while our bisimulations involve both points and intervals, while temporal formulas are evaluated at points only. However, as we will show below, it is precisely this special *two-sorted* character of our notion of bisimulation that allows us to develop the model theory of Since and Until in a direct way (without detours through richer languages).

3.5. 3-BACK-AND-FORTH EQUIVALENCE

The following notion of an equivalence relation on models is taken from (van Benthem, 1991). First, a *partial isomorphism* from M_1 to M_2 is a partial map $\theta : W_1 \to W_2$ such that

– for all proposition letters p and all states w, $w \in V_1(p)$ iff $\theta(w_1) \in V_2(p)$,
– for all states $w_1, v_1 \in W_1$ and all quantifier-free formulas $\alpha(x, y)$ in $<$ and $=$ we have $M_1 \models \alpha[w_1 v_1]$ iff $M_2 \models \alpha[\theta(w_1)\theta(v_1)]$.

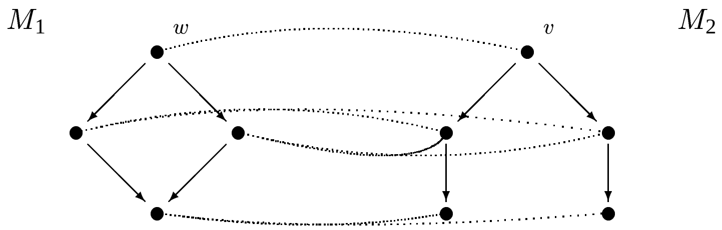


Figure 4. Bisimilar but not 3-back-and-forth-equivalent.

Next, a κ*-back-and-forth system* ($\kappa \leq \omega$) from M_1 to M_2 is a non-empty set $\mathcal{C}$ of partial isomorphisms from M_1 to M_2 such that

1. if $\theta \in \mathcal{C}$ then $|dom(\theta)| \leq \kappa$
2. if $\theta \in \mathcal{C}$ then any restriction of θ to a subset of its domain is also in $\mathcal{C}$
3. if $\theta \in \mathcal{C}$, $w \in W_1 \setminus dom(\theta)$ and $|dom(\theta)| < \kappa$, then there exists θ^+ in $\mathcal{C}$ with $\{w\} \cup dom(\theta) \subseteq dom(\theta^+)$
4. if $\theta \in \mathcal{C}$, $v \in W_2 \setminus rng(\theta)$ and $|dom(\theta)| < \kappa$, then there exists θ^+ in $\mathcal{C}$ with $\{v\} \cup rng(\theta) \subseteq rng(\theta^+)$.

Let $\overline{w} \in M_1$ and $\overline{v} \in M_2$ be tuples of equal length. The structures $(M_1, \overline{w})$ and $(M_2, \overline{v})$ are κ*-back-and-forth equivalent* if there exists a κ-back-and-forth system $\mathcal{C}$ from M_1 to M_2 containing a map θ such that $\theta(\overline{w}) = \overline{v}$; notation $\mathcal{C} : M_1, \overline{w} \simeq_\kappa M_2, \overline{v}$.

Van Benthem (1991) shows that a first-order formula (in $<, =$) can be written with at most three variables iff it is invariant under 3-back-and-forth equivalence. The relevance of this result for temporal logic is that temporal formulas with Since and Until can be translated into the 3-variable fragment of $\mathcal{L}_1$, the first-order correspondence language.

Clearly, $M_1, w \simeq_3 M_2, v$ implies $M_1, w \sim M_2, v$ for all $\sim\, \in \{\underline{\leftrightarrow}_{\mathcal{U}}, \underline{\leftrightarrow}_{\mathcal{B}}, \underline{\leftrightarrow}, \underline{\leftrightarrow}_S, \underline{\leftrightarrow}_{F,P}\}$, but none of the converse implications holds, as is witnessed by the example in Figure 4.

We leave it to the reader to check that $M_1, w \ \underline{\leftrightarrow}_{\mathcal{U}} \ M_2, v$ via the dotted lines (and from this the other bisimilarities follow). However, the single 'end point' in M_1 satisfies the 3-variable statement

$$\exists y \exists z \, (y \neq z \wedge y < x \wedge z < x)$$

which is not satisfied by any node in M_2, so M_1 and M_2 cannot be 3-back-and-forth equivalent.

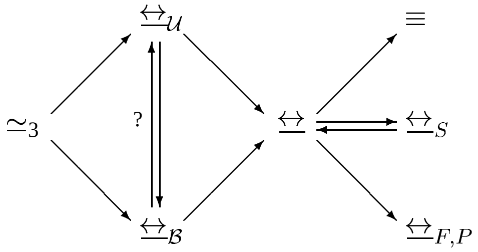


Figure 5. The findings of this section.

PROPOSITION 3.6.

1. $M_1, w \simeq_3 M_2, v$ implies $M_1 \;\underline{\leftrightarrow}\; M_2, v$.
2. $M_1 \;\underline{\leftrightarrow}\; M_2, v$ does not imply $M_1, w \simeq_3 M_2, v$.

3.6. TEMPORAL EQUIVALENCE

Finally, we compare temporal equivalence to bisimilarity.

PROPOSITION 3.7. Let ϕ be a temporal formula, and assume that ϕ cannot distinguish between bisimilar points, that is: if wZ_0v, then ($w \models \phi$ iff $v \models \phi$). If $w_1v_1Z_1w_2v_2$, then $w_1v_1 \models \phi$ implies $w_2v_2 \models \phi$. And if $w_2v_2Z_2w_1v_1$, then $w_2v_2 \models \phi$ implies $w_1v_1 \models \phi$.

Proof. We only prove the first of the two claims. Assume $w_1v_1 \models \phi$ and assume that Z is a bisimulation such that $w_1v_1Z_1w_2v_2$. We have to show that $w_2v_2 \models \phi$. So choose u_2 such that $w_2 <_2 u_2 <_2 v_2$. We need to show that $u_2 \models \phi$. As $w_1v_1Z_1w_2v_2$, there exists u_1 such that $w_1 <_1 u_1 <_1 v_1$ and $u_1Z_0u_2$. Then $u_1 \models \phi$, so by the assumption on ϕ we have $u_2 \models \phi$. □

LEMMA 3.8. If $M_1 = (W_1, <_1, V_1)$ and $M_2 = (W_2, <_2, V_2)$ are two models, and $w_1 \in W_1$, $w_2 \in W_2$, are such that $Z : w_1 \;\underline{\leftrightarrow}\; w_2$, then $w_1 \equiv w_2$. In other words: bisimilarity implies temporal equivalence.

Proof. We argue by induction on the structure of formulas. The atomic and boolean cases are easy. So let us consider the temporal case. Assume $w_1 \models U(\phi, \psi)$ and $Z : w_1 \;\underline{\leftrightarrow}\; w_2$. We need to show that $w_2 \models U(\phi, \psi)$. By definition there exists a v_1 such that (i) $w_1 <_1 v_1$, (ii) $v_1 \models \phi$, and (iii) $w_1v_1 \models \psi$. From (i) and clause 2 of Definition 3.1 we obtain a v_2 with (iv) $w_2 <_2 v_2$, (v) $v_1Z_0v_2$, and (vi) $w_1v_1Z_1w_2v_2$. By the induction hypothesis, (v) and (ii) we get $v_2 \models \phi$. From the induction hypothesis, (iii), (vi), and Proposition 3.7 it follows that $w_2v_2 \models \psi$. By (iv) this implies $w_2 \models U(\phi, \psi)$, as required.

The case for S is proved similarly. □

The converse of the implication proved in Lemma 3.8 (‘Does temporal equivalence imply bisimilarity?’) will be examined in Section 4 below.

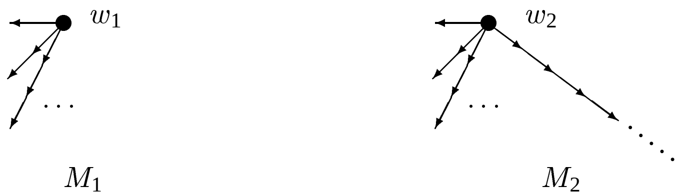


Figure 6. Equivalent but not bisimilar.

Summarizing the findings of this section, we arrive at the diagram of inclusions depicted in Figure 5, where an arrow $\sim \to \approx$ denotes that $\sim$-bisimilarity implies $\approx$-bisimilarity. The upward arrow marked with a question mark represents an open problem due to van Benthem et al. (1994: Open Problem 4.7).

4. Hennessy–Milner Classes

In this section we consider the converse of Lemma 3.8: when does temporal equivalence imply bisimilarity? Using a standard example from the literature on modal logic, it is easily seen that this is not the case in general. The two models in Figure 6 satisfy the same temporal formulas in their root nodes, but there is no bisimulation linking the two root nodes.

To get a handle on situations where temporal equivalence *does* imply bisimilarity, we need the following definition.

DEFINITION 4.1 (Hennessy–Milner class). A class K of models is called a *Hennessy–Milner class* if for M_1, $M_2 \in \mathsf{K}$, and all $w_1 \in M_1$ and $w_2 \in M_2$, $w_1 \leftrightarroweq w_2$ iff $w_1 \equiv w_2$. That is, if temporal equivalence is a bisimulation between M_1 and M_2.

For the standard modal language with $\Diamond$ and $\Box$ the above notion is due to Goldblatt (1995) and Hollenberg (1995). The standard example of a modal Hennessy–Milner class in which modal equivalence and modal bisimilarity coincide, is the class of all image-finite models – models for which the set of $<$-successors is finite for any point in the model.

It turns out that a natural way to determine whether a class of models is a Hennessy–Milner class involves the concept of temporal saturation. Let $\Delta \subseteq_{\mathrm{fin}} \Phi$ denote that Δ is a finite subset of Φ.

DEFINITION 4.2. Let $M = (W, <, V)$ be a model. M is said to be *t-saturated* if it satisfies the following conditions:

If $\forall\Delta \subseteq_{\mathrm{fin}} \Phi \forall\Gamma \subseteq_{\mathrm{fin}} \Psi \exists v \in W$ ($w < v$ and $v \models \bigwedge \Delta$ and $wv \models \bigwedge \Gamma$)
then $\exists v \in W$ ($w < v$ and $v \models \bigwedge \Phi$ and $vw \models \bigwedge \Psi$); and

If $\forall\Gamma \subseteq_{\text{fin}} \Psi \exists u \in W\ (w < u < v \text{ and } u \models \bigwedge \Gamma)$
then $\exists u \in W\ (w < u < v \text{ and } u \models \bigwedge \Psi)$.

(And similarly, with $>$ instead of $<$.) We use $\mathsf{T\text{-}SAT}$ to denote the class of all t-saturated models.

The notion of m-saturation considered in the literature on modal logic arises if one only takes the first condition for $>$ in the definition of t-saturation, with $\Psi = \emptyset$ (see Fine, 1975; Goldblatt, 1995; Hollenberg, 1995).

THEOREM 4.3. $\mathsf{T\text{-}SAT}$ is a Hennessy–Milner class.

Proof. Assume that M_1, M_2 are in $\mathsf{T\text{-}SAT}$. Define Z by putting $w_1 Z_0 w_2$ iff $tp(w_1) = tp(w_2)$; $w_1 v_1 Z_1 w_2 v_2$ iff $tp(w_1 v_1) \subseteq tp(w_2 v_2)$; and, similarly, $w_2 v_2 Z_2 w_1 v_1$ iff $tp(w_2 v_2) \subseteq tp(w_1 v_1)$. We will show that Z is a bisimulation.

The first clause of Definition 3.1 is trivially satisfied. For the second one, assume $tp(w_1) = tp(w_2)$ and $w_1 <_1 v_1$. We need to find a v_2 such that $w_2 <_2 v_2$, $tp(v_1) = tp(v_2)$ and $tp(w_1 v_1) \subseteq tp(w_2 v_2)$. Consider $\Delta \subseteq_{\text{fin}} tp(v_1)$ and $\Gamma \subseteq_{\text{fin}} tp(w_1 v_1)$. Then $w_1 \models U(\bigwedge \Delta, \bigwedge \Gamma)$, and so, as $w_1 \equiv w_2$, we have $w_2 \models U(\bigwedge \Delta, \bigwedge \Gamma)$. Thus, there exists v_2 in M_2 such that $w_2 <_2 v_2$, $v_2 \models \bigwedge \Delta$, and $w_2 v \models \bigwedge \Gamma$. By t-saturation there must be a $w_2 <_2 v_2$ such that $v_2 \models \bigwedge tp(v_1)$ and $v_2 w_2 \models \bigwedge tp(v_1 w_1)$. But then $tp(v_1) = tp(v_2)$ and $tp(w_1 v_1) \subseteq tp(w_2 v_2)$, as required.*

For clause 3 of Definition 3.1, assume that $tp(w_1 v_1) \subseteq tp(w_2 v_2)$ and $w_2 <_2 u_2 <_2 w_2$. We need to find a u_1 such that $w_1 <_1 u_1 <_1 v_1$ and $tp(u_1) = tp(u_2)$. Consider $\Gamma \subseteq_{\text{fin}} tp(u_2)$. Then $w_2 v_2 \not\models \neg \bigwedge \Gamma$, and so, since $tp(w_1 v_1) \subseteq tp(w_2 v_2)$, we find that $w_1 v_1 \not\models \neg \bigwedge \Gamma$. This implies that there exists u in M_1 with $w_1 <_1 u <_1 v_1$ and $u \models \bigwedge \Gamma$. Applying the second clause in the definition of t-saturation, we find a u_1 in M_1 such that $w_1 <_1 u_1 <_1 v_1$ and $tp(u_1) = tp(u_2)$, and we are done.

The remaining clauses may be proved by similar arguments. □

We now give two examples of t-saturated classes of models, the second of which will be used extensively below.

PROPOSITION 4.4. Every finite model is t-saturated.

Proof. Let $w \in |M|$, and consider sets of formulas Φ and Ψ such that for all $\Delta \subseteq_{\text{fin}} \Phi$ and $\Gamma \subseteq_{\text{fin}} \Psi$ there exists a v such that

$$w < v \text{ and } v \models \bigwedge \Delta \text{ and } wv \models \bigwedge \Gamma. \tag{1}$$

We need to show that there exists v such that (1) holds for all of Φ and Ψ. Suppose, for contradiction, that there is no v. Then, for every $v > w$, we find a $\phi_v \in \Phi$ with $v \not\models \phi_v$ or a $\psi_v \in \Psi$ with $wv \not\models \psi_v$. As M is finite, there are only finitely many

* Observe that $v_2 \models tp(v_1)$ implies $tp(v_1) = tp(v_2)$, but $w_2 v_2 \models tp(w_1 v_1)$ only implies $tp(w_1 v_1) \subseteq tp(w_2 v_2)$.

such v; collect the formulas ϕ_v and ψ_v (for $v > w$) together in finite sets $\Delta \subseteq_{\text{fin}} \Phi$, $\Gamma \subseteq_{\text{fin}} \Psi$. For these Δ and Γ (1) does not hold – a contradiction!

To establish the second clause of Definition 4.2, assume $w, v \in |M|$, and consider a set of formulas Ψ such that for every finite $\Gamma \subseteq_{\text{fin}} \Psi$ there exists a $u \in |M|$ such that

$$w < u < v \text{ and } u \models \bigwedge \Gamma. \tag{2}$$

We need to show that there exists u such that (2) holds for all of Ψ. Suppose for contradiction that there is no such u. Then, for every u with $w < u < v$ there is a $\psi_u \in \Psi$ with $u \not\models \psi_u$. Collect these formulas together into a finite set $\Gamma \subseteq_{\text{fin}} \Psi$ (M is finite!). For this Γ (2) fails – a contradiction.

The remaining clauses in Definition 4.2 may be established by similar arguments. □

We need the following form of saturation from first-order logic. Recall first that M_1 is an *elementary extension* of M_2 if $W_1 \supseteq W_2$ and for all $\mathcal{L}_1$-formulas $\alpha(x_1, \ldots, x_n)$ and all tuples $w_1, \ldots, w_n$ of M_2,

$$M_1 \models \alpha(x_1, \ldots, x_n)[w_1, \ldots, w_n] \text{ iff } M_2 \models \alpha(x_1, \ldots, x_n)[w_1, \ldots, w_n].$$

We write $M_2 \preceq M_1$ in this case.

Let κ be a cardinal number. A model M is *κ-saturated* in the sense of first-order logic if whenever Φ is a set of $\mathcal{L}'_1(x)$-formulas, where $\mathcal{L}'_1$ extends $\mathcal{L}_1$ by the addition of fewer than κ many individual constants, and Φ is finitely satisfiable in an $\mathcal{L}'_1$-expansion of M, then Φ itself is satisfiable in this expansion.

To show that M is t-saturated it suffices to show that M is 3-saturated. Below we will need the stronger assumption of ω-saturation.

PROPOSITION 4.5. Every ω-saturated model is t-saturated.

Proof. The proof is similar to the proof of Theorem 4.3. □

One can construe ω-saturated models as ultrapowers over a special kind of ultrafilters. We assume that the reader is familiar with the definition of ultraproducts and ultrapowers of models (consult Hodges, 1993, if necessary). An ultrafilter is called *ω-incomplete* if it is not closed under countable intersections. As a result, if U is an ω-incomplete ultrafilter and M is a model, then the ultrapower $\prod_U M$ is an ω-saturated elementary extension of M.

THEOREM 4.6. Assume that our language is countable. Let M_1, M_2 be two models, and let w_1, w_2 be elements of M_1, M_2, respectively. If $w_1 \equiv w_2$ then M_1 and M_2 have bisimilar ultrapowers.

Proof. The proof is similar to the proof of (de Rijke, 1995b: theorem 5.7). We confine ourselves to a sketch of the proof. Let I be an infinite index set; by

Chang and Keisler (1973: proposition 4.3.5) there is an ω-incomplete ultrafilter U over I. By our previous remarks the ultrapowers $\prod_U(M_1, w_1) =: (M_1', w_1')$ and $\prod_U(M_2, w_2) =: (M_2', w_2')$ are ω-saturated.

Observe that $tp_{M_1'}(w_1') = tp_{M_2'}(w_2') = tp_{M_1}(w_1)$. Hence, $M_1', w_1' \equiv M_2', w_2'$; as M_1', w_1' and M_2', w_2' are ω-saturated, it follows from Proposition 4.5 that $M_1', w_1' \leftrightarrows M_2, w_2'$, as required. □

Thus, temporal equivalence implies that there exist bisimilar ultrapowers. Hennessy–Milner classes can be characterized in terms of a stronger connection between temporal equivalence and bisimilar ultrapowers. We need two lemmas to arrive at this characterization.

LEMMA 4.7. Let I be an index set, and U an ultrafilter over I. Then

1. If for all $i \in I$, $M_i, w_i \leftrightarrows N_i, v_i$, then $\prod_U(M_i, w_i) \leftrightarrows \prod_U(N_i, v_i)$.
2. If $M, w \leftrightarrows N, v$, then $\prod_U(M, w) \leftrightarrows \prod_U(N, v)$.

Proof. We only prove the first item. For each $i \in I$, let $Z^{(i)}$ be a bisimulation linking M_i and N_i: $Z^{(i)} : M_i, w_i \leftrightarrows N_i, v_i$. Define a bisimulation Z between points of $\prod_U(M_i, w_i)$ and $\prod_U(N_i, v_i)$, and pairs of points of $\prod_U(M_i, w_i)$ and $\prod_U(N_i, v_i)$ in the obvious way by putting

$$x_1 Z_0 x_2 \text{ iff } \{i \in I \mid x_1(i) Z_0^{(i)} x_2(i)\} \in U;$$

$$x_1 y_1 Z_1 x_2 y_2 \text{ iff } \{i \in I \mid x_1(i) y_1(i) Z_1^{(i)} x_2(i) y_2(i)\} \in U;$$

$$x_2 y_2 Z_2 x_1 y_1 \text{ iff } \{i \in I \mid x_2(i) y_2(i) Z_2^{(i)} x_1(i) y_1(i)\} \in U.$$

Why is this is a bisimulation? First of all, it is clearly non-empty (take $x_1 : i \mapsto w_i$, and $x_2 : i \mapsto v_i$; then $x_1/U Z_0 x_2/U$). Next, if x in $\prod_U(M_i, w_i)$ has $x \models p$ and $x Z_0 y$, then, by the definition of ultraproducts $\{i \in I \mid x(i) \in V_i(p)\} \in U$. As $x Z_0 y$, this implies

$$X := \{i \in I \mid x(i) \in V_i(p) \text{ and } x(i) Z_0^{(i)} y(i)\} \in U.$$

As each $Z^{(i)}$ is a bisimulation it follows that $X \subseteq \{i \in I \mid y(i) \in V_i(p)\}$, hence the latter set is in U, from which we get $y \models p$, as required.

The remaining clauses may be proved by similar arguments. □

LEMMA 4.8. Let K be a Hennessy–Milner class, and M_1, $M_2 \in K$. Let w_1, w_2 be elements of M_1, M_2, respectively, such that $w_1 \equiv w_2$. Then $\prod_U(M_1, w_1) \leftrightarrows \prod_U(M_2, w_2)$ for all index sets I and ultrafilters U over I.

Proof. From $w_1 \equiv w_2$ and the definition of a Hennessy–Milner class it follows that $w_1 \leftrightarrows w_2$. Applying the second statement of Lemma 4.7 gives the result. □

COROLLARY 4.9. Let K be a class of models. Then K is a Hennessy–Milner class iff the following are equivalent for all models M_1, $M_2 \in \mathsf{K}$ and states $w_1 \in M_1$, $w_2 \in M_2$:

1. $M_1, w_1 \equiv M_2, w_2$, and
2. for all ultrafilters U the ultrapowers of $\prod_U(M_1, w_1)$ and $\prod_U(M_2, w_2)$ are bisimilar.

For the standard modal language with $\diamond$ and $\Box$, Hollenberg (1995) has characterized the *maximal* Hennessy–Milner classes in terms of submodels of canonical models. No such characterization has been obtained for Hennessy–Milner classes for the temporal language with Since and Until; in fact, it is not always clear whether canonical models for Since and Until form a Hennessy–Milner class. For example, the lack of a uniform definition of an accessibility relation in the completeness proofs for logics with Since and Until due to Burgess (1982) and Xu (1988) makes it hard to determine whether their Henkin-style models form a Hennessy–Milner class.

5. Applications to Temporal Model Theory

In this section we apply the tools developed in Sections 3 and 4 to arrive at model-theoretic results for temporal logic on preservation and definability. We give quick proofs of definability, separation, and interpolation theorems, as well as a preservation theorem characterizing the first-order translations of temporal formulas.

To smoothen the presentation of our results, we will be working with so-called *pointed models*; these are structures of the form (M, w), where w lives in the domain of M; w is called the *distinguished point* of (M, w). We will assume that a bisimulation between two pointed models links their distinguished points.

We will also be using the following operations on classes of models: $\mathbf{Pr}$, $\mathbf{Po}$, $\mathbf{B}$. Here $\mathbf{Pr}(\mathsf{K})$ is the class of ultraproducts of models in K; $\mathbf{Po}(\mathsf{K})$ is the class of ultrapowers of models in K; and $\mathbf{B}(\mathsf{K})$ is the class of all models that are bisimilar to a model in K.

LEMMA 5.1. Let K be a class of pointed models.

1. K is closed under bisimulations and ultraproducts iff $\mathsf{K} = \mathbf{BPr}(\mathsf{K})$,
2. K is closed under bisimulations and ultrapowers iff $\mathsf{K} = \mathbf{BPo}(\mathsf{K})$.

Proof. We only prove the first item, and to prove the first item it suffices to show that $\mathbf{PrB}(\mathsf{K}) \subseteq \mathbf{BPr}(\mathsf{K})$. So, assume $(M, w) \in \mathbf{PrB}(\mathsf{K})$. Then there are an index set I, models (M_i, w_i) and (N_i, v_i) $(i \in I)$ such that $(N_i, v_i) \in \mathsf{K}$, $(M_i, w_i) \leftrightarrows (N_i, v_i)$, and $(M, w) = \prod_U(M_i, w_i)$, for some ultrafilter U over I. Trivially, $\prod_U(N_i, v_i) \in \mathbf{Pr}(\mathsf{K})$. By Lemma 4.7, $(M, w) = \prod_U(M_i, w_i) \leftrightarrows$

$\prod_U(N_i, v_i)$. Hence, $(M, w) \in \mathbf{BPr}(\mathsf{K})$, as required. □

We will say that a class K of pointed models is *SU-definable*, or simply *definable*, by means of a set of temporal formulas if there exists a set of temporal formulas T such that $\mathsf{K} = \{(M, w) \mid (M, w) \models T\}$. A class of pointed models K is *definable by means of a single formula* if it is definable by means of a singleton set.

Let K be a class of pointed models; we use $\overline{\mathsf{K}}$ to denote the class of pointed models that are not in K.

THEOREM 5.2. Let K be a class of pointed models. Then

1. K is definable by means of a set of temporal formulas iff $\mathsf{K} = \mathbf{BPr}(\mathsf{K})$ and $\overline{\mathsf{K}} = \mathbf{Po}(\overline{\mathsf{K}})$,
2. K is definable by means of a single temporal formula iff $\mathsf{K} = \mathbf{BPr}(\mathsf{K})$ and $\overline{\mathsf{K}} = \mathbf{Pr}(\overline{\mathsf{K}})$.

Proof. 1. The *only if* direction is easy. For the converse, we can 'bisimulate' familiar arguments from first-order model theory. Assume K is closed under ultraproducts and bisimulations, while $\overline{\mathsf{K}}$ is closed under ultrapowers. Let $T = \bigcap\{tp_{(M,w)}(w) \mid (M, w) \in \mathsf{K}\}$.

We will show that T defines K. First, $\mathsf{K} \models T$. Second, assume that $(M, w) \models T$; we need to show $(M, w) \in \mathsf{K}$. Consider $tp_{(M,w)}(w)$, and define $I = \{\Sigma \subseteq tp_{(M,w)}(w) \mid |\Sigma| < \omega\}$. For each $i = \{\sigma_1, \ldots, \sigma_n\} \in I$ there is a model (M_i, w_i) of i in K. By standard model-theoretic arguments there exists an ultraproduct $\prod_U(M_i, w_i)$ which is a model of $tp_{(M,w)}(w)$; hence $\prod_U(M_i, w_i) \equiv (M, w)$. As $\mathbf{Pr}(\mathsf{K}) \subseteq \mathsf{K}$, $\prod_U(M_i, w_i) \in \mathsf{K}$. By Theorem 4.6 there is an ultrafilter U' such that

$$\prod_{U'}\left(\prod_U(M_i, w_i)\right) \;\underline{\leftrightarrow}\; \prod_{U'}(M, w).$$

Hence, the latter is in K, and, by the closure condition on $\overline{\mathsf{K}}$, this implies $(M, w) \in \mathsf{K}$, as required.

2. Again, the *only if* direction is easy. Assume K, $\overline{\mathsf{K}}$ satisfy the stated conditions. Then both are closed under ultrapowers, hence, by item 1, there are sets of temporal formulas T_1, T_2 defining K and $\overline{\mathsf{K}}$, respectively. Obviously, $T_1 \cup T_2 \models \bot$, so by compactness for some $\phi_1, \ldots, \phi_n \in T_1$, $\psi_1, \ldots, \psi_m \in T_2$, we have $\bigwedge_i \phi_i \models \bigvee_j \neg\psi_j$. Then K is defined by $\bigwedge_i \phi_i$. □

COROLLARY 5.3 (Separation). Let K, L be classes of pointed models such that $\mathsf{K} \cap \mathsf{L} = \emptyset$.

1. If K is closed under bisimulations and ultraproducts, and L is closed under bisimulations and ultrapowers, then there exists a class of models M that is

definable by means of a set of temporal formulas and such that $\mathsf{K} \subseteq \mathsf{M}$ and $\mathsf{L} \cap \mathsf{M} = \emptyset$.

2. If both K and L are closed under bisimulations and ultraproducts, then there exists a class of models M that is definable by means of a single temporal formula and such that $\mathsf{K} \subseteq \mathsf{M}$ and $\mathsf{L} \cap \mathsf{M} = \emptyset$.

Proof. We only prove the first item. Let K' be the class of all pointed models (M, w) such that for some $(N, v) \in \mathsf{K}$, $(M, w) \equiv (N, v)$. Then $\mathsf{K} \subseteq \mathsf{K}'$, and K' is closed under $\equiv$. Moreover, $\mathsf{K}' \cap \mathsf{L} = \emptyset$. For suppose $(M, w) \in \mathsf{K}' \cap \mathsf{L}$; then there exists $(N, v) \in \mathsf{K}$ such that $(N, v) \equiv (M, w)$. By Theorem 4.6 (N, v) and (M, w) have bisimilar ultrapowers $\prod_U (N, v)$ and $\prod_U (M, w)$. As K, L are closed under $\mathbf{B}$ and $\mathbf{Po}$, this implies $\prod_U (N, v) \in \mathsf{K} \cap \mathsf{L}$ – a contradiction.

To complete the proof, let $T = \bigcap \{tp_{(M,w)}(w) \mid (M, w) \in \mathsf{K}'\}$. Then T defines K'. As $\mathsf{K} \subseteq \mathsf{K}'$ and $\mathsf{K}' \cap \mathsf{L} = \emptyset$, we are done. □

Observe that Corollary 5.3, item 2 is a strong form of the Craig interpolation theorem.

To obtain a characterization of the first-order formulas that are equivalent to a temporal formula, we use the following notion. A first-order formula $\alpha(x)$ in $\mathcal{L}_1(x)$ is *invariant for bisimulations* iff for any two pointed models (M, w) and (N, v), any two states $w' \in M$ and $v' \in N$, and any bisimulation Z such that $w' Z v'$, we have that $M \models \alpha[w']$ iff $N \models \alpha[v']$.

COROLLARY 5.4 (Invariance). Let $\alpha(x)$ be an $\mathcal{L}_1(x)$-formula. Then the following are equivalent.

1. $\alpha(x)$ is invariant for bisimulations.
2. $\alpha(x)$ is equivalent to the standard translation of a temporal formula.

Proof. The implication from 2 to 1 is Lemma 3.8. For the converse implication, let $\alpha(x)$ be invariant for bisimulations. Let K be the class of (pointed) models of $\alpha(x)$. Then K and $\overline{\mathsf{K}}$ (being defined by $\neg\alpha(x)$) are closed under ultraproducts. As $\alpha(x)$ is invariant for bisimulations, both K and $\overline{\mathsf{K}}$ must also be closed under bisimulations. Hence, by Theorem 5.2, K must be definable by a single temporal formula ϕ. This means that $\alpha(x)$ is (equivalent to) the standard translation of ϕ. □

6. Concluding Remarks

In this paper we have introduced a notion of bisimulation for temporal logic with Since and Until that allows one to develop the basic model theory for temporal logic. We established a preservation result that characterizes the first-order formulas that correspond to temporal formulas with Since and Until, thereby answering Open

Problem 4.4 from van Benthem et al. (1994). In addition, we proved definability and interpolation results.

A lot remains to be done. First of all, we believe that our notion of bisimulation may be a useful tool in obtaining further results in the model theory of Since and Until. In particular, Kamp's famous result of the expressive completeness of Since and Until over dedekind-complete linear order is an important one, for which multiple proofs should be available. One of the most recent proofs, due to Hodkinson (1995) uses games that seem to be quite close to our notion of bisimulation; it therefore seems feasible to try and prove Kamp's theorem using our bisimulations.

Next, we think that our general methodology of involving more complex patterns of states in the definition of bisimulation for Since and Until also indicates the way to go when attempting to define suitable bisimulations for other complex modal operators whose truth definition involves both universal and existential quantification. In particular, our ideas seem applicable to the *minimality operator* min whose semantics is given by

$$w \models \text{min}(\phi) \text{ iff } \exists y\,(w < y \wedge y \models \phi \wedge \forall z\,(w < z < y \rightarrow z \not\models \phi)).$$

Obviously the min-operator is definable using Since and Until, and as a result we have that states that are bisimilar in our sense agree on formulas involving the min-operator – but what about a notion of bisimulation that exactly characterizes the fragment involving min in the sense of Corollary 5.4? Further examples along these lines could include the temporal operators found in Manna and Pnueli (1992). But more exotic modal operators might also be analyzed using our strategy. A suitable test case would be the binary *interpretability* operator $\triangleright$ whose truth definition is based on a binary relation R and a ternary relation S as follows:

$$w \models \phi \triangleright \psi \text{ iff } \exists y\,(Rwy \wedge y \models \phi \wedge \forall z\,(Swyz \rightarrow z \models \psi)).$$

See Berarducci (1990) for further details on this operator.

In our comparisons in this paper we focused on equivalence relations between models that were defined by fairly simple first-order conditions. De Nicola and Vaandrager (1995) study so-called *branching bisimulations* whose definition involves non first-order definable concepts like 'finitely many silent steps'; they show that on certain transition systems branching bisimulations and several temporal logics induce the same equivalence relations. The exact connection hasn't been determined, though, and to obtain a precise description of the connections one needs other tools than the ones we have used in this paper as these are essentially first-order.

An interesting further point raised by one of the referees is to determine the relation between bisimulation, temporal equivalence and the notion of a Hennessy–Milner class on restricted classes of models, especially on the various classes of linear orders which are most commonly seen in temporal logic.

Finally, in this paper we have given the first notion of bisimulation that allowed for an exact characterization theorem in the sense of Corollary 5.4 of modal operators whose truth definition is not of the simple $\exists \cdots \exists \alpha$ or $\forall \cdots \forall \alpha$ format (for α quantifier-free). Do our ideas of introducing bisimulations that link states to states and sequences to sequences generalize to the extent that we can handle any first-order definable modal operator, no matter how complex its truth definition is? Recent work by Andréka et al. (1995) and by Hollenberg (1996) is relevant here.

Acknowledgments

Part of the research was carried out while the authors were with CWI, Amsterdam; during this period they were supported by the Netherlands Organization for Scientific Research (NWO), project NF 102/62-356.

Maarten de Rijke was partially supported by a grant from the Research and Teaching Innovation Fund at the University of Warwick.

We would like to thank Holger Sturm, Yde Venema, and two anonymous referees for extremely helpful comments on an earlier version of this paper. In particular, Figure 3 is due to Holger Sturm, and it inspired us to weaken the definition of bisimulation to its current version; we are very grateful for his permission to use it.

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