Completeness Results for Two-sorted Metric Temporal Logics*

Angelo Montanari[1] and Maarten de Rijke[2]

[1] Dipartimento di Matematica e Informatica, Università di Udine, Via delle Scienze, 206, 33100 Udine, Italy. E-mail: montana@dimi.uniud.it
[2] Department of Software Engineering, CWI, P.O. Box 94079, 1090 GB Amsterdam, the Netherlands. E-mail: mdr@cwi.nl

Abstract. Temporal logic has been successfully used for modeling and analyzing the behavior of reactive and concurrent systems. One shortcoming of (standard) temporal logic is that it is inadequate for real-time applications, because it only deals with qualitative timing properties. This is overcome by metric temporal logics which offer a uniform logical framework in which both qualitative and quantitative timing properties can be expressed by making use of a parameterized operator of relative temporal realization. We view metric temporal logics as two-sorted formalisms having formulae ranging over time instants and parameters ranging over an (ordered) abelian group of temporal displacements.
In this paper we deal with completeness results for basic systems of metric temporal logic — such issues have largely been ignored in the literature. We first provide an axiomatization of the pure metric fragment of the logic, and prove its soundness and completeness. Then, we show how to obtain the metric temporal logic of linear orders by adding an ordering over displacements.

1 Introduction

Logic-based methods for representing and reasoning about temporal information have proved to be highly beneficial in the area of formal specifications. In this paper we consider their application to the specification of real-time systems. Timing properties play a major role in the specification of reactive and concurrent software systems that operate in real-time. They constrain the interactions between different components of the system as well as between the system and its environment, and minor changes in the precise timing of interactions may lead to radically different behaviors. Temporal logic has been successfully used for modeling and analyzing the behavior of reactive and concurrent systems (see Manna and Pnueli [8] and Ostroff [11]). It supports semantic model checking, in

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order to verify consistency of specifications, and to check positive and negative examples of system behavior against specifications; it also supports pure syntactic deduction, in order to prove properties of systems. Unfortunately, most common representation languages in the area of formal specifications are inadequate for real-time applications, because they lack an explicit and quantitative representation of time. In recent years, some of them have been extended to cope with real-time aspects. In this paper, we focus on *metric temporal logics* which provide a uniform framework in which both qualitative and quantitative timing properties of real-time systems can be expressed.

The idea of a logic of positions (topological, or metric, logic) has originally been formulated by Rescher and Garson [12]. They defined the basic features of the logic, and showed how to give it a temporal interpretation. The logic of positions extends propositional logic with a parametrized operator P_α of positional realization. Such an operator allows one to constrain the truth value of a proposition at position α. The parameter α denotes either (i) an absolute position or (ii) a displacement with respect to the current position which is left implicit. According to interpretation (ii), $P_\alpha p$ is true at the position i if and only if p is true at a position j at distance α from i. In [12], Rescher and Garson introduced two axiomatizations of the logic of positions that differ from each other in the interpretation of parameters. Later, Rescher and Urquhart [13] proved the soundness and completeness of the axiomatization based on an absolute interpretation of parameters through a reduction to monadic quantification theory. A metric temporal logic has been independently developed by Koymans [7] to support the specification and verification of real-time systems. He extended the standard model for temporal logic based on point structures with a distance function that measures, for any pair of time points, how far they are apart in time. He provided the logic with a sound axiomatization, but no proof of completeness is given.

The main issues to confront in developing a metric temporal logic for executable specifications are:

Expressiveness (definability). Is the metric temporal logic powerful enough to express both the properties of the underlying temporal structure and the timing requirements of the specified real-time systems?

Soundness and completeness. Is the metric temporal logic provided with a sound and complete axiomatization?

Decidability. Which properties of the specified real-time system can be automatically verified? Most temporal logics for real-time systems proposed in the literature cannot be decided (see Henzinger [6]). Some of them recover decidability sacrificing completeness.

Executability. How can we prove the consistency and adequacy of specifications? In principle, decidability proof methods (e.g. via Büchi automata) outline an effective procedure to prove the satisfiability and/or validity of a formula. But as soon as certain assumptions about the nature of the temporal domain and the available set of primitive operations are relaxed, the satisfiability/validity problem becomes undecidable (Alur and Henzinger [1]).
An alternative approach consists in looking at metric temporal logics as

particular polymodal logics and supporting derivability by means of proof procedures for nonclassical logics or via translation in first-order theories (see D'Agostino et al [4], and Ohlbach [10]). In this case, providing the logic with a sound and complete axiomatization becomes a central issue.

The aim of this paper is to explore completeness issues of metric temporal logic; we do this by starting with a very basic system, and build on it either by adding axioms or by enriching the underlying structures. We view metric temporal logics as two-sorted logics having both formulae and parameters; formulae are evaluated at time instants while parameters take values in an (ordered) abelian group of temporal displacements. In Section 2, we define a minimal metric logic that can be seen as the metric counterpart of minimal tense logic, and we provide it with a sound and complete axiomatization. In Section 3, we characterize the class of two-sorted frames with a linearly ordered temporal domain.

2 The basic metric logic

In this section we define the minimal metric temporal logic MTL_0, and consider some of its natural extensions.

Language. We define a two-sorted temporal language for our basic calculus MTL_0. First, its algebraic part is built up from a non-empty set of *variables* X. The set of terms over X, $T(X)$, is the smallest set such that (1) $X \subseteq T(X)$, and (2) if α, $\beta \in T(X)$ then $(\alpha + \beta)$, $(-\alpha), 0 \in T(X)$. Next, the temporal part of the language is built from a non-empty set Φ of *proposition letters*. The set of MTL_0-formulae over Φ and X, $F(\Phi, X)$, is the smallest set such that (1) $\Phi \subseteq F(\Phi, X)$, and (2) if ϕ, $\psi \in F(\Phi, X)$ and $\alpha \in T(X)$, then $\neg\phi$, $\phi \wedge \psi$, $\Delta_\alpha \phi$ (and its dual $\nabla_\alpha \phi := \neg\Delta_\alpha\neg\phi$), $\perp \in F(\Phi, X)$. We will adopt the following notational conventions: p, q, ... denote proposition letters; ϕ, ψ, ... denote MTL_0-formulae; Σ, Γ, ... denote sets of MTL_0-formulae; $\alpha, \beta, \ldots$ denote algebraic terms.

Structures. We define a *two-sorted frame* to be a triple $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$, where T is the set of (time) points over which temporal formulae are evaluated, $\mathfrak{D}$ is the algebra of metric displacements in whose domain D terms take their values, and $\mathrm{DIS} \subseteq T \times D \times T$ is an accessibility relation relating pairs of points and displacements.

We require the following properties to hold for the components of two-sorted frames. First, we require $\mathfrak{D}$ to be an abelian group, that is, a 4-tuple $(D, +, -, 0)$ where $+$ is a binary function of ***displacement composition***, $-$ is a unary function of ***inverse displacement***, and 0 is the ***zero displacement*** constant, such that:

(i)	$\alpha + \beta = \beta + \alpha$	(commutativity of $+$)
(ii)	$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$	(associativity of $+$)
(iii)	$\alpha + 0 = \alpha$	(zero element of $+$)
(iv)	$\alpha + (-\alpha) = 0$	(inverse)

Second, we require the displacement relation DIS to respect the converse operation of the abelian group in the following sense: if $\mathrm{DIS}(i, \alpha, j)$ then $\mathrm{DIS}(j, -\alpha, i)$.

We turn a two-sorted frame $\mathfrak{F}$ into a *two-sorted model* by adding an interpretation for our algebraic terms, and a valuation for atomic temporal formulae. An *interpretation* for algebraic terms is given by a function $g : X \to D$ that is automatically extended to all terms from $T(X)$. A *valuation* is simply a function $V : \Phi \to 2^T$. Then, we say that an equation $\alpha = \beta$ is *true* in a model $\mathfrak{M} = (T, \mathfrak{D}; \mathrm{DIS}; g, V)$ whenever $g(\alpha) = g(\beta)$. Next, *truth* of temporal formulae is defined by

$$
\begin{aligned}
\mathfrak{M}, i \Vdash p &\text{ iff } i \in V(p)\\
\mathfrak{M}, i \Vdash \bot &\quad \text{never}\\
\mathfrak{M}, i \Vdash \neg\phi &\text{ iff } \mathfrak{M}, i \nVdash \phi\\
\mathfrak{M}, i \Vdash \phi \wedge \psi &\text{ iff } \mathfrak{M}, i \Vdash \phi \text{ and } \mathfrak{M}, i \Vdash \psi\\
\mathfrak{M}, i \Vdash \Delta_\alpha \phi &\text{ iff there exists } j \text{ such that } \mathrm{DIS}(i, g(\alpha), j) \text{ and } \mathfrak{M}, j \Vdash \phi.
\end{aligned}
$$

To avoid messy complications we only consider one-sorted consequences $\Gamma \models \phi$; for algebraic formulae '$\Gamma \models \phi$' means 'for all two-sorted models $\mathfrak{M}$, if $\mathfrak{M} \models \Gamma$, then $\mathfrak{M} \models \phi$'; for temporal formulae it means 'for all models $\mathfrak{M}$, and times instants i, if $\mathfrak{M}, i \Vdash \Gamma$, then $\mathfrak{M}, i \Vdash \phi$'.

A simple example. Even though the language of MTL_0 is very poor, it already allows us to express conditions on real-time systems. As a first example, consider a communication channel C that outputs each message with a delay δ with respect to its input time, and that neither generates nor loses messages (cf. Montanari et al [9]). C can be specified as follows:

$$out \leftrightarrow \Delta_{-\delta} in$$

This example can easily be generalized to the case of a channel C that collects messages from n different sources $S_1, \ldots, S_n$ and outputs them with a delay δ. To exclude that two input events can occur simultaneously, we add the constraint:

$$\forall i, j\, \neg(in(i) \wedge in(j) \wedge i \neq j),$$

which is shorthand for

$$\neg(in(1) \wedge in(2)) \wedge \ldots \wedge \neg(in(n-1) \wedge in(n)).$$

Then the behavior of C is specified by the formula

$$\forall i\, (out(i) \leftrightarrow \Delta_{-\delta} in(i)),$$

which is shorthand for a finite conjunction.

Notice that preventing input events from occurring simultaneously also guarantees that output events do not occur simultaneously.

Suppose now that C outputs the messages it receives from $S_1, \ldots S_n$ with a (generally different) delay $\delta_1, \ldots, \delta_n$, respectively. Constraining input events not to occur simultaneously no longer guarantees that there are no conflicts at output time. A simple strategy of conflict resolution consists in assigning a different priority to messages coming from different knowledge sources, so that, when a conflict occurs, C only outputs the message with highest priority. Accordingly, the specification of C is modified, preserving the requirement that it does not generate messages, but relaxing the requirement that it does not lose messages.

Assume that $S_1, \ldots, S_n$ are listed in decreasing order of priority. The behavior of C can be specified as follows:

$$\forall i\,(out(i) \leftrightarrow (\Delta_{-\delta_i} in(i) \wedge \neg\exists j\,(\Delta_{-\delta_j} in(j) \wedge j < i))))$$

which is a shorthand for

$$(out(1) \leftrightarrow \Delta_{-\delta_1} in(1)) \wedge (out(2) \leftrightarrow (\Delta_{-\delta_2} in(2) \wedge \neg\Delta_{-\delta_1} in(1))) \wedge \ldots \wedge$$

$$(out(n) \leftrightarrow (\Delta_{-\delta_n} in(n) \wedge (\neg\Delta_{-\delta_1} in(1) \wedge \ldots \wedge \neg\Delta_{-\delta_{n-1}} in(n-1))).$$

More realistic examples are given in the full paper.

Axioms. Our basic calculus MTL_0 has two components. On the one hand it has the usual laws of algebraic logic to deal with the displacements:

(Ref)	$\vdash \alpha = \alpha$	for all terms α (reflexivity)
(Sym)	$\vdash \alpha = \beta \implies \vdash \beta = \alpha$	(symmetry)
(Tra)	$\vdash \delta = \alpha,\ \alpha = \beta \implies \vdash \delta = \beta$	(transitivity)
(Rep)	$\vdash \alpha = \beta \implies \vdash [\alpha/x]\delta = [\beta/x]\delta$	(replacement)
(Sub)	$\vdash \alpha = \beta \implies \vdash [\delta/x]\alpha = [\delta/x]\beta$	(substitution),

as well as the above axioms (i)–(iv) for abelian groups. Here $[\alpha/x]\beta$ denotes the result of substituting α for all occurrences of x in β.

The second component of MTL_0 governs the temporal aspect of our structures; its axioms are the usual axioms of propositional logic plus

(Ax1)	$\nabla_\alpha(p \to q) \to (\nabla_\alpha p \to \nabla_\alpha q)$	(normality of ∇_α)
(Ax2)	$p \to \nabla_\alpha \Delta_{-\alpha} p,$	

and its rules are modus ponens and

(NEC)	$\vdash \phi \implies \vdash \nabla_\alpha \phi$	(necessitation rule for ∇_α)
(SUB)	$\vdash \phi \leftrightarrow \psi \implies \vdash \chi(\phi/p) \leftrightarrow \chi(\psi/p)$	(uniform substitution)
	where (ϕ/p) denotes substitution of ϕ for the variable p	
(LIFT)	$\vdash \alpha = \beta \implies \vdash \nabla_\alpha \phi \leftrightarrow \nabla_\beta \phi$	(transfer of identities).

Axiom (Ax1) is the usual distribution axiom; axiom (Ax2) expresses that a displacement α is the converse of a displacement $-\alpha$. The rules (NEC) and (SUB) are familiar from modal logic, and the rule (LIFT) allows us to transfer provable algebraic identities from the displacement domain to the temporal domain.

A *derivation in* MTL_0 is a sequence of terms and/or formulae $\sigma_1, \ldots, \sigma_n$ such that each σ_i $(1 \le i \le n)$ is either an axiom, or obtained from $\sigma_1, \ldots, \sigma_{n-1}$

by applying one of the derivation rules of MTL_0. We write $\vdash_{MTL_0} \sigma$ to denote that there is a derivation in MTL_0 that ends in σ.It is an immediate consequence of this definition that $\vdash_{MTL_0} \alpha = \beta$ iff $\alpha = \beta$ is provable (in algebraic logic) from the axioms of abelian groups only: whereas we can lift algebraic information from the displacement domain to the temporal domain using the (LIFT) rule, there is no way in which we can import temporal information into the displacement domain. As with consequences, we only consider one-sorted inferences '$\Gamma \vdash \phi$'.

Completeness. In this subsection we prove completeness for the basic calculus MTL_0. Our strategy will be to construct a canonical-like model by taking the free abelian group over our algebraic variables as the displacement component, by taking the familiar canonical model as the temporal component, and by linking the two in a suitable way.

The displacement domain. Recall that $T(X)$ is the collection of all algebraic terms built up from the variables in the set X. Define a congruence relation θ on $T(X)$ by taking

$$(\alpha, \beta) \in \theta \;\text{ iff }\; \vdash_{MTL_0} \alpha = \beta.$$

Then the *canonical displacement domain* $\mathfrak{D}^0$ is constructed by taking

$$\begin{aligned} D^0 &= T(X)/\theta \\ \alpha/\theta + \beta/\theta &= (\alpha+\beta)/\theta \\ -\alpha/\theta &= (-\alpha)/\theta \\ 0 &= 0/\theta. \end{aligned}$$

That $\mathfrak{D}^0$ is indeed an abelian group is easily shown using the defining axioms and rules of MTL_0. The group $\mathfrak{D}^0$ is known as the *free abelian group over* X (cf. Burris and Sankappanavar [3]).

We interpret our terms using the canonical mapping $g : T(X) \to \mathfrak{D}^0$ defined by $\alpha \mapsto \alpha/\theta$.

The temporal domain. A set of MTL_0-formulae is *maximal MTL_0-consistent* (or: an MCS) if it is MTL_0-consistent and it does not have proper MTL_0-consistent extensions. The *canonical temporal domain* T^0 is constructed by taking

$$T^0 = \{\Sigma \mid \Sigma \text{ is maximal } MTL_0\text{-consistent }\}.$$

Define a *canonical valuation* V^0 by putting $V^0(p) = \{\Sigma \mid p \in \Sigma\}$.

The canonical model for MTL_0. We almost have all the ingredients to define a canonical model for MTL_0; we only need to define a displacement relation $\mathrm{DIS}^0 \subseteq T^0 \times D^0 \times T^0$. This is done as follows:

$\mathrm{DIS}^0(\Sigma, \alpha/\theta, \Gamma)$ iff for every formula γ, $\gamma \in \Gamma$ implies $\Delta_\alpha\gamma \in \Sigma$
(equivalently: for all formulae σ, if $\nabla_\alpha\sigma \in \Sigma$ then $\sigma \in \Gamma$).

Note that if $(\alpha, \beta) \in \theta$, then $\vdash \alpha = \beta$, hence $\vdash \nabla_\alpha \phi \leftrightarrow \nabla_\beta \phi$ by the (LIFT) rule, for all formulae ϕ. From this one easily derives that the definition of DIS^0 does not depend on the representative we take for α/θ.

Also, $\mathrm{DIS}^0(\Sigma, \alpha/\theta, \Gamma)$ implies $\mathrm{DIS}^0(\Gamma, -\alpha/\theta, \Sigma)$: if $\mathrm{DIS}^0(\Sigma, \alpha/\theta, \Gamma)$ and $\sigma \in \Sigma$, then $\nabla_\alpha \Delta_{-\alpha} \sigma \in \Sigma$ by axiom (Ax2), hence $\Delta_{-\alpha}\sigma \in \Gamma$.

Then, the *canonical model* for MTL_0 is the model $\mathfrak{M}^0 = (T^0, \mathfrak{D}^0; \mathrm{DIS}^0; g, V^0)$.

Theorem 1. *MTL_0 is sound and complete for the class of all MTL_0-frames.*

Proof. Proving soundness is left to the reader. To prove completeness we show that every consistent set of MTL_0-formulae is satisfiable in a model based on a two-sorted frame.

Let Σ be a MTL_0-consistent set of formulae; by standard techniques we can extend it to a maximal MTL_0-consistent set Σ^+ that lives somewhere in the canonical model $\mathfrak{M}^0$ for MTL_0. To complete the proof of the theorem it suffices to establish the following Truth Lemma. For all MTL_0-formulae ϕ and all $\Sigma \in T^0$:

$$\phi \in \Sigma \text{ iff } \mathfrak{M}^0, \Sigma \Vdash \phi.$$

This can be done using standard arguments from modal logic. $\dashv$

Imposing additional constraints. For many purposes two-sorted frames as we have studied them so far are too simple. In particular, they don't satisfy all the natural conditions one may want to impose on the displacement relation. Examples of such properties that arise in application areas such as real-time system specification include

Transitivity:	$\forall i, j, k, \alpha, \beta (\mathrm{DIS}(i, \alpha, j) \wedge \mathrm{DIS}(j, \beta, k) \to \mathrm{DIS}(i, \alpha + \beta, k))$
Quasi-functionality:	$\forall i, j, j', \alpha\, (\mathrm{DIS}(i, \alpha, j) \wedge \mathrm{DIS}(i, \alpha, j') \to j = j')$
Reflexivity:	$\forall i\, \mathrm{DIS}(i, 0, i)$
Antisymmetry:	$\forall i, j, \alpha\, (\mathrm{DIS}(i, \alpha, j) \wedge \mathrm{DIS}(j, \alpha, i) \to i = j \wedge \alpha = 0)$.

As in standard modal and temporal logic only some of the natural properties we want to impose on structures are expressible. In particular, the first three of the above properties are expressible in metric temporal logic, as follows (see Montanari et al [9]):

(Ax3)	$\nabla_{\alpha+\beta} p \to \nabla_\alpha \nabla_\beta p$	(transitivity)
(Ax4)	$\Delta_\alpha p \to \nabla_\alpha p$	(quasi-functionality w.r.t. the 3rd argument)
(Ax5)	$\nabla_0 p \to p$	(reflexivity)

In the case of Transitivity, Quasi-functionality, and Reflexivity we are able to extend the basic completeness result fairly effortlessly because each of the corresponding temporal formulae is a Sahlqvist formula. And the important feature of Sahlqvist formulae is that they are *canonical* in the sense that they are validated by the frame underlying the canonical model defined in the proof of Theorem 1 (see Goldblatt [5] for analogous arguments in standard modal and temporal logic, or De Rijke and Venema [15] for the general picture). As a consequence we have the following:

Theorem 2. *Let $X \subseteq \{\text{Ax3}, \text{Ax4}, \text{Ax5}\}$. Then MTL_0X is complete with respect to the class of frames satisfying the properties expressed by the axioms in X.*

Further natural properties like Euclidicity $(\forall i, j, k, \alpha, \beta((\text{DIS}(i, \alpha, j) \wedge \text{DIS}(i, \alpha + \beta, k)) \rightarrow \text{DIS}(j, \beta, k)))$ can already be derived from $MTL_0\text{Ax3}$.

In the case of Antisymmetry, we have to do more work. First of all, Antisymmetry is not expressible in the basic metric language. One can use a standard unfolding argument to prove this claim (as in ordinary modal logic). Despite the undefinability of Antisymmetry, we can prove a completeness result for the class of antisymmetric two-sorted frames. Using a technique which is based on Burgess' chronicle construction (see Burgess [2]) it is indeed possible to prove the following theorem.

Theorem 3. *MTL_0 is complete with respect to the class of all antisymmetric two-sorted frames.*

3 Two-sorted frames based on ordered groups

For a variety of application purposes, our basic calculus and its semantics need to be extended with orderings. In particular, a linear order on the temporal domain is needed in many application areas; for instance, in real-time specification we want to guarantee that between any two time instants there is a unique displacement. In the following, we achieve this by adding a total ordering on the displacement domain D.

In the definition of a two-sorted frame we replace the abelian component by an *ordered* abelian group. That is, by a structure $\mathfrak{D} = (D, +, -, 0, <)$, where $(D, +, -, 0)$ is an abelian group, and $<$ is an irreflexive, asymmetric, transitive and linear relation that satisfies the comparability property (*viii*) below:

(*v*)	$\neg(\alpha < \alpha)$
(*vi*)	$\neg(\alpha < \beta \wedge \beta < \alpha)$
(*vii*)	$\alpha < \beta \wedge \beta < \gamma \rightarrow \alpha < \gamma$
(*viii*)	$\alpha < \beta \vee \alpha = \beta \vee \beta < \alpha.$

Next, there are two axioms expressing the relation between $+$ and $-$, and $<$:

(*ix*)	$\alpha < \beta \rightarrow \alpha + \gamma < \beta + \gamma$
(*x*)	$\alpha < \beta \rightarrow -\beta < -\alpha.$

One can use various languages to talk about ordered abelian groups. We do not have any clear preference, as long as the language used can be equipped with a complete axiomatization. We will simply use full first-order logic over $=$, $<$ to reason about the ordered abelian component of our two-sorted frames.

To be precise, our metric temporal language for talking about two-sorted frames based on an ordered abelian group, has a first-order component built up from terms in $T(X)$ and predicates $=$ and $<$; its temporal component is as before.

The interpretation of this language on two-sorted frames based on an ordered abelian group is fairly straightforward: the first-order component is interpreted on the group, and the temporal component on the temporal domain. Validity in this language is easily axiomatized; for the displacement component we take the axioms and rules of identity, ordered abelian groups, strict linear order together with any complete calculus for first-order logic; and for the temporal component we take the same axioms as in the case of MTL_0: axioms (Ax1), (Ax2) and the rules modus ponens, (NEC), (SUB) and (LIFT). Let MTL_1 denote the resulting two-sorted calculus.

Theorem 4. *MTL_1 is complete with respect to the class of two-sorted frames based on ordered abelian groups.*

Proof. We can simply repeat the proof of Theorem 1 here, and replace the free algebra construction of the displacement domain by a Henkin construction for first-order logic. $\dashv$

3.1 Deriving a temporal ordering

Given that we have an ordering $<$ on the algebraic component of our frames, a natural definition for an ordering $\ll$ on the temporal frame suggests itself:

$$i \ll j \text{ iff for some } \alpha > 0, \mathrm{DIS}(i, \alpha, j). \tag{1}$$

So i and j are $\ll$-related if there exists a positive displacement between them. Using the relation $\ll$, we can define the qualitative operators F, P of non-metric temporal logic as follows:

$$\mathfrak{M}, i \Vdash F\phi := \exists j\,(i \ll j \wedge j \Vdash \phi) \text{ and } \mathfrak{M}, i \Vdash P\phi := \exists j\,(j \ll i \wedge j \Vdash \phi).$$

However, we will not consider this extension in this abstract.

Additional properties. The definition of $\ll$ given in (1) does not produce a temporal ordering with all the natural properties that we usually expect it to have. In particular, unless we put further restrictions on the relation of temporal displacement, $\ll$ will not be a strict linear order, and there may be time instants without a unique temporal distance between them.

To repair this situation, we assume that the displacement relation DIS satisfies the following properties: transitivity, quasi-functionality, reflexivity (as defined in Section 2), and total connectedness and quasi-functionality w.r.t. the second argument:

(xi) $\forall i, j \exists \alpha\, \mathrm{DIS}(i, \alpha, j)$ (total connectedness)

(xii) $\forall i, j, \alpha, \beta\, (\mathrm{DIS}(i, \alpha, j) \wedge \mathrm{DIS}(i, \beta, j) \rightarrow \alpha = \beta)$
(quasi-functionality w.r.t. the 2nd argument).

Given these assumptions on the displacement relation, we can show that the temporal relation $\ll$ as defined in (1) is a strict linear order. To see that $\ll$ is transitive, assume that $i \ll j \ll k$. Then there exist α, β with $\mathrm{DIS}(i, \alpha, j)$ and $\mathrm{DIS}(j, \beta, k)$. Hence $\mathrm{DIS}(i, \alpha + \beta, k)$ and $i \ll k$.

For irreflexivity, assume $i \ll i$. Then $\mathrm{DIS}(i, \alpha, i)$ for some $\alpha > 0$. By reflexivity of DIS, $\mathrm{DIS}(i, 0, i)$, hence, by quasi-functionality of the second argument, $\alpha = 0$ — a contradiction.

For asymmetry, assume $i \ll j \ll i$. Then $\mathrm{DIS}(i, \alpha, j)$ and $\mathrm{DIS}(j, \beta, i)$ for some $\alpha, \beta > 0$. Then $\mathrm{DIS}(j, -\alpha, i)$ and so $\beta = -\alpha$, by quasi-functionality of the second argument again, which yields a contradiction.

Finally, to prove totality, take any two i, j. By total connectedness there exists α such that $\mathrm{DIS}(i, \alpha, j)$. By axiom $(viii)$, $\alpha > 0 \vee \alpha = 0 \vee 0 > \alpha$. If $\alpha > 0$, then $i \ll j$. If $\alpha = 0$, then by quasi-functionality and reflexivity of DIS, $i = j$. And if $\alpha < 0$, then $-\alpha > 0$ and $\mathrm{DIS}(j, -\alpha, i)$, so $j \ll i$.

Let us call a two-sorted frame *nice* if it is transitive, reflexive, totally-connected, and quasi-functional in both the 2nd and 3rd argument of its displacement relation; a model is *nice* if it is based on an nice frame.

The next obvious question is: can we characterize the nice frames in the language of MTL_1? The answer is 'no'. To see this, we quickly adapt two truth preserving constructions from standard modal logic to the present setting. Due to space limitations we confine ourselves to frames that share the same displacement domain; however, the definitions are easily generalized to the general case.

Definition 5. Let $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ and $\mathfrak{F}' = (T', \mathfrak{D}; \mathrm{DIS}')$ be two-sorted frames. The *disjoint union* of $\mathfrak{F}$ and $\mathfrak{F}'$ is the two-sorted frame $\mathfrak{F} \uplus \mathfrak{F}' = (T'', \mathfrak{D}, \mathrm{DIS}'')$. Here, T'' is the disjoint union of T and T', while the displacement relation DIS'' is just the disjoint union of DIS and DIS'.

Theorem 6. *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be two-sorted frames, and $\mathfrak{F} \uplus \mathfrak{F}'$ their disjoint union. For all algebraic terms α, β, if $\mathfrak{F} \models \alpha = \beta$ and $\mathfrak{F}' \models \alpha = \beta$, then $\mathfrak{F} \uplus \mathfrak{F}' \models \alpha = \beta$. And, for all formulae ϕ, if $\mathfrak{F} \models \phi$ and $\mathfrak{F}' \models \phi$, then $\mathfrak{F} \uplus \mathfrak{F}' \models \phi$.*

Theorem 7. *There is no modal formula ϕ that expresses total connectedness of two-sorted frames.*

Proof. We prove the claim by showing that the existence of such a formula would violate preservation of truth under disjoint union. An intuitive account of this negative conclusion can be given noticing that disjoint unions are not totally connected frames "by definition".

Suppose that there exists a formula ϕ expressing total connectedness. By Theorem 6, it follows that ϕ is valid in the disjoint union $\mathfrak{F} \uplus \mathfrak{F}' = (T'', \mathfrak{D}; \mathrm{DIS}'')$ of any two frames $\mathfrak{F}$ and $\mathfrak{F}'$ validating ϕ. Take $i \in \mathfrak{F}$ and $j \in \mathfrak{F}'$; by definition of $\mathfrak{F} \uplus \mathfrak{F}'$, it follows that there exists no $\alpha \in \mathfrak{D}$ such that $\mathrm{DIS}''(i, \alpha, j)$. $\dashv$

Definition 8. Let $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ and $\mathfrak{F}' = (T', \mathfrak{D}; \mathrm{DIS}')$ be two-sorted frames. A *bounded morphism* from $\mathfrak{F}$ to $\mathfrak{F}'$ is a mapping $f : T \to T'$ such that:

1. if $\mathrm{DIS}(i,\alpha,j)$, then $\mathrm{DIS}'(f(i),\alpha,f(j))$;
2. if $\mathrm{DIS}'(f(i),\alpha,j')$, then for some $j \in T$ both $f(j)=j'$ and $\mathrm{DIS}(i,\alpha,j)$ hold.

Theorem 9. *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be two-sorted frames, and f a surjective bounded morphism from $\mathfrak{F}$ to $\mathfrak{F}'$. For all algebraic terms α, β, if $\mathfrak{F} \models \alpha = \beta$, then $\mathfrak{F}' \models \alpha = \beta$. And, for all formulae ϕ, if $\mathfrak{F} \models \phi$, then $\mathfrak{F}' \models \phi$.*

Theorem 10. *There is no modal formula ϕ that expresses quasi-functionality w.r.t. the second argument of the displacement relation.*

Proof. We prove the claim by showing that the existence of such a formula would violate preservation of truth under bounded morphisms. Suppose that there exists a formula ϕ expressing quasi-functionality with respect to the second argument of the accessibility relation.

Consider the two-sorted frames $\mathfrak{F} = (T,\mathfrak{D};\mathrm{DIS})$ and $\mathfrak{F}' = (T',\mathfrak{D};\mathrm{DIS}')$ such that $T = \{i_1,i_2,i_3,i_4,j_1,j_2,j_3,j_4\}$, $T' = \{i',j'\}$, DIS contains $(i_1,1,j_1)$, $(i_1,2,j_3)$, $(i_2,2,j_1)$, $(i_2,1,j_3)$, $(i_3,1,j_2)$, $(i_3,2,j_4)$, $(i_4,1,j_4)$, and $(i_4,2,j_2)$, together with the converse triplets $(j_1,-1,i_1)$, $(j_3,-2,i_1)$, and so on, while $\mathrm{DIS}' = \{(i',1,j'), (i',2,j'), (j',-2,i'), (j',-1,i')\}$. Clearly, $\mathfrak{F}$ satisfies the requirement of quasi-functionality, while $\mathfrak{F}'$ does not.

Now, consider the mapping $f : T \to T'$ defined by $f(i_1) = f(i_2) = f(i_3) = f(i_4) = i'$, $f(j_1) = f(j_2) = f(j_3) = f(j_4) = j'$. It is easy to verify that f is a surjective bounded morphism. Then, from $\mathfrak{F} \models \phi$ Theorem 9 allows us to infer that $\mathfrak{F}' \models \phi$, and we have a contradiction. $\dashv$

Enriching the language. Given that nice frames cannot be characterized in the language of MTL_1, a possible way out consists in enriching the language to make it possible to express the two properties of total connectedness and quasi-functionality of the displacement relation in its 2nd argument. We briefly show that those properties can actually be expressed by adding to the language the future and past operators F, P, the difference operator $\mathcal{D}$, and by allowing information to be lifted from the temporal domain to the displacement domain by permitting the two languages to be mixed.

First, the *difference operator* (De Rijke [14]) is a unary modal operator $\mathcal{D}$ that allows us to model unbounded jumps. Its semantic interpretation is defined as follows:

$$(\mathfrak{F},V),i \Vdash \mathcal{D}\phi \;\text{ iff }\; \exists j\,(j \neq i \wedge (\mathfrak{F},V),j \Vdash \phi)$$

with dual $\overline{\mathcal{D}}$:

$$(\mathfrak{F},V),i \Vdash \overline{\mathcal{D}}\phi \;\text{ iff }\; \forall j\,(j \neq i \rightarrow (\mathfrak{F},V),j \Vdash \phi).$$

The difference operator and its dual allow us to define three derived unary operators $\mathcal{E}$, its dual $\mathcal{A}$, and $\mathcal{U}$ that respectively model truth in at least one world, truth in all worlds, and truth in one and only one world:

$$\mathcal{E}\phi \equiv \mathcal{D}\phi \vee \phi,\; \mathcal{A}\phi \equiv \overline{\mathcal{D}}\phi \wedge \phi,\; \text{and } \mathcal{U}\phi \equiv \mathcal{E}(\phi \wedge \neg\mathcal{D}\phi).$$

In a language in which the algebraic and temporal formulas may be mixed, properties *(xi)* and *(xii)* can be axiomatized by means of the qualitative operators F, P and $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{U}$ as follows:

(Ax6) $\mathcal{D}p \rightarrow Fp \vee Pp$ (total connectedness of DIS)
(Ax7) $\mathcal{U}p \wedge \mathcal{U}q \rightarrow (\mathcal{E}(p \wedge \Delta_\alpha q) \wedge \mathcal{E}(p \wedge \Delta_\beta q) \rightarrow \alpha = \beta)$
(quasi-functionality of DIS w.r.t. the 2nd argument).

Details are supplied in the full paper.

However, we prefer to remain within the original language of MTL_1 and reason about nice frames there, mainly because adding the axioms Ax6 and Ax7 forces us to give up the simplicity of the basic calculus and to include non-standard derivation rules to govern the difference operator. As we will show below, the logic of nice frames can be captured in the original language.

Completeness for nice frames. Instead of increasing the expressive power of metric temporal logic, we can leave it as it stands, and prove a completeness result for nice frames in the old language. We will do this in two steps. We first prove completeness with respect to totally connected frames via some sort of generated submodel construction, and then we prove the full result.

Here's the idea for the case of total connectedness. Let $\mathfrak{F} = (T, \mathfrak{D}; \mathrm{DIS})$ be a two-sorted frame. The *master relation* on $\mathfrak{F}$ is defined by

$$(i,j) \in \mathit{Master} \text{ iff } (i,j) \in (\ll \cup \gg)^*.$$

Thus i, j are in the master relation iff there exists a zig zag path along the displacement relation from i to j in the following sense:

$$\mathrm{DIS}(i, \alpha_1, j_1), \mathrm{DIS}(j_1, \alpha_2, j_2), \ldots, \mathrm{DIS}(j_n, \alpha_{n+1}, j),$$

where $\alpha_1, \ldots, \alpha_n \in D$, and $j_1, \ldots, j_n \in T$.

A *point-generated component* of a model $\mathfrak{M} = (T, \mathfrak{D}; \mathrm{DIS}; g, V)$ is a model $(T', \mathfrak{D}; \mathrm{DIS}'; g, V')$ such that for some $i \in T$,

- $T' = \{j \in T \mid (i,j) \in \mathit{Master}\}$
- $\mathrm{DIS}' = \mathrm{DIS} \cap (T' \times D \times T')$
- $V'(p) = V(p) \cap T'$, for all p.

Proposition 11. *Let $\mathfrak{M}'$ be a point-generated component of a model $\mathfrak{M}$ based on a two-sorted frame with ordered abelian group. If $\mathfrak{M}$ has a transitive displacement relation, then $\mathfrak{M}'$ has a transitive and totally connected displacement relation.*

Lemma 12. *Let $\mathfrak{M}'$ be a point-generated component of a two-sorted model $\mathfrak{M}$. Then $\mathfrak{M}'$ satisfies exactly the same algebraic formulae as $\mathfrak{M}$. Moreover, for all $i \in T'$ and for all temporal formulae ϕ we have $\mathfrak{M}, i \Vdash \phi$ iff $\mathfrak{M}', i \Vdash \phi$.*

MTL_1Ax3 extends MTL_1 with the transitivity axiom $\nabla_{\alpha+\beta} p \rightarrow \nabla_\alpha \nabla_\beta p$.

Theorem 13. *MTL_1Ax3 is sound and complete with respect to the class of two-sorted frames based on ordered abelian groups whose displacement relation is transitive and totally connected.*

Proof. We only prove completeness, and to establish this it suffices to show that every MTL_1Ax3-consistent set of formulae is satisfiable in a model based on a frame of the right kind.

Let Γ be a MTL_1Ax3-consistent set of formulae. By a Sahlqvist style argument (see Theorem 2) it is easily seen that Γ is satisfiable in a model $\mathfrak{M}$ based on a two-sorted frame with a transitive displacement relation, say at a time instant i. Let $\mathfrak{M}'$ be a point-generated component of $\mathfrak{M}$ that contains i. By Proposition 11 $\mathfrak{M}'$ has a transitive and totally connected displacement relation, and by Lemma 12 we have $\mathfrak{M}', i \Vdash \Gamma$, as required. $\dashv$

To prove completeness w.r.t. the class of nice frames, we need to carry out a second construction. First, call a two-sorted frame *almost nice* if it is transitive, reflexive, totally-connected, and quasi-functional in the 3rd argument of its displacement relation; a model is *almost nice* if it is based on an almost nice frame. So a frame is nice if it is almost nice and quasi-functional in the 2nd argument of its displacement relation.

Now, to build a nice model we will take an almost nice model and carefully unfold it. To be precise, let $\mathfrak{M} = (T, \mathfrak{D}; \mathrm{DIS}; g, V)$ be an almost nice model, and let $i \in T$. The *i-stratification of $\mathfrak{M}$* is the model $\mathfrak{M}' = (T', \mathfrak{D}; \mathrm{DIS}'; g, V')$ which is defined as follows:

$$\begin{aligned}
T' &= \{(0,i)\} \cup \{(\alpha, j) \mid \mathrm{DIS}(i, \alpha, j) \text{ in } \mathfrak{M}\}\\
\mathrm{DIS}_0 &= \{((0,i), \alpha, (\alpha, j)) \mid (\alpha, j) \in T'\} \cup \{((\alpha, j), -\alpha, (0,i)) \mid (\alpha, j) \in T'\}\\
\mathrm{DIS}_1 &= \{((\alpha, j), \beta - \alpha, (\beta, k)) \mid (\alpha, j), (\beta, k) \in T'\}\\
\mathrm{DIS}' &= \mathrm{DIS}_0 \cup \mathrm{DIS}_1\\
V'(p) &= \{(\alpha, j) \in T' \mid j \in V(p)\}.
\end{aligned}$$

Observe that $\mathrm{DIS}_0 \subseteq \mathrm{DIS}_1$.

Proposition 14. *Let $\mathfrak{M}$ be an almost nice model, and let $i \in \mathfrak{M}$. The i-stratification of $\mathfrak{M}$ is nice.*

Proof. We first observe first that for any pairs (α, j), $(\gamma, k) \in T'$, and $\beta \in \mathfrak{D}$, if $\mathrm{DIS}'((\alpha, j), \beta, (\gamma, k))$ holds then $\beta = \gamma - \alpha$.

Now, to prove the proposition, we have to check the nice-ness properties. First of all, we show that $\mathrm{DIS}'((\alpha, j), \beta, (\gamma, k))$ implies $\mathrm{DIS}'((\gamma, k), -\beta, (\alpha, j))$. By the observation $\beta = \gamma - \alpha$. Also, (α, j), $(\gamma, k) \in T'$ implies $\mathrm{DIS}'((\gamma, k), \alpha - \gamma, (\alpha, j))$, that is, $\mathrm{DIS}'((\gamma, k), -\beta, (\alpha, j))$.

Next, we show that DIS' is reflexive. As $\mathfrak{M}$ is assumed to be reflexive, we have $\mathrm{DIS}(i, 0, i)$, hence $\mathrm{DIS}((0, i), 0, (0, i))$. As to other points $(\alpha, j) \in T'$, $\mathrm{DIS}_0((0, i), \alpha, (\alpha, j))$ and $\mathrm{DIS}_0((\alpha, j), -\alpha, (0, i))$ imply $\mathrm{DIS}'((\alpha, j), 0, (\alpha, j))$.

To see that DIS' is quasi-functional with respect to its 3rd argument, assume $\mathrm{DIS}'((\alpha, j), \beta, (\gamma, k))$ and $\mathrm{DIS}'((\alpha, j), \beta, (\gamma', k'))$. We need to show that $\gamma = \gamma'$

and $k = k'$. First of all, $\beta = \gamma - \alpha = \gamma' - \alpha$, hence $\gamma = \gamma'$. Therefore, $\mathrm{DIS}(i, \gamma, k)$ and $\mathrm{DIS}(i, \gamma, k')$. So by the assumption that DIS is quasi-functional in its 3rd argument, $k = k'$.

Given that $\mathfrak{M}$ is total, the totality of its i-stratifications is immediate.

Transitivity of $\mathfrak{M}'$ may be established as follows: assume $\mathrm{DIS}'((\alpha, j), \beta, (\gamma, k))$ and $\mathrm{DIS}'((\gamma, k), \beta', (\delta, l))$. Then $\mathrm{DIS}'((\alpha, j), \delta - \alpha, (\delta, l))$. As $\beta + \beta' = (\gamma - \alpha) + (\delta - \gamma)$, we are done.

Finally, to prove quasi-functionality of DIS$'$ in its 2nd argument, assume $\mathrm{DIS}'((\alpha, j), \beta, (\gamma, k))$ and $\mathrm{DIS}'((\alpha, j), \beta', (\gamma, k))$. It follows that $\beta = \gamma - \alpha = \beta'$. $\dashv$

Proposition 15. *Let $\mathfrak{M}$ be an almost nice model, and let $\mathfrak{M}'$ be an i-stratification of $\mathfrak{M}$. For all formulae ϕ, j in $\mathfrak{M}$, and (α, j) in $\mathfrak{M}'$, we have $\mathfrak{M}, j \Vdash \phi$ iff $\mathfrak{M}', (\alpha, j) \Vdash \phi$.*

Proof. This is by induction on ϕ. The base case and the boolean cases are trivial. So consider a temporal formula $\Delta_\gamma \psi$. Assume first that $j \Vdash \Delta_\gamma \psi$. Then there exists k with $\mathrm{DIS}(j, \gamma, k)$. Now, let α be such that $(\alpha, j) \in T'$. Then $\mathrm{DIS}(i, \alpha, j)$, and hence $\mathrm{DIS}(i, \alpha + \gamma, k)$ and $(\alpha + \gamma, k) \in T'$. By definition, $\mathrm{DIS}_0((0, i), \alpha, (\alpha, j))$ and $\mathrm{DIS}_0((0, i), \alpha + \gamma, (\alpha + \gamma, k))$. But then $\mathrm{DIS}'((\alpha, j), \gamma, (\alpha + \gamma, k))$. By induction hypothesis, $(\alpha + \gamma, k) \Vdash \psi$, hence $(\alpha, j) \Vdash \Delta_\gamma \psi$.

Conversely, assume that $(\alpha, j) \Vdash \Delta_\gamma \psi$. Then there exists $(\beta, k) \in T'$ with $\mathrm{DIS}'((\alpha, j), \gamma, (\beta, k))$ and $(\beta, k) \Vdash \psi$. Hence $\gamma = \beta - \alpha$. By construction we must have $\mathrm{DIS}(i, \alpha, j)$ and $\mathrm{DIS}(i, \beta, k)$ and hence $\mathrm{DIS}(j, \beta - \alpha, k)$. As $k \Vdash \psi$ (by induction hypothesis) and $\gamma = \beta - \alpha$, this implies $j \Vdash \Delta_\gamma \psi$, as required. $\dashv$

We are ready now for a completeness result for the class of nice frames. Let MTL_2 denote the extension of MTL_1 with axioms Ax3, Ax4 and Ax5 (expressing transitivity, quasi-functionality of DIS in its 3rd argument, and reflexivity, respectively). By an easy adaptation of the proof of Theorem 13, MTL_2 is sound and complete w.r.t. the class of almost nice frames.

Theorem 16. *MTL_2 is sound and complete with respect to the class of nice frames.*

Proof. We only show that every MTL_2-consistent set of temporal formulae is satisfiable on a nice model. Let Γ be such a set. By earlier remarks Γ is satisfiable on an almost nice model at some time instant i. Let $\mathfrak{M}'$ be the i-stratification of $\mathfrak{M}$. By Propositions 14 and 15 $\mathfrak{M}'$ is a nice model that satisfies Γ at i. $\dashv$

Conclusion

In this paper we have proved completeness results for basic systems of metric temporal logic. We started with the minimal calculus and showed how to extend it to obtain the logic of two-sorted frames with a linear temporal order in which there exists a unique temporal distance between any two time instants.

So far we have only considered simple languages that do not allow us to lift information from the temporal domain to the algebraic domain. Obviously, for application purposes they have to be extended. In particular, we are considering the possibility of a restricted form of mixing temporal and displacement formulae, so as to enable more complex ways of interaction between the two domains.

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