Hints for Exercises

Chapter 3


Hint. 3. Use induction on $n$ for the stages $TC(a, n)$.

4. Show that for all $b \subseteq TC(a), TC(b) \subseteq TC(a)$.

$\therefore$ show that $a \cup \bigcup_{b \in a} TC(b)$ is transitive. $\square$

39 Show that $x \in TC(a)$ if $x \in^* a$.

Hint. Show that $xRy \equiv \text{def } x \in y$ and $R^* \equiv \text{def } x \in TC(y)$ satisfy parts 1-3 of Lemma 3.21.

Alternatively, define $R_n$ and use induction on $n$ to show that $x \in TC(a, n)$ if $x \in_n a$. $\square$

43 $\mathbb{Z}$ is the set of integers. Define $H : \wp(\mathbb{Z}) \to \wp(\mathbb{Z})$ by $H(X) \equiv \text{def } \{0\} \cup \{S(x) \mid x \in X\}$. Identify the fixed points of $H$.

Hint. Show that for any fixed point $K$ of $H$, $-1 \in K$ if for all $n \in \omega$, $-(1+n) \in K$. Use this to show that $H$ has exactly two fixed points. $\square$

44 Prove Theorem 3.27.

Hint. Do not use Theorem 3.24. Show that $n < m \Rightarrow H|n \subset H|m$. Show: if $H(X) \subset X$, then, for all $n$, $H|n \subset X$. Finally, show that $H(H|\omega) \subset H|\omega$. (For this, you will need the fact that if $Y$ is finite and $Y \subset \bigcup_{n \in \omega} H|n$, then for some $m \in \omega$, $Y \subset H|m$. This is shown by induction w.r.t. the number of elements of $Y$, cf. Definition 3.15, p.17. $\square$

45 Let $A = \omega \cup \{\omega\}$ and define $H : \wp(A) \to \wp(A)$ by $H(X) = \{0\} \cup \{S(x) \mid x \in X\}$ if $\omega \not\subset X$, and $H(X) = A$ otherwise. Show: $H$ is monotone, $H$ is not finite, $H|A = A$, $\forall n \in \omega H|n = n$. Thus, $H|A \neq \bigcup_n H|n$.

Hint. To show that $H|A = A$, first show that for all $n \in \omega$, $n \in A$. $\square$

51 (Simultaneous inductive definitions.) Suppose that $\Pi, \Delta : \wp(A) \times \wp(A) \to \wp(A)$ are monotone operators in the sense that if $X_1, Y_1, X_2, Y_2 \subset A$ are such that $X_1 \subset X_2$ and $Y_1 \subset Y_2$, then $\Pi(X_1, Y_1) \subset \Pi(X_2, Y_2)$ (and similarly for $\Delta$). Show that $K, L$ exist such that

1. $\Pi(K, L) \subset K$, $\Delta(K, L) \subset L$; in fact, $\Pi(K, L) = K$, $\Delta(K, L) = L$,

2. if $\Pi(X, Y) \subset X$ and $\Delta(X, Y) \subset Y$, then $K \subset X$ and $L \subset Y$.

Show that, similarly, greatest (post-) fixed points exist. Generalize to more operators.

Hint. Consider the operator $H : \wp(A \times A) \to \wp(A \times A)$ defined by $H(Z) = \Pi(\pi_1[Z], \pi_2[Z]) \times \Delta(\pi_1[Z], \pi_2[Z])$ (where, as usual, $\pi_1$ and $\pi_2$ denote the projection onto the first and second coordinates). Show that $H$ has at least one fixed point of the form $K \times L$, and that $K$ and $L$ satisfy the given conditions. $\square$