Hints to Exercises

Chapter 4

54. Assume that the set $a$ is transitive. Show:

1. $a \subseteq G$ if $\in$ is well-founded on $a$,
2. $a \subseteq TR$ if $\in$ is transitive on $a$.

Thus, an ordinal is the same as a transitive set on which $\in$ is a transitive and well-founded relation. (This is the standard definition of the notion.)

Hint.
1. If $b \ni a$ witnesses $a \notin G$, show that $a \cap b$ has no $\in$-minimal element. Find a way to also do this the other way around.
2. Rewrite both as properties of $x, y, z$ with $x \in y \in z \in a$.

58. Show:

1. $\mathbb{V} \ni \mathbb{V}$, $\mathbb{V} \ni \mathbb{V}$,
2. if $K$ is a non-empty class of ordinals, then $\bigcap K$ is the least element of $K$,
3. if $A$ is a set of ordinals, then $\bigcup A$ is an ordinal that is the $\sup$ of $A$ (the least ordinal $\geq$ every $\alpha \in A$).

Hint.
1. Use $\alpha \neq \beta$ for one of the implications.
2. Show that $\bigcup A$ is an ordinal, and then use (1).

61. Assume that $(A, \prec)$ is a well-ordering and $B \subseteq A$.
Show that $\text{type}(B, \prec) \leq \text{type}(A, \prec)$.

Hint. Assume otherwise, and construct an order-preserving injection from an ordinal into a lesser ordinal to derive a contradiction.

64. Prove Theorem 4.13: suppose that $\varepsilon$ is a well-founded relation on the class $U$ such that for all $a \in U$, $\{b \in U \mid b \varepsilon a\}$ is a set, then for every operation $H : V \rightarrow V$ there is a unique operation $F : U \rightarrow V$ such that for all $a \in U$:

$$F(a) = H(F(\{b \in U \mid b \varepsilon a\})).$$

Hint
Ignore the hint from the syllabus. Use a proof analogous to that of Theorem 4.10, but requiring that a good function $f$ has a domain satisfying $\{b \in U \mid b \varepsilon a\} \subseteq \text{Dom}(f)$ for all $a \in \text{Dom}(f)$, and using $\varepsilon$-wellfoundedness instead of transfinite induction to show that good functions agree on their domain. Note that in the given conditions, $\{b \in U^* \mid b \varepsilon a\}$ is a set for all $a \in U$.

65. Let $a_0 \in V$ be a set and $G : V \rightarrow V$ an operation. Show: there exists a unique operation $F : \text{OR} \rightarrow V$ on $\text{OR}$ such that

- $F(0) = a_0$.
Hint
Applying the Recursion Theorem on OR to a suitable operation $H$.

70. Show that the single recursion equation $H[\alpha] = \bigcup_{\xi \leq \alpha} H(H[\xi])$ defines the same operation as the one defined in Definition 4.14 by three equations. (And, of course, $H[\alpha] = \bigcap_{\xi \leq \alpha} H(H[\xi])$ is a single equation defining the greatest fixed point hierarchy — cf. Exercise 72.)

Hint.
Use that the least fixed point hierarchy is cumulative, i.e. $\alpha < \beta \Rightarrow H[\alpha] \subset H[\beta]$.

72 Let $H$ be a monotone operator over a set $U$. The greatest fixed point hierarchy is the sequence $\{H[\alpha]\}_\alpha$ recursively defined by

- $H[0] = U$,
- $H[\alpha + 1] = H(H[\alpha])$,
- $H[\gamma] = \bigcap_{\xi < \gamma} H[\xi]$ (for limits $\gamma$).

Show that:

1. the hierarchy is descending, i.e., that $\alpha < \beta \Rightarrow H[\beta] \subset H[\alpha]$.
2. some stage $H[\alpha_0]$ is a fixed point of $H$.
3. $H[\alpha_0] = \bigcap_\alpha H[\alpha]$ is the greatest fixed point of $H$.

Try to generalize for the case where $U$ may be a proper class.

Hint.
Consider the dual operator $H^d$ from exercise 52.