Hints for Exercises

Chapter 6

111. (AC) Suppose that $p, q, r, s$ are cardinals such that $p < q$ and $r < s$. Show that $p + r < q + s$.

*Hint.* Use Lemma 6.8.

114. Prove Lemma 6.22:

1. $\gamma^{\text{cf}(\kappa)} > \kappa$,
2. $\text{cf}(2^\kappa) > \kappa$.

*Hint.* Use Theorem 6.15.

116. (Hausdorff) Prove that $\kappa^+ = \kappa^\kappa^+$.

*Hint.* Distinguish as to whether $\kappa^+$ is $< \kappa$ than $\kappa^\kappa^+$.

120. Show: if $\kappa$ is a strongly inaccessible initial number, then

1. $\beta < \kappa \Rightarrow V_\beta <_1 \kappa$,
2. $V_\kappa = \kappa$,  
3. $(V_\kappa, \in)$ satisfies all ZFC Axioms,
4. if $\kappa$ is the least strong inaccessible, then $(V_\kappa, \in) \models \text{"there is no strong inaccessible"}$. 

*Hint.*

1. Straightforward by induction on $\beta$.
2. Use $|V_\kappa| = |\bigcup_{\beta < \kappa} V_\beta| \leq \sum_{\beta < \kappa} |V_\beta|$.
3. To show that $V_\kappa$ satisfies Substitution, show that for any $a \in V_\kappa$ and any operator $F$ with $F[a] \subset V_\kappa$, $\rho(F[a]) < \kappa$. The other axioms are straightforward for any limit ordinal.
4. You may assume that for any 'bound' formula $\phi$ (where all quantifiers are of the form $\exists x \in y$ or $\forall x \in y$) and any $\vec{x} \in V_\kappa$, $(V_\kappa \models \phi(\vec{x}) \iff \phi(\vec{x}))$. Show that if $\alpha < \kappa$ does not satisfy one of the conditions for being strongly inaccessible, then there exists a 'witness' for this in $V_\kappa$, and hence $\alpha$ fails this same condition in $V_\kappa$.

122. Show:

1. Suppose that $X \subset \alpha$ is cofinal in $\alpha$. Show that $\alpha$ has a cofinal subset $Y$ of type $\text{cf}(\alpha)$ such that $Y \subset X$.
2. Show: if $\alpha$ and $\beta$ have cofinal subsets of the same type, then $\text{cf}(\alpha) = \text{cf}(\beta)$.
3. Show: if $\alpha$ is a limit, then $\text{cf}(\omega_\alpha) = \text{cf}(\alpha)$.

*Hint.*

1. Suppose that $f : \text{cf}(\alpha) \to \alpha$ has Ran$(f)$ cofinal in $\alpha$. Define $g : \text{cf}(\alpha) \to X$ by $g(\xi) = \bigcap \{\delta \in X \mid f(\xi) \leq \delta\}$, and apply Lemma 6.26.
2. Use that a cofinal subset of a cofinal subset is a cofinal subset.

3. Show that $\omega_\alpha$ and $\alpha$ have cofinal subsets of the same type.