Solutions To Exercises

Chapter 3


Solution.
1. \( a = TC(a, 0) \subseteq TC(a) \).
2. If \( x \in y \subseteq TC(a) \), then, for some \( n \in \omega \), \( y \subseteq TC(a, n) \). Thus, \( x \notin \bigcup TC(a, n) = TC(a, S(n)) \subseteq TC(a) \).
3. Suppose \( b \supseteq a \) is transitive. Then \( a = TC(a, 0) \subseteq b \), and if for some \( n \), \( TC(a, n) \subseteq b \), then \( TC(a, Sn) = \bigcup TC(a, n) \subseteq b \subseteq b \). Thus, by induction on \( n \), \( TC(a, n) \subseteq b \) for all \( n \), and therefore \( TC(a) \subseteq b \).
4. \( \supseteq \): First, \( a \subseteq TC(a) \). Next, if \( b \subseteq a \), then \( b \subseteq TC(a) \) (since \( a \subseteq TC(a) \)) and \( TC(b) \subseteq TC(a) \) (by property 3).
\( \subseteq \): By property 3, it suffices to show that \( a \cup \bigcup_{b \in a} TC(b) \) is a transitive superset of \( a \). Transitivity: if \( x \in y \in a \cup \bigcup_{b \in a} TC(b) \), then \( y \in a \), or \( b \in a \) exists such that \( y \in TC(b) \). In the first case, \( x \in TC(y) \subseteq a \cup \bigcup_{b \in a} TC(b) \). In the second, \( x \in TC(b) \subseteq a \cup \bigcup_{b \in a} TC(b) \).

39 Show that \( x \in TC(a) \Leftrightarrow x \in a \).

Solution 1.
Define \( Rx y \equiv \text{def} x \in y \) and \( R' = \text{def} x \in TC(y) \). Now \( R \) and \( R' \) satisfy parts 1-3 of Lemma 3.21:
1. if \( a \subseteq b \), then \( a \subseteq TC(b) \), since \( b \subseteq TC(b) \).
2. if \( a \subseteq TC(b) \) and \( b \subseteq TC(c) \), then by transitivity of \( TC(c) \) we have that \( b \subseteq TC(c) \) and \( TC(b) \subseteq TC(c) \), and therefore \( a \subseteq TC(c) \).
3. Assume \( R \subseteq S \) and \( S \) is transitive. It suffices to show that for any \( b \), \( \{ x \mid xR' b \} \subseteq \{ x \mid xS b \} \). This follows from \( \{ x \mid xR' b \} = TC(b) \) and the observation that \( \{ x \mid xS b \} \) is a transitive set (since for all \( x, y \), if \( yS b \) and \( x \in y \), then \( xSy \) and \( xS b \)).
It follows that \( R' = R^* \).

Solution 2.
Define \( R_n \) by
\[
\text{def} \quad aR_n b \equiv \exists f \{ \text{Dom}(f) = n + 2 \land f(0) = a \land f(n + 1) = b \land \forall i < n + 1 (f(i)Rf(i + 1)) \}
\]
It can easily be seen that \( R_0 = R \), that for all \( a, b \) and \( n \), \( aR_{n+1} b \iff \exists c[aRc \land cR_n b] \), and that for all \( a \) and \( b \), \( aR^* b \iff \exists n \in \omega : aR_n b \). Now we can show by induction on \( n \) that for all \( n \), if and only if \( x \in TC(a, n) \).

If we assume that for a given \( n \) and \( a \) and for all \( y \), \( y \subseteq a \Leftrightarrow y \subseteq TC(a, n) \), then for all \( x \),
\[
\begin{align*}
0 \in a & \iff x \in a \iff x \in TC(a, 0) \\
\forall n (a \subseteq a) & \iff \exists y \{ x \in y \in a \} \iff \exists y \{ x \in y \in TC(a, n) \} \iff x \in \bigcup TC(a, n) = TC(a, n + 1)
\end{align*}
\]

Therefore \( x \in a \iff \exists x \in a \iff \exists x \in TC(a, n) \iff x \in TC(a) \).

43 \( \mathbb{Z} \) is the set of integers. Define \( H : \wp(\mathbb{Z}) \to \wp(\mathbb{Z}) \) by \( H(X) = \text{def} \{0\} \cup \{ S(x) \mid x \in X \} \). Identify the fixed points of \( H \).
Solution.

$H$ is a finite operator, so $H| = \mathbb{N}$ is the least fixed point of $H$. For any fixed point $K$ of $H$, by induction on $n$, $-1 \in K$ iff for all $n \in \omega$, $-(1+n) \in K$. It follows that the only other fixed point of $H$ is $\mathbb{Z}$.

44 Prove Theorem 3.27.

Solution.

1. We prove the equivalent statement that for all $n, m$, $H[n] \subset H[(n+m)]$, by induction w.r.t. $n$:
   - Basis $n = 0$: $H[n] = \emptyset \subset H[(n+m)]$ is obvious.
   - Induction step: if $H[n] \subset H[(n+m)]$, then $H[(n+1)] = H(H[n]) \subset H(H[(n+m)]) = H[(n+1+m)]$.

2. Suppose that $H(X) \subset X$. By induction on $n$, it follows that $H[n] \subset X$:
   - Basis $n = 0$: $H[0] = \emptyset \subset X$ is obvious.
   - Induction step: if $H[n] \subset X$, then $H[(n+1)] = H(H[n]) \subset H(X) \subset X$.

3. If $Y \subset H[\omega] = \bigcup_n H[n]$ is finite, then $n$ exists s.t. $Y \subset H[n]$: induction w.r.t. $n$ of elements of $Y$.
   - Basis, $Y = \emptyset$. Then $Y \subset H[0]$.
   - Induction step. IH: for $n$-element $Y$, the statement holds. Now let $Y \subset \bigcup_n H[n]$ have $n+1$ elements. For instance, $Y = Y' \cup \{y\}$, where $Y'$ has $n$ elements. By IH, $n_1$ exists s.t. $Y \subset H[n_1]$. Furthermore, $n_2$ exists s.t. $y \in H[n_2]$. Let $m = \max(n_1, n_2)$. Then clearly (by 1), $Y \subset H[m]$.

4. $H(H[\omega]) \subset H[\omega]$:
   - Assume that $a \in H(H[\omega])$. By finiteness, a finite $Y \subset H[\omega]$ exists s.t. $a \in H(Y)$. By 3 we can assume that for some $n$, $Y \subset H[n]$. Then $a \in H(Y) \subset H(H[n]) = H[n] + 1 \subset H[\omega]$.

45 Let $A = \omega \cup \{\omega\}$ and define $H : \wp(A) \to \wp(A)$ by $H(X) = \{0\} \cup \{S(x) \mid x \in X\} \cap A$ if $\omega \not\subset X$, and $H(X) = A$ otherwise. Show: $H$ is monotone, $H$ is not finite, $H[A] = A$, $\forall n \in \omega \ H[n] = n$. Thus, $H[A] \neq \bigcup_n H[n]$.

Solution.

$H$ is monotone: Let $X \subset Y \subset A$. If $\omega \subset Y$, then $H(X) \subset A = H(Y)$. Otherwise, $\omega \not\subset X, Y$, so $H(X) = \{0\} \cup \{S(x) \mid x \in X\} \subset \{0\} \cup \{S(x) \mid x \in X\} = H(Y)$.

$H$ is not finite: Since $\omega \in H(\omega)$, and for all finite sets $X \subset \omega$, $\omega \not\subset H(X)$, we see that $H$ is not finite.

For all $n$, $H[n] = n$, by induction on $n$: For $n = 0$, $H[0] = \emptyset = 0$. If $H[n] = n$, then $H[n+1] = H(H[n]) = H(n) = \{0\} \cup \{S(x) \mid x \in n\} = n+1$.

$H[A] = A$: Studying the proof of Theorem 3.24 it is apparent that $\bigcup_n H[n]$ is inductive even if $H$ is not finite. So $\omega = \bigcup_n H[n] \subset H[\omega]$. Therefore $A = H(\omega) \subset H(H[\omega]) = H[\omega]$. We conclude that $H[A] = A$.

51 (Simultaneous inductive definitions.) Suppose that $\Pi, \Delta : \wp(A) \times \wp(A) \to \wp(A)$ are monotone operators in the sense that if $X, Y_1, X_2, Y_2 \subset A$ are such that $X_1 \subset X_2$ and $Y_1 \subset Y_2$, then $\Pi(X_1, Y_1) \subset \Pi(X_2, Y_2)$ (and similarly for $\Delta$). Show that $K, L$ exist such that

1. $\Pi(K, L) \subset K, \Delta(K, L) \subset L$; in fact, $\Pi(K, L) = K, \Delta(K, L) = L$.

2. if $\Pi(X, Y) \subset X$ and $\Delta(X, Y) \subset Y$, then $K \subset X$ and $L \subset Y$.

Show that, similarly, greatest (post-) fixed points exist. Generalize to more operators.

Solution.

Consider the operator $H : \wp(A \times A) \to \wp(A \times A)$ defined by $H(Z) = \Pi(\pi_1[Z], \pi_2[Z]) \times \Delta(\pi_1[Z], \pi_2[Z])$ (where, as usual, $\pi_1$ and $\pi_2$ denote the projection onto the first and second coordinates).

$H$ is monotone: assume that $Z \subset Z' \subset (A \times A)$. Then $\pi_1[Z] \subset \pi_1[Z']$ and $\pi_2[Z] \subset \pi_2[Z']$. So by our assumption for $\Pi$, $\Pi(\pi_1[Z], \pi_2[Z]) \subset \Pi(\pi_1[Z'], \pi_2[Z'])$, and analogously for $\Delta$. It follows that $H(Z) \subset H(Z')$.

Since $H$ is monotone, it has a least fixed point $H[\|]$. Setting $K = \pi_1[H[\|]], L = \pi_2[H[\|]]$, we have that $H[\|] = H(H[\|]) = \Pi(K, L) \times \Delta(K, L)$, so $K = \pi_1[H[\|]] = \Pi(K, L)$ and $L = \pi_2[H[\|]] = \Delta(K, L)$ (and $H[\|] = K \times L$).

For the second part, assume that for $X, Y \subset A$, $\Pi(X, Y) \subset X$ and $\Delta(X, Y) \subset Y$. Then $H(X \times Y) = \Pi(X, Y) \times \Delta(X, Y) \subset X \times Y$, and hence $K \times L = H[\|] \subset X \times Y$. Therefore $K \subset X$ and $L \subset Y$. 
