Chapter 7

125. Recall (Exercise 16 p. 12 and Theorem 4.18 p. 34) that \(G\) is the least class such that

\[ \varphi(G) = G. \]

Show, in ZF minus Foundation:

1. If \(K\) is a class such that \(\varphi(K) = K\), then every ZF axiom — with the possible exception of Foundation — holds in \(K\).

2. Foundation is true in \(G\).

Thus, \(G\) is an inner model for ZF including Foundation in ZF minus Foundation. Therefore, if the latter theory is consistent, then so is the former.

Solution.

1. Assume that \(K = \varphi(K)\), i.e. for all \(a, b \in K \iff a \subset K\). We will prove that the relativization of all the axioms with respect to \(K\) holds. In most cases, it suffices to show that the set whose existence is postulated by the axiom is in \(K\), since the defining formula is bounded or \(\Pi^0_1\), and \(K\) is a transitive inner model.

   **Extensionality** For all \(a, b \in K\), if \(\forall x \in K : [x \in a \iff x \in b]\), then \(a \cap K = b \cap K\). Now since also \(a, b \subset K\), it follows that \(a = b\).

   **Separation** Let \(a \in K\), and let \(\Phi\) be a formula that doesn’t contain \(b\) freely. Set \(b = \{x \in a \mid \Phi^K(x, a)\}\). Then \(b \subset a \subset K\), so \(b \in K\).

   **Pairing** For all \(a, b \in K\), \(\{a, b\} \subset K\), so \(\{a, b\} \in K\).

   **Sumset** Let \(a \in K\). For all \(b \in a, b \subset K\), so \(\bigcup a \subset K\), so \(\bigcup a \in K\).

   **Powerset** Let \(a \in K\). Then \(a \subset K\), so \(\varphi(a) \subset \varphi(K) = K\), so \(\varphi(a) \in K\).

   **Substitution** Let \(a \in K\), and let \(\Psi\) be a formula, not containing \(b\) freely, such that \(\forall x \in a : [x \in K \iff \exists y \in a : y \in a \Psi^K]\). Then by Substitution \(b = \{y \mid y \in K \wedge \exists x \in a : y \in a \Psi^K\}\) is a set.

   Since \(b \subset K\), we have \(b \in K\), and \(\forall y \in K : [y \in b \iff \exists x \in a : y \in a \Psi^K]\).

   **Infinity** Since \(T(K) \subset \varphi(K) = K\), we have \(\text{OR} \subset K\), so \(\omega \in K\).

2. Let \(a \in G\). Since \(a \in a \cup \{a\}\), we can find an \(x \in a \cup \{a\}\) such that \(x \cap (a \cup \{a\}) = \emptyset\). It follows that either \(x = a = \emptyset\), or \(x \in a \subset G\) and \(x \cap a = \emptyset\). Thus, Foundation holds in \(G\).

127 Show that the following ZF axioms cannot be deduced from the others (modulo a consistency assumption):

1. Infinity,
2. Powerset,
3. Substitution (e.g., existence of \(\omega + \omega\) is unprovable),
4. Sumsets.
Solution.

1. The class $K_1 = \{x \in G \mid \forall y \in^* \{x\} \exists y \text{ is finite}\}$ of hereditary finite sets satisfies all axioms of ZFC except Infinity. All the axioms follow easily from the property that $\forall x[x \in K_1 \iff (x \subset K_1 \land x \text{ is finite})]$. Note that it can be shown that $K_1 = V_{\omega_1}$. Since $\omega \not\in K_1$, Infinity does not hold in $K_1$.

2. For any cardinal $\aleph_\alpha$, the class $K_2 = \{x \in G \mid \forall y \in^* \{x\} \exists y \leq \aleph_\alpha\}$ of hereditary $\aleph_\alpha$-cardinality satisfies all axioms of ZFC except Powerset. All the axioms follow easily from the property that $\forall x[x \in K_2 \iff (x \subset K_2 \land |x| \leq \aleph_\alpha)]$. Note that it can be shown that $K_2 \subset V_{\omega_{\alpha+1}}$.

Since $\varphi(\omega_\alpha) \subset K_2$ and $\varphi(\omega_\alpha) \not\subset K_2$, Powerset does not hold in $K_2$.

3. For any limit ordinal $\alpha > \omega$ (and in particular, for $\alpha = \omega + \omega$), the class $K_3 = V_{\omega_\alpha}$ satisfies all axioms except Substitution. All the axioms follow easily from the property that for all $\beta < \alpha$ and $n \in \omega$, $\beta + n < \alpha$ (and for Infinity, that $\alpha > \omega$).

For $\alpha = \omega + \omega$, Substitution does not hold in $K_3$: the set $\{\omega, \omega + 1, \omega + 2, \ldots\}$ is not in $K_3$, even though it is constructible from $\omega$ using Substitution with the operator $(\omega + \cdot)$.

4. For any strong limit cardinal $\aleph_\alpha$ (for instance, $\aleph_\omega$ under GCH) the class $K_4 = \{x \in G \mid \forall y \in^* \{x\} : |y| < \aleph_\alpha\}$ of hereditary cardinality-less-than-$\aleph_\alpha$ sets satisfies all axioms of ZFC except Sumset. All the axioms follow easily from the property that $\forall x[x \in K_4 \iff (x \subset K_4 \land |x| < \aleph_\alpha)]$ (and for Powerset, that $\alpha$ is a limit ordinal). Note that it can be shown that $K_4 \subset V_{\varphi(\aleph_\alpha)}$.

If $\aleph_\alpha$ is singular (as $\aleph_\omega$ is), then there exists a cofinal subset $B \subset \omega_\alpha$ with $|B| < \aleph_\alpha$ (and hence $B \in K_4$), and since $\bigcup B = \omega_\alpha \not\in K_4$, Sumset does not hold in $K_4$. $\square$

134 Prove a few items of Lemma 7.12: give bounded formulas expressing the following.

Solution.

1. $x = \emptyset \iff \forall u \in x (u \neq u)$

2. $x \subset y \iff \forall u \in x (u \in y)$

3. $z = \{x\} \iff x \in z \land \forall u \in z (z = x)\\ z = \{x\} \iff x \in z \land \forall u \in z (z = x \lor z = y)$

4. $z = x \cup y \iff x \subset z \land y \subset z \land \forall u \in z (u \in x \lor u \in y)\\ z = x \cup \{y\} \iff x \subset z \land y \in z \land \forall u \in z (u \in x \lor u = y)$

5. $x = 0, x = 1, x = 2, x = 3, \ldots, x = n + 1 \iff \exists y \in x (x = y \cup \{y\} \land y = n)$.

6. $x = V_0, x = V_1, x = V_2, x = V_3, \ldots, x = V_{n+1} \iff \exists y \in x (y = V_n \land \forall z \in x (z \subset y) \land \emptyset \in x \land \forall z \in x \forall i \in y (z \cup \{i\} \in x)$.

7. $x$ is $0, \text{S-closed} \iff \emptyset \in x \land \forall y \in x \exists z \in x (z = y \cup \{y\})$.

8. $x \in \text{OR} \iff x$ is a transitive set of transitive sets $\iff \forall y \in x \forall z \in y (z \in x \land \forall u \in z (u \in y))$ (cf. Exercise 130).

9. $\alpha$ is a successor ordinal $\iff \alpha = \text{OR} \land \exists x \in \alpha (\alpha = x \cup \{x\})$ $\alpha$ is a limit ordinal $\iff \alpha \in \text{OR} \land \lnot (\alpha = \emptyset) \land \lnot (\alpha$ is a successor ordinal)

10. $x \in \omega \iff x \in \text{OR} \land \lnot \text{lim}(x) \land \forall y \in x \lnot \text{lim}(y)$

$x = \omega \iff x \in \text{OR} \land \text{lim}(x) \land \forall y \in x \lnot \text{lim}(y)$

11. $y = \bigcup x \iff \forall z \in x \forall u \in z (u \in y) \land \forall u \in y \exists z \in x (u \in z)$

12. $z = (x, y) \iff \exists u \in z \exists v \in z (u = \{x, x\} \land v = \{x, y\} \land z = \{u, v\})$
13. \( p \) is an ordered pair \( \iff \exists u \exists x \in u \exists y \in u (p = (x, y)) \)

14. \( R \) is a relation \( \iff \forall p \in R (p \) is an ordered pair \( ) \)

\[ xRy \iff \exists z \in R (z = (x, y)) \]

15. \( f \) is a function \( \iff f \) is a relation \( \land \forall p \in f \forall u \in p \forall x \in u \forall y \in u \forall p' \in f \forall u' \in p' \forall x' \in u' \forall y' \in u' \)

\[ ((p = (x, y) \land p' = (x', y') \land x = x') \rightarrow y = y') \]

\[ f(x) = y \iff \exists z \in R (z = (x, y)) \]

\( f \) is an injection \( \iff \) \( f \) is a function \( \land \forall p \in f \forall u \in p \forall x \in u \forall y \in u \forall p' \in f \forall u' \in p' \forall x' \in u' \forall y' \in u' \)

\[ ((p = (x, y) \land p' = (x', y') \land y = y') \rightarrow x = x') \]

\( f \) is a surjection onto \( Y \iff \) \( f \) is a function \( \land Y = \text{Ran}(f) \),

\( f \) is a bijection \( X \rightarrow Y \iff f \) is an injective function \( \land X = \text{Dom}(f) \land Y = \text{Ran}(f) \)

16. \( X = \text{Dom}(f) \iff \forall x \in X \exists p \in f \exists u \in p \exists y \in u (p = (x, y)) \land \forall p \in f \forall u \in p \forall x \in u \forall y \in u \)

\[ (p = (x, y) \rightarrow x \in X) \]

\[ Y = \text{Ran}(f) \iff \forall y \in Y \exists p \in f \exists u \in p \exists x \in u (p = (x, y)) \land \forall p \in f \forall u \in p \forall x \in u \forall y \in u \]

\[ (p = (x, y) \rightarrow y \in Y) \]

\[ g = f[A] \iff g \subset f \land \forall p \in f \forall u \in p \forall x \in u \forall y \in u (p = (x, y) \land x \in A) \rightarrow p \in g) \]

136

1. Decide which ZF Axioms/Axiom schemas hold in \( V_\omega \), and which are false.

2. Same question for \( V_{\omega+\omega} \).

3. Same question for \( V_{\omega_1} \).

4. Obtain some relative consistency results from 1–3.

5. What about the truth of Theorem 4.10 (p. 29) (\textit{every well-ordering has a type}) in the above models?

6. Suppose that Theorem 4.10 holds in \( V_\alpha \) and \( \alpha > \omega \). Can you give lower bounds for \( \alpha \)? And if AC holds in \( V_\alpha \)?

\textit{Solution.}

See also exercise 127.

1. All ZF Axioms except Infinity hold in \( V_\omega \).

2. All ZF Axioms except Substitution hold in \( V_{\omega+\omega} \).

3. All ZF Axioms except Substitution hold in \( V_{\omega_1} \) (note that \( V_{\omega+4} \) contains a well-ordering of type \( \omega_1 \), namely the well-ordering of all the well-orderings (modulo order-isomorphism) of \( \omega \), ordered by inclusion).

4. If ZF is consistent, then so are (ZF-Infinity)+−Infinity and (ZF-Substitution)+−Substitution.

5. In \( V_\omega \), every set is finite, so every well-ordering has a finite type, which is in \( V_\omega \). In \( V_{\omega+\omega} \) and \( V_{\omega_1} \), the aforementioned set of all well-orderings of \( \omega \) has no type.

6. If \( V_\beta \) contains a well-ordering for a set \( x \), then \( V_{\beta+4} \) contains a well-ordering of type \( \Gamma(x) \) (namely the well-ordering of all the well-orderings (modulo order-isomorphism) of \( x \)). By induction on \( \beta \), this implies that for all \( \beta, V_{\omega+4, \beta+1} \) contains a well-ordering of type \( \omega_\beta \). Thus, \( \alpha \) must satisfy \( \forall \beta (\omega_4 + 4 \cdot \beta < \alpha \rightarrow \omega_\beta < \alpha) \). It follows that \( \alpha \) is an initial and \( \omega_\alpha = \alpha \). The smallest \( \alpha \) which satisfies this is \( \alpha = \bigcup \{\omega, \omega_\omega, \omega_{\omega_\omega}, \ldots\} \). Note that \( \alpha \) might need to satisfy other contraints as well, so this is just a lower bound.

If we assume AC, then we also have that for all \( \beta < \alpha, V_\beta \) has a well-ordering with type in \( \alpha \), so \( |V_\beta| < |\alpha| \). It follows that \( \alpha = |V_\alpha| \). The smallest \( \alpha > \omega \) which satisfies this is \( \alpha = \bigcup \{\omega + 1, |V_{\omega+1}|, |V_{\omega_\omega+1}|, \ldots\} \). For this \( \alpha \), if \( x \in V_\alpha \), then \( x \in V_\beta \) for some \( \beta < \alpha \), and then \( |x| \leq |V_\beta| < |V_\alpha| = \alpha \), thus any well-orderings of \( x \) have type in \( V_\alpha \): it follows that this is not merely a lower bound but also a valid choice for \( \alpha \).

Note that under GCH, these two choices for (the lower bound of) \( \alpha \) are of equal value.