Probability Theory and Related Fields manuscript No.
(will be inserted by the editor)

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Reinforced Random Walks

September 6, 2002

Abstract. This article is about recurrence in reinforced random walks, where edges in a graph are traversed with probabilities that may be different (reinforced) at second, third etc. traversals. After a brief review of general theory, we focus on the case where the probability for any edge only changes once, after its first traversal. As a special case, we show that the once-reinforced random walk on the infinite ladder is almost surely recurrent if reinforcement is small, extending a result by Thomas Sellke from an at this time unpublished article[7]. Then we adapt these techniques to show that for a class of graphs which generalizes the infinite ladder, recurrence also holds for sufficiently *large* reinforcements.

Key words. random walks – reinforcement

1. Introduction

In the orthodox random walk, the probability of traversing a specific street from a specific intersection is always the same, unaffected by anything that happened before. In this paper, we will study *reinforced* random walks, where traversing a street changes the probabilities for that street. In terms of the Drunkard's Walk example, the drunkard vaguely recognizes streets he has walked before, and is either more likely to traverse them (as he considers them safe) or less likely (as he considers them boring), depending on the conditions of the reinforcement. Reinforced random walks were first introduced by Diaconis and Coppersmith[1], and later generalized by Davis[2] and Pemantle[6]

This paper presumes some knowledge of graph theory and probability theory, although an effort has been made to make it as self-contained as possible.

The paper has the following structure. We start in Section 2 by characterizing recurrence of non-reinforced random walks. The basic concepts and techniques introduced in this section will be used throughout the rest of this paper, as will the results themselves. In Section 3 we will introduce reinforced random walks, and review some of their general properties. Section 4 will focus on a more specific reinforcement scheme, where traversing an edge more than once does not

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I would like to express my gratitude to Gertjan van der Hoeven, Mike Keane, Judith Keijsper and Michiel van Lambalgen for their contributions to this paper. This research was supported by NWO-PIONIER project PGS 22-262.

cause additional reinforcement. Random walks on several classes of graphs will be considered, including the square lattice graph on $\mathbb{Z} \times \{1, \ldots, n\}$. In Section 5, we adapt these techniques to show recurrence on a particular class of graphs for random walks with *large* reinforcements.

2. The Non-Reinforced Random Walk

In this section we will consider the non-reinforced random walk. We will start by reviewing notational conventions, defining the non-reinforced random walk and giving several characterizations of recurrence for this walk. Next we will introduce such basic concepts as martingales, stopping times and harmonic functions, and show how martingales naturally arise from random walks. Then we will characterize recurrence of non-reinforced random walks on graphs in terms of the existence of certain superharmonic functions on the vertices of these graphs, and give several examples of the application of these theorems to specific graphs. Finally, we will use a correspondence with electrical circuit equations to derive some results about recurrence on subgraphs of a known graph. As all but the final results are common to many random walk papers, proofs will be omitted.
The random walks considered in this paper are walks on the edges of weighted graphs with countably infinitely many vertices.

Remark 1. We will assume that any given graph is connected, that there are no 'degenerate' edges of weight 0, and that each vertex has only finitely many neighbors. We will also assume that any given graph is simple (i.e. without loops or parallel edges) unless explicitly stated otherwise. The reader is invited to verify for him- or herself that all definitions, proofs and results in this paper can easily be extended to non-simple graphs. Indeed, the generalization to non-simple graphs of Lemma 11 will be used in the proof of Theorem 8. However, since this extension does not add anything conceptually, and since it is convenient to be able to denote edges and arcs by their endpoints, we will concern ourselves with simple graphs, and postulate generalizations to non-simple graphs when necessary.

Notation. We denote a weighted graph G as $G = (V, E, w)$, where V and E are the sets of vertices and edges of G, and $w : E \to \mathbb{R}_{>0}$ is its weight function. Edges are denoted by their endpoints, as in 'the edge uv'. Note that uv and vu denote the same edge. Whenever the order of the vertices is important (for instance, when we want to indicate the direction in which an edge has been traversed), we use *arcs* (oriented edges), denoted as in 'the arc $\overrightarrow{uv}$', instead of edges. u and v are called the *tail* and *head* of $\overrightarrow{uv}$, respectively.
Some other notational conventions:

– $\#S$ denotes the size of a set S.
– v and u are used for vertices.
– $N_G(v)$ denotes the *neighbor set* of a vertex v in a graph $G = (V, E, w)$, i.e. the set of vertices u such that $uv \in E$.
– $w_G(v)$ denotes the total weight $\sum_{u \in N(v)} w(vu)$ of the edges adjacent to v.
– $\rho_G(v)$ denotes the *degree* of v in G, i.e. the number of adjacent edges.

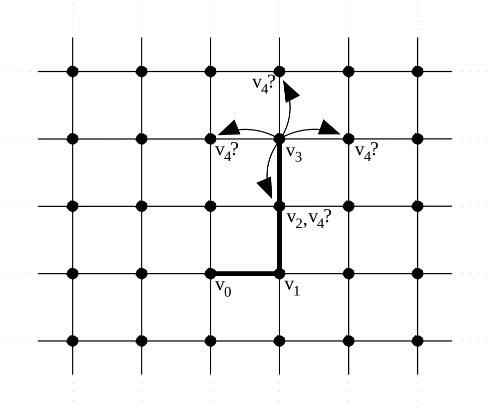

Fig. 1. The first few steps of a random walk on the square lattice on $\mathbb{Z}^2$.

- $d_G(v,u)$ denotes the *distance* in G between the vertex v and the vertex u (i.e. the number of edges contained in the shortest $v-u$ path in G).
- $d_G(v,F)$ denotes the distance in G between a vertex v and a vertex set F.

The index G is omitted when no confusion is possible.

A random walk is a stochastic process of traversing the edges of a graph, where, each time a vertex is reached, the random walk continues over a randomly selected adjacent edge. Specifically, the *non-reinforced random walk* on a graph $G=(V,E,w)$ starting at a vertex $v_0 \in V$, is the following stochastic process:

- We start with the vertex v_0.
- Next, we randomly pick an edge $v_0v_1 \in E$ that connects v_0 with some other vertex $v_1 \in V$. All candidate edges have a probability of being picked proportional to their weight. The random walk is said to *traverse* the edge v_0v_1, and to *visit* the vertex v_1 at *time* 1.
- Next, we randomly pick an edge $v_1v_2 \in E$ that connects v_1 with some other vertex $v_2 \in V$, in the same manner as in the previous step.
- Continuing in this manner, we obtain an infinite path $v_0v_1v_2v_3\ldots$.

More formally,

Definition 1. *A* non-reinforced random walk *on a weighted graph* $G=(V,E,w)$ *is a series of stochastic variables* $v_0,v_1,\ldots \in V$ *such that for any* time $t \in \mathbb{N}$,

$$P(v_{t+1}=u \mid \mathcal{F}_t) = \begin{cases} \frac{w(v_t u)}{w(v_t)} & \textit{if } u \in N(v_t) \\ 0 & \textit{otherwise} \end{cases} \tag{1}$$

where $\mathcal{F}_t$ *denotes the* σ*-algebra of the history up to time* t*. Note that by our assumptions on graphs,* $N(v) \neq \emptyset$ *for all* $v \in V$*.*

Notation. v_t always denotes the location of the random walk at time t. Sometimes we write $v_0v_1v_2\ldots$ for the random walk itself. Throughout this paper s and t are used for (integer) times, and the use of t as a subscript indicates a (stochastic) variable whose contents changes over time (such as v_t).

Definition 2. *A realization of a random walk is said to be* recurrent *if every vertex is visited infinitely often, and* transient *if every vertex is visited only finitely many times.*

The question we are mainly concerned with in this paper, is under what conditions a random walk is recurrent almost surely (i.e. with probability 1). For non-reinforced random walks we have the following observations:

Lemma 1. *Let $G{=}(V, E, w)$ be a weighted graph, and consider the non-reinforced random walk on G starting in a vertex v_0. Then, depending on G, the random walk is either almost surely recurrent or almost surely transient.*

Lemma 2. *Let $G = (V, E, w)$ be a weighted graph, $F \subset V$ a finite set of vertices of G, and $v \in F$. Then the following are equivalent:*

(i) Any non-reinforced random walk on G is almost surely recurrent.
(ii) The non-reinforced random walk on G starting in v is almost surely recurrent.
(iii) The non-reinforced random walk on G starting in v returns to v almost surely.
(iv) Any non-reinforced random walk on G visits F almost surely.

Our main tools for showing recurrence of random walks will be the concept of *martingales* and the Optional Stopping Theorem.

Definition 3. *A series of stochastic variables $(M_t)_{t\in\mathbb{N}}$ is called a* martingale *if for all $t \in \mathbb{N}$,*

$$M_t = E(M_{t+1} \mid \mathcal{F}_t) \tag{2}$$

where $E(M_{t+1} \mid \mathcal{F}_t)$ denotes the expectation, at time t, of the value of M_{t+1}.

Definition 4. *A series of stochastic variables $(M_t)_{t\in\mathbb{N}}$ is called a* supermartingale *[*submartingale*] if for all $t \in \mathbb{N}$,*

$$M_t \geq [\leq]\ E(M_{t+1} \mid \mathcal{F}_t) \tag{3}$$

It is easy to see that if M is a martingale, then $E(M_t) = M_0$ for any time $t \in \mathbb{N}$. The Optional Stopping Theorem for Martingales basically states that the same holds for the expectation of the value of the martingale at times which are defined in terms of states or conditions, such as the first time at which the value of the martingale is < 0 or > 100. To state the theorem, we need the concept of *stopping times*.

Definition 5. *A* stopping time *is a stochastic variable τ, taking values in $\mathbb{N} \cup \{\infty\}$, such that for all $t \in \mathbb{N}$, the history up to time t completely determines whether $\tau = t$ or not.*

Stopping times are usually defined in the manner of 'let τ be the first time at which some condition holds'. Often we are only interested in the course of a random walk up to a certain event, such as its first visit to some given vertex v. In that case we write 'the random walk which stops at time τ', 'the random walk which stops as soon as some condition holds', or even 'the random walk which stops at v'.

Theorem 1 (Optional Stopping Theorem for Martingales). *Let M_t be a martingale [supermartingale, submartingale] and τ a stopping time such that $\tau < \infty$ almost surely. If M_t is bounded [bounded from below, bounded from above] for $t < \tau$, then*

$$M_0 = [\geq, \leq] E(M_\tau) \tag{4}$$

and more generally

$$M_{t_0} = [\geq, \leq] E(M_\tau \mid \mathcal{F}_{t_0}) \tag{5}$$

if $t_0 \leq \tau$.

Kakutani[5] found that random walks give rise to martingales naturally, if we can find a function on the vertex set of the graph with the property of *harmonicity*:

Definition 6. *Let $G = (V, E, w)$ be a weighted graph, and let $h : V \to \mathbb{R}$ be a function. We say that h is* harmonic *[*superharmonic, subharmonic*] on a vertexset $V' \subset V$ if for all $v \in V'$,*

$$h(v) = [\geq, \leq] \sum_{u \in N(v)} h(u) \frac{w(vu)}{w(v)} \tag{6}$$

or, equivalently,

$$\sum_{u \in N(v)} w(vu) \Delta_h(\overrightarrow{vu}) = [\leq, \geq]\ 0 \tag{7}$$

where $\Delta_h(\overrightarrow{vu})$ denotes $h(u) - h(v)$.

Lemma 3. *Let $G = (V, E, w)$ be a weighted graph, and let $h : V \to \mathbb{R}$ be a harmonic [superharmonic,subharmonic] function on a subset $V' \subset V$. Consider a non-reinforced random walk on G, and define*

$$M_t = \sum_{t'=0}^{t} \begin{cases} \Delta_h(\overrightarrow{v_{t'} v_{t'+1}}) & \textit{if } v_{t'} \in V' \\ 0 & \textit{otherwise} \end{cases} \tag{8}$$

for $t \in \mathbb{N}$. Then M is a martingale [supermartingale, submartingale]. Furthermore, as long as $V - V'$ has not yet been visited,

$$M_t = h(v_t) - h(v_0) \tag{9}$$

More about martingales may be found in [3].
Now, if h is a superharmonic function on (a subset of) the vertex set V of a graph G then the Optional Stopping Theorem for Martingales places bounds on the expected values of $h(v_t)$. We can use this to characterize recurrence of random walks in terms of the existence of superharmonic functions with certain properties.

Definition 7. *Let $G = (V, E, w)$ be a weighted graph, and let $h : V \to \mathbb{R}$ be a function. We say that $h(v)$* goes to infinity if v goes to infinity *if*

$$\forall r \in \mathbb{R}\ \exists n \in \mathbb{N}\ \forall v \in V\ (d_G(v_0, v) > n \ \Rightarrow\ h(v) > r) \tag{10}$$

For the graphs we are concerned about, in which no vertex has infinitely many neighbors, this is equivalent to the condition that

$$\textit{for each } r \in \mathbb{R}, \{v \in V \mid h(v) < r\} \textit{ is finite} \tag{11}$$

Theorem 2. *Let $G = (V, E, w)$ be a weighted graph. Then non-reinforced random walks on G are almost surely recurrent if there exists a function $h : V \to \mathbb{R}$ satisfying*

1. *h is superharmonic everywhere except on some finite set F.*
2. *$h(v)$ goes to infinity if v goes to infinity.*

Conversely, if non-reinforced random walks on G are almost surely recurrent, then a function h as above exists, and F may be chosen to be an arbitrary non-empty finite set.

Example 1. The random walk on the square lattice graph on $\mathbb{Z}^2$ with unit weights is almost surely recurrent.

Proof
Let $h : \mathbb{Z}^2 \to \mathbb{R}$ be defined by

$$h(x,y) = \begin{cases} \log(1/12) & \text{if } (x,y) = (0,0) \\ \log(1/4) & \text{if } (x,y) = (0,\pm 1) \text{ or } (x,y) = (\pm 1, 0) \\ \log(x^2+y^2-1) & \text{otherwise} \end{cases} \tag{12}$$

Then h satisfies the conditions of theorem 2, with $F = \{(0,0)\}$.

□

Example 2. For any $n \in \mathbb{N}_{>0}$, the random walks on the square lattice graphs on $\mathbb{Z} \times \{1, \ldots, n\}$ and $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$ with unit weights are almost surely recurrent.

Proof
Let $h : (\mathbb{Z} \times \{1 \ldots n\}) \to \mathbb{R}$ be defined by

$$h(x,y) = |x| \tag{13}$$

Then h satisfies the conditions of theorem 2, with $F = \{(0,y) \mid 1 \leq y \leq n\}$. The proof for the cylinder lattice $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$ is completely analogous.

□

Interestingly enough, the non-recurrence of random walks on a graph can *also* be characterized in terms of the existence of certain superharmonic functions.

Theorem 3. *Let $G = (V, E, w)$ be a weighted graph. Then non-reinforced random walks on G are not almost surely recurrent if and only if there exists a bounded non-constant function $h : V \to \mathbb{R}$ that is superharmonic on V.*

Example 3. The random walk on the cubic lattice graph on $\mathbb{Z}^3$ is not almost surely recurrent.

Proof
Let $h : \mathbb{Z}^3 \to [0, 6^{-1/2}]$ be defined by

$$h(x,y,z) = \frac{1}{(x^2+y^2+z^2+6)^{1/2}} \tag{14}$$

Using a truncated Taylor Series expansion of h, we can show that for all $x, y, z \in \mathbb{Z}$,

$$h(x+1, y, z) + h(x-1, y, z) \leq 2h(x, y, z) + \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2 + 6)^{5/2}} \tag{15}$$

Analogous inequalities hold for $h(x, y+1, z) + h(x, y-1, z)$ and $h(x, y, z+1) + h(x, y, z-1)$. Taking the sum of these inequalities yields the superharmonicity inequality.

□

Example 4. Let $G = (V, E, w)$ be a weighted graph with $V = \{v^n \mid n \in \mathbb{Z}\}$, and $E = \{v^n v^{n+1} \mid n \in \mathbb{Z}\}$.[1] Then random walks on G are almost surely recurrent if and only if $\sum_{n=-\infty}^{-1}(1/w(v^n v^{n+1}))$ and $\sum_{n=0}^{\infty}(1/w(v^n v^{n+1}))$ both diverge.

Proof
If $\sum_{n=0}^{\infty}(1/w(v^n v^{n+1}))$ converges to $c \in \mathbb{R}$, then define $h : V \to [0, c]$ by setting $h(v^n) = \max(c, \sum_{k=n}^{\infty}(1/w(v^k v^{k+1})))$ for $n \in \mathbb{Z}$. It is easily verified that h is non-constant, harmonic on $V - \{v^0\}$ and superharmonic on $\{v^0\}$, fulfilling the conditions of Theorem 3. Likewise for the case that $\sum_{n=-\infty}^{-1}(1/w(v^n v^{n+1}))$ converges.
Now suppose $\sum_{n=-\infty}^{-1}(1/w(v^n v^{n+1}))$ and $\sum_{n=0}^{\infty}(1/w(v^n v^{n+1}))$ both diverge to ∞. Then define $h : V \to \mathbb{R}_{\geq 0}$ by setting $h(v^n) = \sum_{k=0}^{n-1}(1/w(v^k v^{k+1}))$ for $n \geq 0$ and $h(v^n) = \sum_{k=-n}^{-1}(1/w(v^k v^{k+1}))$ for $n < 0$. Again it is easily verified that h is non-constant, $h(v^n) \to \infty$ if $n \to \infty$ or $n \to -\infty$, and h is harmonic on $V - \{v^0\}$, fulfilling the conditions of Theorem 2

□

The harmonicity equations are the same as the equations governing electrical current and potential levels in networks of resistors. This is not surprising, since electrical current can be viewed as a statistical universe's worth of electrons traversing a network randomly. This connection was explored by Doyle and Snell [4]:

Lemma 4. *Let $G = (V, E, w)$ be a weighted graph, let $F \subset V$ be a nonempty set of stopping positions, and consider the non-reinforced random walk on G starting in a vertex v_0, and stopping as soon as a vertex in F is reached. Define the function $H : V \to \mathbb{R}$ by*

$$H(v) = \frac{1}{w(v)} E(\#\{t < \tau \mid v_t = v\}) \tag{16}$$

where τ denotes the stopping time of the walk. Then H is well-defined, and satisfies

$$\sum_{u \in N(v)} w(vu)(H(v) - H(u)) = \begin{cases} 1 \text{ if } v = v_0 \\ 0 \text{ if } v \neq v_0 \end{cases} \tag{17}$$

on $V - F$, i.e. H is harmonic on $V - F - \{v_0\}$ and superharmonic on $\{v_0\}$.
Now, we can convert G to an electrical circuit, with each edge e corresponding to a resistor of resistance $1/w(e)$, and electrical poles connected to v_0 and F. If we

[1] The superscript index v^n is used here to avoid confusion with the temporal index v_t.

apply sufficient potential difference to generate a current of 1 unit, then at each node $v \in V$, the potential difference with F will be equal to $H(v)$.
As a consequence, $H(v_0)$ is the maximum of H. Furthermore, since $H(v_0)$ is the potential difference between v_0 and F necessary to generate a current of 1 unit, it is also equal to the resistance of the electrical circuit between v and F. Finally, for each edge vu, the current flowing through vu is at most 1 (the total current flowing through the circuit), and thus the potential difference $|H(v) - H(u)|$ is at most $1/w(vu)$.

Proof
See Doyle and Snell [4]. Note that for each vertex v there is at least one path from v to F, which will be traversed with positive probability each time v is visited, and therefore the expected number of visits to v will be finite and $H(v)$ is well-defined.
□

For random walks without stopping positions, there is no direct correspondence with electrical networks, but we have the following result, also by Doyle and Snell:

Lemma 5. *Let G=(V, E, w) be a weighted graph, and consider the non-reinforced random walk on G starting in a vertex v_0. For $n \in \mathbb{N}$, let $F_n \subset V$ be the set of vertices at distance n from v_0, and let H_n be the corresponding function from Lemma 4. Then the non-reinforced random walk on G is recurrent if and only if $\lim_{n\to\infty} H_n(v_0) = \infty$.*

Now taking a subgraph corresponds to removing some connections in the electrical circuit and increasing the resistance of other connections, both of which increase the electrical resistance of the circuit as a whole. Hence, this perspective yields the following results:

Corollary 1. *Let $G = (V, E, w)$ be a weighted graph, let $G' = (V', E', w')$ be a subgraph (i.e. $V' \subset V$, $E' \subset E$ and for all $e \in E'$, $w'(e) \le w(e)$), and let $F \subset V'$ be a nonempty set of stopping positions. Consider the non-reinforced random walks on G and G' starting in a vertex $v_0 \in V'$, and stopping as soon as a vertex in F is reached. Let H and H' be the corresponding functions from Lemma 4. Then $H(v_0) \le H'(v_0)$.*

Theorem 4. *Let $G = (V, E, w)$ be a weighted graph, such that non-reinforced random walks on G are almost surely recurrent. If $G' = (V', E', w')$ is a subgraph of G, then random walks on G' are almost surely recurrent.*

3. Reinforced Random Walks

In this section, we will introduce the concept of reinforced random walks. We will compare them with non-reinforced random walks, give analogs of results and techniques from the previous section, and show that under some very general conditions reinforced random walks on trees are almost surely recurrent. Finally we will give a sufficient condition for recurrence of reinforced random walks on general graphs, which we will use in later sections.

In a reinforced random walk, when an edge has been traversed we change the probability that it will be traversed again, by increasing or decreasing the *weight* of the edge. In general reinforced random walks, the new weight may depend on many things, such as the edge in question, the number of times it has been traversed before, the time of traversal and the pattern formed by edges traversed at previous times, etc. etc. Davis [2] defines the category of reinforced random walks of *matrix* type, where for each edge vu, the current weight of vu is determined solely by the number of times $k_t(vu)$ it has been traversed up to then, and is not influenced by anything that has happened to any other edge. Note that in general walks of matrix type, the relationship between current weight and number of traversals may be different for each edge. In this paper we concern ourselves with a specific subclass of walks of matrix type, where a sequence $(\delta_k)_{k\in\mathbb{N}}$ is given which is the same for all edges, and the current weight of an edge at any given time is determined by multiplying its original weight by $\delta_{k_t(vu)}$. A formal definition:

Definition 8. *Let $(\delta_k)_{k\in\mathbb{N}}$ be a sequence of strictly positive real numbers, Set the* weight of vu at time t *to*

$$w_t(vu) = \delta_{k_t(vu)} w(vu) \tag{18}$$

where $k_t(vu)$ denotes the number of traversals of vu up to time t, i.e.

$$k_t(vu) = \#\left\{t' < t \mid v_{t'}v_{t'+1} = vu\right\} \tag{19}$$

A reinforced random walk *on a graph $G = (V, E, w)$ with reinforcement sequence $(\delta_k)_{k\in\mathbb{N}}$, is a series of stochastic variables $v_0, v_1, \ldots \in V$ such that for all $t \in \mathbb{N}$,*

$$P(v_{t+1} = u \mid \mathcal{F}_t) = \begin{cases} \frac{w_t(v_t u)}{w_t(v_t)} & \text{if } u \in N(v_t) \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

Recurrence and transience are defined in the same manner as before.

Remark 2. The random walks defined above are similar but not identical to Davis' random walks of *sequence* type, where the current weight of an edge vu is defined as $w_t(vu) = w_0(vu) + \delta_{k_t(vu)}$ for some non-descending sequence $(\delta_k)_{k\in\mathbb{N}}$ [2]. Davis gave many results for walks of this type on the linear lattice $\mathbb{Z}$, most of which also hold for the random walks defined above. In this paper, we focus on more general classes of graphs.

There are a number of differences between a reinforced and a non-reinforced random walk. For instance, a reinforced random walk is influenced by its history, and hence we might want to consider random walks with initial states in which some edges are considered to have been traversed already. Another difference is that if the reinforcement increases sharply enough, the random walk might get 'stuck' on an edge:

Theorem 5. *Let $G = (V, E, w)$ be a weighted graph, and let $(\delta_k)_{k\in\mathbb{N}}$ be such that*

$$\sum_{k=0}^{\infty} \frac{1}{\delta_k} \textit{ converges.} \tag{21}$$

Then for any edge $vu \in E$, and all random walks on G, there exists a $t_0 \geq 0$ such that

$$P(\forall t > t_0 : v_t \in \{v, u\}) > 0 \tag{22}$$

Proof

Since G is connected, every point is reachable, and hence there exists a $t_0 \in \mathbb{N}$ such that with non-zero probability v is visited at time t_0. Assume that it has. Then the probability that from time t_0 on, the random walk will keep traveling from v to u and back again, is

$$\prod_{i=0}^{\infty} \frac{w_{t_0+i}(vu)}{w_{t_0+i}(v_{t_0+i})} \tag{23}$$

$$\geq \prod_{i=0}^{\infty} \frac{\delta_{k_{t_0}(vu)+i} w(vu)}{c + \delta_{k_{t_0}(vu)+i} w(vu)} \tag{24}$$

$$\geq \prod_{k=k_{t_0}(vu)}^{\infty} e^{-c/(\delta_k w(vu))} \tag{25}$$

$$= e^{-c/w(vu)\cdot \sum_{k=k_{t_0}(vu)}^{\infty} (1/\delta_k)} \tag{26}$$

$$> 0 \tag{27}$$

where c is the total weight assigned at time t to edges other than vu that are incident with v or u.

□

The converse implication, that if $\sum_{k\in\mathbb{N}} \delta_k$ diverges, the random walk will almost surely not get 'stuck', does not hold in general.[2]However, it *does* hold for non-descending sequences, and for general sequences it is possible to come close, as the following analogs of Lemma's 1 and 2 show:

Lemma 6. *Let $G = (V, E, w)$ be a weighted graph, and let $(\delta_k)_{k\in\mathbb{N}}$ be such that*

$$\sum_{j=0}^{\infty} \frac{1}{\max(\delta_0, \delta_1, \ldots, \delta_j)} \textit{ diverges.} \tag{28}$$

Then a reinforced random walk on G starting from any initial state will almost surely visit infinitely many vertices, and

$$P(\textit{the walk is transient}) + P(\textit{the walk is recurrent}) = 1 \tag{29}$$

[2] For instance, if G is a tree with unit weights on which non-reinforced random walks are almost surely recurrent, then it can be shown that the reinforced random walk on G with reinforcement sequence $(\delta_k)_{k\in\mathbb{N}} = (1, 2, 1, 4, 1, 8, 1, 16, \ldots)$ starting from a vertex v_0 almost surely eventually stays within $\{v_0\} \cup N(v_0)$.

Proof
The first assertion follows from the second, since both transient and recurrent walks visit infinitely many vertices. To prove the second assertion it suffices to show that for all $v, u \in V$

$$P(v \text{ is visited infinitely often and } u \text{ only finitely often}) = 0 \tag{30}$$

If we can show that the above holds for vertices $v, u \in V$ with $vu \in E$, then the general result follows by induction on the distance $d_G(v, u)$. So let $v, u \in V$ with $vu \in E$. Fix $t_0 \in \mathbb{N}$, and suppose that u has not been visited since time $t_0 \in \mathbb{N}$, and at some time $t > t_0$ v is visited again for the k-th time. Then $w_t(v) \leq w(v) \max\{\delta_0, \dots, \delta_{2k}\}$, and since vu has been traversed at most t_0 times, $w_t(vu) \geq w(vu) \min\{\delta_0, \dots, \delta_{t_0}\}$. Hence, the probability of *not* immediately traversing vu in this situation is at most

$$1 - c/\max\{\delta_0, \dots, \delta_{2k}\} < e^{-c/\max\{\delta_0,\dots,\delta_{2k}\}} \tag{31}$$

where $c = w(vu) \min\{\delta_0, \delta_1, \dots, \delta_{t_0}\}/w(v)$.
Therefore, applying induction on k, we have that for all $k \geq 1$,

$$P(u \text{ is not visited between } t_0 \text{ and the } k+1\text{-th visit to } v) \tag{32}$$

$$\leq \prod_{k'=1}^{k} e^{-c/\max(\delta_0,\dots,\delta_{2k'})} \tag{33}$$

$$= e^{-c\cdot\sum_{k'=1}^{k}(1/\max(\delta_0,\dots,\delta_{2k'}))} \tag{34}$$

Consequently

$$P(v \text{ is visited infinitely often and } u \text{ never after time } t_0)$$

$$\leq e^{-c\sum_{k=1}^{\infty}(1/\max(\delta_0,\dots,\delta_{2k}))} \tag{35}$$

$$= 0 \tag{36}$$

Summing over all times $t_0 \in \mathbb{N}$ gives the desired result.

□

Lemma 7. *Let $G = (V, E, w)$ be a weighted graph, $F \subset V$ a finite set of vertices of G, and $v_0 \in V$. Let $(\delta_k)_{k \in \mathbb{N}}$ be such that equation (28) holds. Then for the reinforced random walk on G starting from v_0, the following are equivalent:*

(i) The reinforced random walk on G starting from v_0 is almost surely recurrent.
(ii) For any $t_0 \in \mathbb{N}$, and any history up to time t_0, F will be (re)visited at some time at or after time t_0 almost surely.

Proof
$(i) \Rightarrow (ii)$ is trivial. If (ii) holds, then by applying it repeatedly we find that the reinforced random walk on G starting from v_0 will almost surely visit F infinitely often. Then the random walk is almost surely not transient, and by the previous Lemma, this implies it is almost surely recurrent.

□

Remark 3. In condition (ii) of Lemma 7, conceptually we *restart* the walk at time t_0. i.e. we look at a walk which starts at time t_0, with t_0 traversals part of a 'fixed' history up to time t_0 (as opposed to starting at time 0 with a blank initial state). If all such restarted walks can be shown to visit F almost surely, Lemma 7 states that the original reinforced random walk is almost surely recurrent.

Lemma 8. *For random walks on weighted trees, the direction in which an edge is traversed is the same at all odd-numbered traversals (and opposite to the direction of traversal at all even-numbered traversals). This allows us to replace, for reinforced random walks on weighted trees, the condition of Lemma's 6 and 7 by the condition that*

$$\sum_{k=0}^{\infty}(1/\delta_{2k}) \textit{ and } \sum_{k=0}^{\infty}(1/\delta_{2k+1}) \textit{ both diverge.} \tag{37}$$

Proof
Consider a random walk on a weighted tree $G = (V, E, w)$, and assume that equation (37) holds. In order to show that the conclusions of Lemma's 6 and 7 hold, it suffices to show that for all vertices $v, u \in V$ with $vu \in E$,

$$P(v \text{ is visited infinitely often and } u \text{ only finitely often}) = 0 \tag{38}$$

So let $v \in V$, and let $u^0, u^1, \ldots, u^m$ be the neighbors of v in G, with u^0 being the unique neighbor of v that is on a path between v and v_0 if $v \neq v_0$. Set, for $i \leq m, k \in I\!N$,

$$R^i_k = \delta_{2k+1} w(vu^i) \text{ if } i = 0 \text{ and } v \neq v_0,\ R^i_k = \delta_{2k} w(vu^i) \text{ otherwise} \tag{39}$$

Then R^i_k is the weight of the edge vu^i if v is visited and the arc vu^i has been traversed (in that direction) k times before.
The next part of the proof is based on a proof of Herman Rubin concerning a generalized Polya Urn problem [2]. Let Y^i_k be independent exponential random variables such that $E(Y^i_k) = 1/R^i_k$,[3] and put

$$A^i = \left\{ \sum_{k'=0}^{k} Y^i_{k'}, k' \geq 0 \right\} \text{ for } i \leq m \tag{40}$$

□

Define a sequence of edges vu^i by making the k-th element of the sequence vu^i if the k-th smallest element of $A_0 \cup \ldots \cup A_m$ is from A_i. Now since by equation (37)

$$\text{for all } i \leq m, \sum_{k=0}^{\infty} \frac{1}{R^i_k} \text{ diverges.} \tag{41}$$

we have that almost surely

$$\text{for all } i \leq m, \sum_{k=0}^{\infty} Y^i_{k'} \text{ diverges.} \tag{42}$$

[3] I.e. the probability distribution of Y^i_k is given by $P(Y^i_k > r) = e^{-rR^i_k}$ for all $r \in I\!R$

and hence almost surely vu^i will appear infinitely often in the sequence for all $i \leq m$.
As it turns out, this sequence has exactly the same probability distribution as the sequence of edges traversed from v in the reinforced random walk. In other words, we may decide that at visits to v we traverse successive arcs of the sequence, without changing any probabilities. The proof of this relies on properties of exponential random variables, and is straightforward but cumbersome. Interested readers are referred to Rubin's proof [2]. We conclude that equation (38) holds.

□

Now let us consider recurrence for reinforced random walks. The proofs given in the previous section used the fact that, if a function h on the vertexset of a weighted graph G is harmonic, then in a *non-reinforced* random walk, $h(v_t)$ behaves like a martingale. This does not in general hold for *reinforced* random walks. If h is a harmonic function, then a vertex which has neighbors with higher h-values will also have neighbors with lower h-values, but the probabilities of the corresponding edges being traversed are not necessarily balanced, or even constant over time. In order to find an analog of Lemma 3, we will need to compensate for the difference in probabilities.

Lemma 9. *Let $G = (V, E, w)$ be a weighted graph, and let $h : V \to \mathbb{R}$ be a harmonic [superharmonic, subharmonic] function on a subset $V' \subset V$. Consider the reinforced random walk with reinforcement sequence $(\delta_k)_{k \in \mathbb{N}}$ and define*

$$M_t = \sum_{t'=0}^{t} \begin{cases} \frac{\Delta_h(\overrightarrow{v_{t'} v_{t'+1}})}{\delta_{k_{t'}(v_{t'} v_{t'+1})}} & \text{if } v_{t'} \in V' \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

for $t \in \mathbb{N}$, where (as before) $\Delta_h(\overrightarrow{vu})$ denotes $h(u) - h(v)$. Then M is a martingale [supermartingale,submartingale].

Proof
If $v_t \in V - V'$, then $M_{t+1} = M_t$, otherwise

$$M_t = [\geq, \leq] \quad M_t + \frac{1}{w_t(v_t)} \cdot \sum_{u \in N(v_t)} w(v_t u) \Delta_h(\overrightarrow{v_t u}) \tag{44}$$

$$= M_t + \sum_{u \in N(v_t)} \frac{w_t(v_t u)}{w_t(v_t)} \cdot \frac{\Delta_h(\overrightarrow{v_t u})}{\delta_{k_t(v_t u)}} \tag{45}$$

$$= M_t + \sum_{u \in N(v_t)} P(v_{t+1} = u \mid \mathcal{F}_t) \frac{\Delta_h(\overrightarrow{v_t u})}{\delta_{k_t(v_t u)}} \tag{46}$$

$$= E(M_{t+1} \mid \mathcal{F}_t) \tag{47}$$

□

As an application of the above martingale, we will show that if non-reinforced random walks on a weighted tree are almost surely recurrent, then for reinforced random walks on that tree, a very weak condition on the reinforcement sequence suffices to show recurrence.

Theorem 6. *Let $G = (V, E, w)$ be a weighted tree, with the property that non-reinforced random walks on G are almost surely recurrent. Let $(\delta_k)_{k\in\mathbb{N}}$ be a non-descending reinforcement sequence that satisfies the condition of Lemma 7 (or that of Lemma 8). Furthermore, assume either that $(\delta_k)_{k\in\mathbb{N}}$ is bounded, or that $\delta_{k+1} > \delta_k$ for some even $k \in \mathbb{N}$. Then the reinforced random walk with reinforcement sequence $(\delta_k)_{k\in\mathbb{N}}$ is almost surely recurrent.*

Proof

Consider a reinforced random walk on G starting from some vertex $v_0 \in V$. By Lemma 7 (or Lemma 8), to show recurrence, it suffices to show for all $t_0 \in \mathbb{N}$, and any history up to time t_0, that v_0 will be revisited almost surely at some time at or after time t_0. So let $t_0 \in \mathbb{N}$, and fix the history up to time t_0.

First, we need a function h on V that is superharmonic on $V - \{v_0\}$. Since non-reinforced random walks on G are almost surely recurrent, such a function h exists by Theorem 2. For $r \in \mathbb{R}$, define the stopping time τ_r as the first time $t \geq t_0$ at which $v_t = v_0$ or $h(v_t) > r$. By Lemma 6, the random walk will almost surely leave the finite set of vertices $\{v \in V \mid h(v) \leq r\}$. Hence $\tau_r < \infty$ almost surely.

Next, let M_t be the martingale of Lemma 9. For walks on weighted trees, the direction of traversal of an edge is the same for all odd-numbered traversals, and opposite to the direction for all even-numbered traversals. Furthermore, all odd-numbered traversals are traversals going from the lower to higher h-value, for otherwise it would be possible to construct an infinite sequence of vertices of decreasing h-value, which would contradict the fact that $h \to \infty$ if $v \to \infty$. Hence, an edge vu which has been traversed k times at time t contributes

$$|\varDelta_h(\overrightarrow{vu})| \cdot \sum_{j=0}^{k-1} \begin{cases} 1/\delta_j & \text{if } j \text{ is even and } vu \text{ is not incident with } v_0 \\ 0 & \text{if } j \text{ is even and } vu \text{ is incident with } v_0 \\ -1/\delta_j & \text{if } j \text{ is odd} \end{cases} \tag{48}$$

to the value of the martingale. Now by the conditions on $(\delta_k)_{k\in\mathbb{N}}$, there exists a $c > 0$ such that either $1/\delta_k > c$ for all $k \in \mathbb{N}$, or $1/\delta_k - 1/\delta_{k+1} > c$ for some even $k \in \mathbb{N}$. We can use either property, together with the monotonicity of $(\delta_k)_{k\in\mathbb{N}}$, to obtain the following lower bound on the above contribution:

$$|\varDelta_h(\overrightarrow{vu})| \cdot \left(\begin{cases} c \text{ if } k \text{ is odd} \\ 0 \text{ if } k \text{ is even} \end{cases} - \begin{cases} \lceil k/2 \rceil/\delta_1 & \text{if } vu \text{ is incident with } v_0 \\ 0 & \text{otherwise} \end{cases} \right) \tag{49}$$

At any time t, the edges of G that have been traversed an odd number of times are *exactly* the edges of the unique path in G between v_0 and v_t. Furthermore, between times t_0 and τ, there will be no traversals of edges incident with v_0, except for a possible traversal *to* v_0 at time τ. Hence the martingale M_t satisfies

$$M_t \geq c(h(v_t) - h(v_0)) - c' \tag{50}$$

where $c' = \sum_{u\in N(v_0)} |\varDelta_h(\overrightarrow{v_0 u})| \lceil k_{t_0}(v_0 u)/2 \rceil/\delta_1$. Now we can apply the Optional Stopping Times Theorem to obtain

$$M_{t_0} \geq E(M_{\tau_r}) \geq (1 - P(v_{\tau_r} = v_0))c(r - h(v_0)) - c' \tag{51}$$

We conclude that $P(v_{\tau_r} = v_0) \geq 1 - (M_{t_0} + c')/c(r - h(v_0))$ for all $r > h(v_0)$, and hence v_0 is almost surely revisited at some time after time t_0.

$\square$

Remark 4. For the proof of the above theorem, we can weaken the conditions on the reinforcement sequence to the conditions of Lemma 8 and, for some $c > 0$,[4]the inequality

$$\sum_{j=0}^{k-1} (-1)^j/\delta_k > 0 \text{ for } k \text{ even },\ > c \text{ for } k \text{ odd} \tag{52}$$

4. Once-Reinforced Random Walks

In this section we will consider the once-reinforced random walk, where the weight of an edge only changes the *first* time it is traversed, and afterwards remains constant. For this walk, the martingale M_t defined in the previous section can be expressed as $h(v_t)$ plus a certain (bounded) *bias*. If the expectation of the bias is small enough, we will be able to show recurrence in a similar manner as in Section 2.

Definition 9. *Let $\delta > 0$. The* once-reinforced *random walk with* reinforcement factor δ *is the reinforced random walk with reinforcement sequence*

$$(\delta_k)_{k=0}^{\infty} = (1, \delta, \delta, \delta, \delta, \ldots) \tag{53}$$

Definition 10. *Define the stochastic variables E_t and A_t, for $t \in \mathbb{N}$, by setting*

$$E_t = \{v_s v_{s+1} \mid s < t\} \tag{54}$$

$$A_t = \{\overrightarrow{vu} \mid vu \in E_t, \overrightarrow{vu} = \overrightarrow{v_s v_{s+1}} \text{ for } s = \min\{s' < t \mid v_{s'} v_{s'+1} = vu\}\} \tag{55}$$

i.e. E_t is an edgeset containing the edges that have been traversed up to time t, and A_t is an arcset obtained from E_t by orienting each edge in the direction that it was first traversed.

Lemma 10. *In a once-reinforced random walk with reinforcement factor $\delta > 0$, let $t_0 \in \mathbb{N}$, and let M_t be as in Lemma 9 for some function $h : V \to \mathbb{R}$ which is (super/sub)harmonic on $V' \subset V$. Then for $t \geq t_0$,*

$$\delta(M_t - M_{t_0}) = h(v_t) - h(v_{t_0}) + (\delta - 1) \sum_{\overrightarrow{vu} \in A_t - A_{t_0}} \Delta_h(\overrightarrow{vu}) \tag{56}$$

as long as $V - V'$ has not been visited at any time between t_0 and t (including t_0 and excluding t).

[4] Regrettably, the constant $c > 0$ cannot be replaced by 0. A counterexample is given by the reinforced random walk with reinforcement sequence $(\delta_k)_{k \in \mathbb{N}} = (1, 1, 2, 2, 3, 3, \ldots)$ on the linear lattice graph $G = (V, E, w)$ with $V = \{v^n \mid n \in \mathbb{N}\}$, $E = \{v^n v^{n+1} \mid n \in \mathbb{N}\}$ and $w(v^n v^{n+1}) = n + 1$. Starting from v^0, this walk is almost surely transient.

Proof
At time $t = t_0$, the equality holds. If an arc $\overrightarrow{vu}$ is traversed that has been traversed before, then M_t changes by $\Delta_h(\overrightarrow{vu})/\delta$, $h(v_t)$ changes by $\Delta_h(\overrightarrow{vu})$, and A_t does not change, so equality is preserved. If an arc vu is traversed that has not been traversed before, then M_t changes by $\Delta_h(\overrightarrow{vu})$, $h(v_t)$ changes by $\Delta_h(\overrightarrow{vu})$, and $\overrightarrow{vu}$ is added to A_t, so equality is again preserved.

□

Now, in our proof of the recurrence of non-reinforced random walks, a key point was that when we moved farther away from F, the value of the martingale increased as well. Since the expectation of the martingale was bounded, this implied that the probability of reaching a border decreased if we moved the border further away. In order to use similar reasoning here, we will need the bias $(\delta-1)\sum_{\overrightarrow{vu}\in A_t}\Delta_h(\overrightarrow{vu})$ to be positive in the long run, or at least not *too* negative.

Lemma 11. *Let $G = (V, E, w)$ be a weighted graph. Let $h : V \to \mathbb{R}$ be a function satisfying*

1. *h is superharmonic everywhere except on a finite subset $F \subset V$.*
2. *h goes to infinity if v goes to infinity.*

Consider the once-reinforced random walk on G with reinforcement factor δ starting at some vertex v_0. Suppose that for some $\epsilon > 0$, the following holds for any time t_0 and any history up to time t_0:

> *There exists a $c \in \mathbb{R}$ such that for all $r_0 \in \mathbb{R}$ we can find $r > r_0$ with*
>
> $$(\delta-1)E\left(\sum_{\overrightarrow{vu}\in A_{\tau_r}}\Delta_h(\overrightarrow{vu})\,\middle|\,\mathcal{F}_{t_0}\right) \geq -(1-\epsilon)r - c \qquad (57)$$
>
> *(where the stopping time τ_r is the first time at or after t_0 that F is visited or $h(v_t) \geq r$).*

Then the once-reinforced random walk on G with reinforcement factor δ starting at v_0 is almost surely recurrent.

Proof
Without loss of generality we may assume that $h \geq 0$. Note that the reinforcement sequence satisfies the condition of Lemma's 6 and 7. Therefore it suffices to show for all $t_0 \in \mathbb{N}$, and any history up to time t_0, that F will be revisited almost surely at some time at or after time t_0. So let $t_0 \in \mathbb{N}$, and fix the history up to time t_0.
Let M_t be the supermartingale of Lemma 9, and let $r \in \mathbb{R}$. For any $t \leq \tau_r$, the set $A_t - A_{t_0}$ is contained in the finite set $\{\overrightarrow{vu} \in V \mid v \notin F \wedge h(v) < r\}$. So M_t is bounded for $t \leq \tau_r$, and furthermore $\tau_r < \infty$ almost surely by Lemma 6. Hence we can apply the Optional Stopping Times Theorem to obtain $E(\delta M_{\tau_r}) \leq \delta M_{t_0}$, which by Lemma 10 is equivalent to

$$E(h(v_t)) \leq h(v_0) + (\delta-1)\sum_{\overrightarrow{vu}\in A_{t_0}}\Delta_h(\overrightarrow{vu}) - (\delta-1)E\left(\sum_{\overrightarrow{vu}\in A_{\tau_r}}\Delta_h(\overrightarrow{vu})\right) \qquad (58)$$

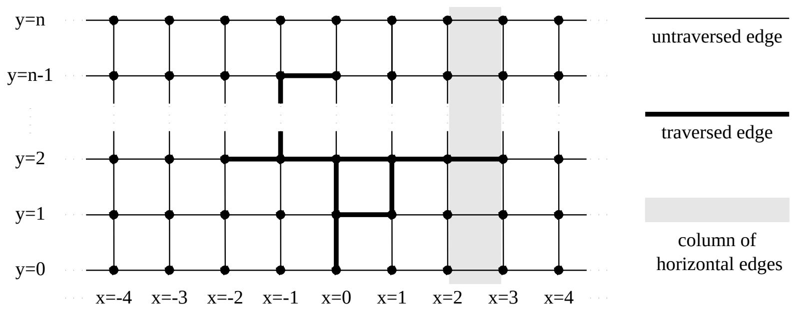

Fig. 2. The square lattice graph on $\mathbb{Z} \times \{1, \ldots, n\}$.

Combining this with the formula $E(h(v_t)) \geq (1 - P(v_{\tau_r} \in F \mid \mathcal{F}_{t_0}))r$, we obtain

$$P(v_{\tau_r} \in F \mid \mathcal{F}_{t_0}) \geq 1 - \frac{h(v_0)}{r} - \frac{\delta - 1}{r} \sum_{\overrightarrow{vu} \in A_{t_0}} \Delta_h(\overrightarrow{vu}) + \frac{\delta - 1}{r} E\left(\sum_{\overrightarrow{vu} \in A_{\tau_r}} \Delta_h(\overrightarrow{vu}) \right) \tag{59}$$

By assumption we can find $c, r \in \mathbb{R}$ such that

$$\frac{\epsilon}{2} r > c + h(v_0) + (\delta - 1) \sum_{\overrightarrow{vu} \in A_{t_0}} \Delta_h(\overrightarrow{vu})) \tag{60}$$

and (57) holds. Then

$$P(v_{\tau_r} \in F \mid \mathcal{F}_{t_0}) \geq 1 - \epsilon/2 + \frac{c}{r} - \frac{(1 - \epsilon)r + c}{r} = \epsilon/2 \tag{61}$$

So there is at least a chance of $\epsilon/2$ of coming back to F at time $t = \tau_r$. In the event that this does not happen, we start over at time $\tau_r + 1$, and each time we have a chance of $\epsilon/2$ of visiting F. It follows that the random walk will visit F almost surely.

□

Next are two applications of this lemma. In both cases, we will write the bias as the sum of 'local' biases in order to estimate it. The first application demonstrates how to use absolute bounds on $\sum_{\overrightarrow{vu} \in A_{\tau_r}} \Delta_h(\overrightarrow{vu})$, to show recurrence for δ close to 1, the second application uses more probabilistic methods.

Theorem 7. *Let $n \geq 1$, and let $G = (V, E, w)$ be the square lattice graph on $\mathbb{Z} \times \{1, \ldots, n\}$ or on $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$. If $1 - \frac{1}{n} < \delta < 1 + \frac{1}{n-2}$ (for $n \geq 3$), or $1 - \frac{1}{n} < \delta$ (for $n = 1, 2$), then the once-reinforced random walk on G with reinforcement factor δ is almost surely recurrent* [5].

Proof

[5] Recurrence for $1 \leq \delta < 1 + \frac{1}{n-2}$ was first proven by Sellke in [7], using different methods.

First assume that G is the square lattice graph on $\mathbb{Z}\times\{1,\ldots,n\}$. With each vertex v of G we can associate coordinates x_v, y_v with $x_v \in \mathbb{Z}, y_v \in \{1,\ldots,n\}$, in the obvious fashion. We may assume that the random walk starts at a point v_0 with $x_{v_0} = 0$. For our superharmonic function h we will use $h(v) = |x_v|$, which is easily seen to be harmonic everywhere except on the finite set $F = \{v \in V \mid x_v = 0\}$.

With this function h, the only edges that contribute to the bias are horizontal edges. For any $c \in \mathbb{Z}$, consider the *column* C_c of n horizontal edges connecting points v with $x_v = c$ to points u with $x_u = c+1$. We need to estimate the number of edges of this column that, at first traversal, are traversed going from the lower to the higher h-value. This number is obviously at most n, and unless the column has not been traversed at all, it is at least 1 (since the random walk cannot reach the side of the column with higher h-values without crossing the column at least once). Similarly, the number of edges that, at first traversal, are traversed going from the higher to the lower h-value, is at least 0 and at most $n-1$. So the contribution of the column to the bias satisfies

$$(\delta-1)\sum_{\overrightarrow{vu}\in A_t, vu\in C_c} \Delta_h(\overrightarrow{vu}) \geq \begin{cases} (\delta-1)\max(0,n-2) & \text{if } \delta \geq 1 \\ (1-\delta)n & \text{if } \delta < 1 \end{cases} = -(1-\epsilon) \quad (62)$$

where $\epsilon = 1-(\delta-1)\max(0,n-2) > 0$ if $1 \leq \delta < 1 + 1/\max(0,n-2)$ and $\epsilon = 1-(1-\delta)n > 0$ if $1-1/n < \delta < 1$.

Now for any t_0 and any $r > t_0/\epsilon$, if τ_r is the first time at or after t_0 that F is visited or $h(v_t) > r$, then the horizontal edges in A_{τ_r} are all contained in the $r+t_0$ columns with x-coordinates between $-t_0$ and r (in the case that $x_{v_{t_0}} > 0$) or between $-r$ and t_0 (in the case that $x_{v_{t_0}} > 0$). Summing all columns, we obtain

$$(\delta-1)\sum_{\overrightarrow{vu}\in A_\tau} \Delta_h(\overrightarrow{vu}) \geq -(1-\epsilon)r - (1-\epsilon)t_0 \quad (63)$$

Hence the conditions of Lemma 11 are satisfied, and the reinforced random walk is almost surely recurrent.

The proof for the square lattice graph on the cylinder $\mathbb{Z}\times(\mathbb{Z}/n\mathbb{Z})$ is identical. □

Remark 5. Of course, using the absolute bound on $\sum\{\Delta_h(\overrightarrow{vu}) \mid \overrightarrow{vu} \in A_{\tau_r}\}$ is a very unsophisticated method of obtaining a bound on the *expected* value of the bias. In the above case, we could improve the bounds on the expected value of the bias with a few simple probabilistic calculations, resulting in

Corollary 2. *Let $n \geq 1$, and let $G = (V,E,w)$ be the square lattice graph on $\mathbb{Z}\times\{1,\ldots,n\}$ or on $\mathbb{Z}\times(\mathbb{Z}/n\mathbb{Z})$. Then there exist $\delta_{min}, \delta_{max} > 0$ with $\delta_{min} < 1-\frac{1}{n} < 1+\frac{1}{n-2} < \delta_{max}$ such that for all δ with $\delta_{min} < \delta < \delta_{max}$, the once-reinforced random walk on G with reinforcement factor δ is almost surely recurrent*

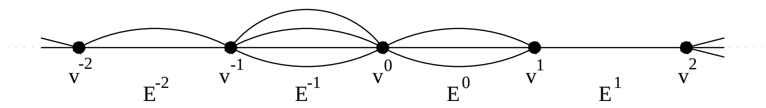

Fig. 3. The graph of Theorem 8

with the actual values of δ_{min} and δ_{max} depending on the particular bound chosen for the expected value of the bias. Unfortunately in practice this seens to be a lot of effort, yielding only an insignificant improvement over the previous result. In the next section a proof will be given of recurrence for *large* values of δ. No proof is yet known for intermediate values of δ.

It is also not yet known whether the once-reinforced random walk on the square lattice graph $\mathbb{Z}^2$ is recurrent for any reinforcement factor $\delta \neq 1$. The above method can be adapted to show recurrence for a variant once-reinforced random walk, where the reinforcement factor converges to 1 if you move away to infinity. Similarly, it is possible to adapt the proof given in the next two sections to show recurrence for a variant once-reinforced random walk where the reinforcement factor converges to ∞ if you move away to infinity. But except for these marginal results, nothing is known yet for reinforced random walks on the square lattice $\mathbb{Z}^2$.

Theorem 8. *Let $G = (V, E, w)$ be a weighted graph with vertices $V = \{v^i \mid i \in \mathbb{Z}\}$ and for any n, a finite non-zero number of parallel edges between the vertices v^n and v^{n+1}. If the non-reinforced random walk on G is almost surely recurrent, then the reinforced random walk on G is almost surely recurrent for any reinforcement factor $\delta > 0$.*

Proof

Note that, to prove this, we actually will need the generalization of Lemma 11 to graphs with parallel edges. As stated in Remark 1, we will simply postulate this generalization and proceed.

If the non-reinforced random walk on G is almost surely recurrent, then by Theorem 2 there exists a function $h : V \to \mathbb{R}$ such that h is superharmonic on $V - \{v^0\}$ and $h(v^n) \to \infty$ if $n \to \infty$ or $n \to -\infty$. It is easily seen that $h(v^n) > h(v^m)$ if $n > m > 0$ or $n < m < 0$.

Now for $n \in \mathbb{Z}$, let $G^n = (V^n, E^n, w^n)$ be the subgraph induced by $\{v^n, v^{n+1}\}$ (i.e. $V^n = \{v^n, v^{n+1}\}$, E^n is the set of edges between v^n and v^{n+1}, and $w^n = w_{|E^n}$). Although events outside G^n may effect *whether* and *when* an edge of G^n is traversed, *which* edge of G^n is traversed is only dependent on the relative current weights of the edges of G^n. So we can estimate the expected contribution to the bias of each set E^n separately, and take the sum to arrive at an estimate for the total expected bias. The possibility that at some point the walk in G will no longer return tot G_n can be simulated by a stopping time for the walk in G_n.

So fix $n \in \mathbb{Z}$ and consider the reinforced random walk on the finite graph G^n, starting in v^n if $n \geq 0$, and in v^{n+1} otherwise. Note that in both cases the random walk will start at the vertex with the lower h-value and then alternate between the two vertices. Set $c = |h(v^{n+1}) - h(v^n)|$. If for G^n we define A^n_t and E^n_t as usual, we have for any $t \in I\!N$

$$E\left(\sum_{\mathbf{a}\in A^n_{t+1}} \Delta_h(\mathbf{a}) \middle| \mathcal{F}_t\right) = \sum_{\mathbf{a}\in A^n_t} \Delta_h(\mathbf{a}) + (-1)^t \frac{w(E^n)-w(E^n_t)}{w(E^n)+(\delta-1)w(E^n_t)} \cdot c \quad (64)$$

where $w(X)$ denotes $\sum_{e\in X} w(e)$. This implies that for any stopping time τ

$$E\left(\sum_{\mathbf{a}\in A^n_\tau} \Delta_h(\mathbf{a})\right) = E\left(\sum_{t=0}^{\tau-1} (-1)^t \frac{w(E^n)-w(E^n_t)}{w(E^n)+(\delta-1)w(E^n_t)}\right) \cdot c \quad (65)$$

For all t, $w(E^n_t) \le w(E^n_{t+1}) \le w(E^n)$, and hence

$$\frac{w(E^n)-w(E^n_t)}{w(E^n)+(\delta-1)w(E^n_t)} \le \frac{w(E^n)-w(E^n_{t+1})}{w(E^n)+(\delta-1)w(E^n_{t+1})} \quad (66)$$

We conclude that for any stopping time τ,

$$0 \le E\left(\sum_{\mathbf{a}\in A^n_\tau} \Delta_h(\mathbf{a})\right) \le c \quad (67)$$

Now let us return to the random walk on G. Fix $t_0 \in I\!N$ and the history up to time t_0. Then all the vertices that have been visited up to time t_0 have indices between $-t_0$ and t_0. Furthermore, all the vertices that can be visited after time t_0 are on the same side of v^0 until the first visit to v^0; without loss of generality we may assume that this is the side of the vertices with positive indices. If we transfer the results we obtained for the walks on the graphs G^n to the random walk on the graph G, and take the sum of the inequalities over all edgesets E^n with $n \ge t_0$, then we obtain

$$-c' \le E\left(\sum_{\mathbf{a}\in A_\tau} \Delta_h(\mathbf{a}) \middle| \mathcal{F}_{t_0}\right) \le E(\max\{h(v_t) \mid t \le \tau\}) - h(v^{t_0}) + c' \quad (68)$$

where τ is any stopping time such that the walk does not leave the set of vertices with positive indices, and $c' = \sum_{n=-t_0}^{t_0-1} (\#E_n)|h(v^{n+1}) - h(v^n)|$.
This implies the condition of (the generalization of) Lemma 11 for all $\delta > 0$.

□

Remark 6. Recurrence for $\delta \ge 1$ can be proven for all weighted trees (provided non-reinforced random walks on the tree are recurrent): this already follows from Theorem 6. The same doesn't hold for $\delta < 1$: it can be shown, for instance, that the once-reinforced random walk on the tree given in figure 4 starting from v_0 is not almost surely recurrent if the reinforcement factor δ is less than $1/4$.

To prove recurrence for *all* $\delta \ge 1$, in any graph on which non-reinforced random walks are recurrent, one could use something like

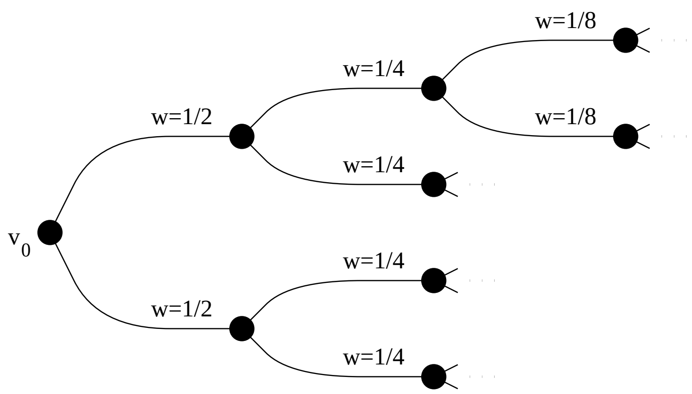

Fig. 4. A tree on which the once-reinforced random walk with $\delta < 1/4$ is not almost surely recurrent.

Proposition 1. For any edge vu that is 'far away' from all edges traversed so far, if v is closer than u to the origin of the walk (in the sense that $h(v) < h(u)$), then vu has at least as much chance of being traversed (the first time it is traversed) from v to u as it has of being traversed (the first time it is traversed) from u to v.

or, very loosely formulated, closer vertices are visited earlier. Alas, so far this proposition has neither been proved nor refuted.

5. Large Reinforcements

In this section we will consider recurrence of once-reinforced random walks with very large reinforcement factors. We will first show that if the reinforcement factor δ is very large, then the 'growth process' of the subgraph of traversed edges will not only be very slow, but also very uniform. Later in the section we will use this to show that on some graphs, this implies that the estimated bias of the random walk is nonnegative. enabling us to use Lemma 11 to show recurrence for δ large enough.

So let us start by considering what happens if the reinforcement δ is very large. Then an edge that has never been traversed before has a very small chance of being selected, compared to an edge that *has* been walked before. As a result, the once-reinforced random walk will remain in the subgraph of traversed edges for (on average) very long periods. Each such period ends when the walk leaves the subgraph (thereby extending it with a new edge) and during each period the walk behaves like a non-reinforced random walk.

Now consider this non-reinforced random walk. Generalizing the situation, we have a graph and a subgraph, with the property that the weight of edges outside the subgraph is very small compared to the weight of edges in the subgraph, and we are considering the non-reinforced random walk which stops when it leaves this subgraph. If we compare this walk to the non-reinforced random walk in the subgraph, then we see that if the latter is almost surely recurrent, then the former will visit each vertex of the subgraph, on average, a very large number of times.

Furthermore, this number will be proportional to the total weight of edges adjacent to that vertex. This is enough to show that the probability that the walk ends by leaving the subgraph by a particular edge (and hence the probability that in the original once-reinforced random walk that edge will be added to the subgraph of traversed edges), is nearly the same for edges that are close together, relative to their weight.
A necessary requirement for this is that non-reinforced random walks on the subgraph are recurrent, since otherwise each vertex will only get a bounded number of visits (in an approximation of transience). Furthermore, in order to be able to put uniform bounds on 'very small' and 'nearly the same', we will require the graph to be a subgraph of a finitely-patterned graph:

Definition 11. *A weighted graph $G = (V, E, w)$ is called* finitely-patterned *if there is a finite set $F \subset V$ such that for any vertex $v \in V$, there is a graph-automorphism ϕ of G with $\phi(v) \in F$.*

Lemma 12. *Let $G^* = (V^*, E^*, w^*)$ be a finitely-patterned weighted graph on which non-reinforced random walks are almost surely recurrent. Then for any $\eta > 0$ there exist c_dist, $\delta > 0$ such that the following holds:*

Let $G = (V, E, w)$ be a subgraph of G^* (possibly with lesser weights), $v \in V$, $V' \subset V$ a (possibly empty) set of stopping positions, and $E' \subset E$ a set of edges. Consider the random walk in G starting from v_0 and continuing until an edge in $E - E'$ is traversed. Now, if $v_a u_a$ and $v_b u_b$ are edges in $E - E'$, then

$$1 - \eta d \leq \frac{P(\text{the walk ends by walking from } v_b \text{ to } u_b)/w(v_a u_a)}{P(\text{the walk ends by walking from } v_a \text{ to } u_a)/w(v_b u_b)} \leq \frac{1}{1 - \eta d} \tag{69}$$

provided that

1. *for any edge $vu \in E - E'$, $w(vu) \leq w^*(vu)/\delta$*
2. *v_a and v_b are in G^* at least distance c_dist away from any vertex $v' \in V'$*
3. *v_a and v_b and connected by a path P in E' with $d = \sum_{e \in P}(1/w(e)) \leq 1/\eta$*

This also holds if we add additional stopping positions V', provided v_a and v_b are not within G^*-distance c_{dist} of V'.

Proof
Let G^* be as stated, and let c_dist be such that for any $v \in V^*$, the random walk on G^* starting at v will on average revisit v at least $2w^*(v)/\eta$ times before time $t = c_dist$. Since G^* is finitely-patterned, we need only consider finitely many $v \in V^*$, and hence we can take c_dist finite. Let $\delta = 1/(2^{1/M} - 1)$.
Now let G, V', E', v_0, v_a, v_b be as stated. Note that the random walk cannot end by traversing $\overrightarrow{v_a u_a}$ or $\overrightarrow{v_b u_b}$ without first visiting v_a or v_b. So if the random walk does not start at v_a or v_b, then we can walk until the first visit to v_a or v_b (or the walk ends by traversing some other edge $E - E'$), and therefore we can write the probabilities to be calculated as linear combinations of the probabilities for the

cases where the random walk *does* start at v_a or v_b. Hence it suffices to prove the Lemma for the case where $v_0 = v_a$.
Let G_s be the graph obtained from G by adding another stopping position s to G, and replacing all edges $vu \in E - E'$ by edges vs, us with weight $w_s(vs) = w_s(us) = w(vu)$, Then the random walk under consideration is equivalent to a random walk in G_s which stops at $V' \cup \{s\}$. Now, let H be the function from Lemma 4. Then we have that, for $vu \in E - E'$,

$$P(\text{the walk ends with } vu) = E(\text{the number of visits to } v) \cdot \frac{w(vu)}{w(v)} = w(vu)H(v) \tag{70}$$

Furthermore, by Lemma 4 H is maximal at v_a, and for any edge vu, $|H(v) - H(u)| \leq 1/w(vu)$. We conclude

$$H(v_a) - d \leq H(v_a) - \sum_{e \in P}(1/w(e)) \leq H(v_b) \leq H(v_a) \tag{71}$$

and therefore

$$1 - \frac{d}{H(v_a)} \leq \frac{H(v_b)}{H(v_a)} = \frac{P(\text{the walk ends by walking from } v_b \text{ to } u_b)/w(v_a u_a)}{P(\text{the walk ends by walking from } v_a \text{ to } u_a)/w(v_b u_b)} \leq 1 \tag{72}$$

We want to show that $H(v_a) > 1/\eta$. So consider the graph G^*_s obtained from G^* by adding a stopping position s to G and edges vs and us for each edge $vu \in E^*$, with weight $w^*_s(vs) = w^*_s(us) = w^*(vu)/\delta$. Note that G_s is a subgraph of G^*_s. Now in the random walk in G^*_s starting at v_a and stopping at s, at each node there is a probability $\frac{1/\delta}{1+1/\delta}$ of traversing to s and stopping. Since it is impossible to visit V' starting from v_a before time $c_d ist$ the probability of *not* stopping before time $c_d ist$ is $(\frac{1}{1+1/\delta})^M = 1/2$, and the expected number of visits to v_a before stopping is at least $2w^*(v_a)/\eta \cdot 1/2 = w^*(v_a)/\eta$. If H^* is the function from Lemma 4 for G^*_s, then this can be written as

$$H^*(v_a) \geq \frac{1}{w^*(v_a)}(w^*(v_a)/\eta) = 1/\eta \tag{73}$$

which together with Corollary 1 implies

$$H(v_a) \geq H^*(v_a) \geq 1/\eta \tag{74}$$

□

Now let us apply this lemma to the reinforced walk with large reinforcement factor δ. As before, let E_t denote the set of edges already-traversed at time t, and A_t the set of arcs (oriented edges) obtained by orienting all edges of E_t in the direction of first traversal. We say that an arc $\overrightarrow{vu}$ *extends* A_t if it could be added to A_t, i.e. if A_t does not contain $\overrightarrow{vu}$ but does contain an edge incident with v. Then the above lemma can be used to show that all edges that can extend A_t and are close together, have approximately the same probability of being added to A_t, relative to their weight. Note that, in order to be able to apply the result later, we will allow stopping times satisfying certain conditions.

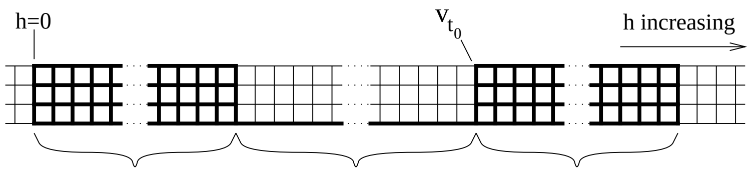

Fig. 5. An anomalous situation

Corollary 3. *Let $G = (V, E, w)$ be a weighted subgraph (possibly with lesser weights) of a finitely-patterned graph G^* on which non-reinforced random walks are recurrent. Then for any $\eta > 0$ there exist $c_{dist}, \delta > 0$ such that the following holds:*

> *Consider the once-reinforced random walk on G with reinforcement factor δ, let $t_0 \in \mathbb{N}$, fix the history up to time t_0. let $0 < d < 1/\eta$, and let B be a set of arcs extending E_{t_0}. Then for any arc $\overrightarrow{vu} \in B$,*
>
> $$\frac{P(\textit{the first arc added to } A_t \textit{ is } \overrightarrow{vu})/w(vu)}{P(\textit{the first arc added to } A_t \textit{ is from } B)/w(B)} \in [1-\eta d, 1/(1-\eta d)] \quad (75)$$
>
> *provided the tails of B are all pairwise connected by E_{t_0}-paths P with $\sum_{e \in P}(1/w(e)) \leq d$. Here $w(B)$ denotes the sum of the weights of all arcs of B.*
> *This also holds if we add additional stopping positions V', provided we add the condition that no arc of B is within G^*-distance c_{dist} of V'.*

Proof
Given the history up to t_0, let G_2 be obtained from G by assigning each edge the 'current' weight at time t_0, divided by δ. Then G_2 is a subgraph of G and G^*. Set $E' = E_{t_0}$, then all the conditions of Lemma 12 are satisfied w.r.t. G_2 for any pair of arcs $\overrightarrow{vu}, \overrightarrow{v'u'} \in B$. The result follows.

□

If the previous corollary would hold for some fixed d, with tails whose arcs where arbitrarily far apart. then we would be done. For then the growth of A_t would always be uniform, and the expected change to the bias would be proportional to the weighted average of $\Delta_h(\overrightarrow{vu})$ over all arcs $\overrightarrow{vu}$ extending A_t. For any (super)harmonic function h this can be shown to be positive, and hence we would then be able to apply Lemma 11 to show recurrence.
Unfortunately, Corollary 3 is not that strong. In order to be able to use the corollary, we need to divide the graph into *smaller* areas, such that the arcs in that area extending A_t are close together and have, on average, nonnegative $\Delta_h(\overrightarrow{vu})$. Then we can take the sum over the areas, and arrive at a nonnegative expected change to the total bias.
As it turns out, this does not always work. Figure 5 shows a situation where it doesn't, arising in the random walk on the ladder $\mathbb{Z} \times \{1, 2, 3, 4\}$ with $h(v) = |x(v)|$. In this situation, all edges extending A_t that are not far away from v_{t_0} are contained in an extremely-wide 'room' R. If the next horizontal edge added to A_t

is from the right side of the room, then it will be traversed right-to-left, and will contribute a negative Δ_h to the bias. If it is from the left side of the room, then it will be traversed left-to-right, and will contribute a positive Δ_h. $h(v) = |x(v)|$ is a harmonic function, and the harmonicity equations ensure that these potential contributions have a zero sum. So all would be well if the edges would have equal probabilities of being added to A_t. But if the width of the room is large enough (relative to δ), then in the reinforced random walk starting at v_{t_0}, the next arc added to A_t will be much more likely to come from the right side of the room than from the left side. In the given situation therefore, the expectation of the next change to the bias is negative.
So we need to make sure that such anomalous situations occur so rarely that they will have a negligible effect on the expectation of the change to the bias. In the case of the next theorem, this problem is addressed by considering only only 'strip-like' graphs that can be viewed as rows of connected vertexsets V_i.[6]

Definition 12. *Let $G = (V, E, w)$ be a weighted graph. We say that G is* strip-like *if w has upper and lower bounds w_{max} and w_{min}, and there is a partition $(V^i)_{i\in\mathbb{Z}}$ of V satisfying the following:*

1. *$|V^i|$ is bounded by some $k \in \mathbb{Z}$.*
2. *All edges in G are between vertices of V^i, or between vertices of V^i and V^{i+1}, for some $i \in \mathbb{Z}$.*
3. *For all i, $G_{|V^i}$ is connected.*

At any time there will be sets V_i such that all edges in the subgraph V_i have been traversed, which will act as 'walls', dividing the graph into 'rooms' similar to the one shown in figure 5. Since the size of the sets V_i is bounded, so is the average distance between the walls. Anomalous situations can only occur in rooms with extremely large width, much greater than average, but these situations will be extremely rare (unless they are part of a given initial situation) and will have little effect on the expectation of the bias.
This structure also solves an additional problem, namely that in general edges that are close together in G are not guaranteed to be close together in the subgraph of traversed edges, and if they are not, A_t has no uniform growth there. This cannot happen in rooms of small width, since the walls 'short-circuit' long paths.
If G is strip-like, and k and w_{max} are as in definition 12, we can define the graph $G^* = (V^*, E^*, w^*))$ by setting $V^* = \mathbb{Z} \times \{1, \ldots, k\}$, $E^* = \{((i_1, j_1), (i_2, j_2)) \mid |i_1 - i_2| \leq 1\}$ and $w^*(e) = w_{max}$. Now G^* is finitely-patterned, and G is a subgraph of G^* (possibly with lesser weights). Moreover, it is easily seen that non-reinforced random walks on G^* are almost surely recurrent, and hence so are non-reinforced walks on G.
Lemma 2 guarantees the existance of a superharmonic function on $V - V^0$ witnessing this recurrence, which we can use when applying Lemma 11. But using the given restrictions on G, we can construct a function h with some additional nice properties, which will simplify later calculations considerably:

[6] This class of graphs may be viewed as a generalization of the infinite ladders $\mathbb{Z} \times \{1, \ldots, n\}$ and cylinders $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$ of Theorem 7.

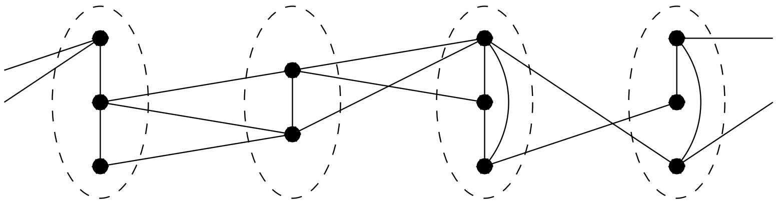

Fig. 6. An example of a strip-like graph for Theorem 9, with $k = 3$.

Lemma 13. *Let $G = (V, E, w)$ be a strip-like weighted graph, and let w_{max}, w_{min}, k and $(V^i)_{i\in\mathbb{Z}}$ be as in 12. Then there exists a function $h : V \to \mathbb{R}$ satisfying*

1. *h is harmonic on $V - V_0$ and 0 on V_0.*
2. *$h \to \infty$ if $v \to \infty$.*
3. *For all vertices $vu \in E$,*

$$|\Delta_h(\overrightarrow{vu})| \leq 1/w(vu) \leq 1/w_{min} \tag{76}$$

4. *For all $i \in \mathbb{Z}$,*

$$\sum\{w(vu)\Delta_h(\overrightarrow{vu}) \mid v \in V^i, u \in V^{i+1}, vu \in E\} = \begin{cases} 1 & \text{if } i \geq 0 \\ -1 & \text{if } i < 0 \end{cases} \tag{77}$$

Proof
We can construct the function h on $V^{>0}$ and $V^{<0}$ separately. So for purposes of this proof, we will ignore $V^{<0}$. For $v_0 \in V^{>0}$, let H_{v_0} be the function from Lemma 4 with respect to the random walk starting at v_0 and stopping at V_0. Then for any v_0, H_{v_0} is harmonic on $V - V^0 - v_0$, satisfies 76, and satisfies 77 for $i = 0$. Now 76 implies that for any $v \in V^{\geq 0}$, $\{H_{v_0}(v) \mid v_0 \in V^{\geq 0}\}$ is bounded, and therefore has a limit point. We can inductively construct a 'common limit point' in the form of a function h such that for any finite set $F \subset V^{\geq 0}$, $h_{|F}$ is a limit point of $\{h_{v_0})_{|F} \mid v_0 \in V^{\geq 0}\}$. Then it follows that h is harmonic on $V - V^0$, satisfies 76, and satisfies 77 for $i = 0$. By taking linear combinations of the harmonicity equality, we obtain 77 for $i \geq 0$.
Finally, let $r \geq 0$, and set $U_r = \{v \in V^{\geq 0} \mid h(v) \leq r\}$. If U_r were to have an infinite connected component, then for some $i_0 > 1$, U_r would intersect all V^i with $i \geq i_0$, and hence by the connectedness of the sets V_i, $h(v) < r + kw_{min}$ for all $v \in V^{\geq i_0}$. Then $-h$ would be a bounded non-constant superharmonic function, contradicting the fact that non-reinforced random walks on G are almost surely recurrent. On the other hand, if U_r were to have a finite non-empty component F disjoint from V^0, we could take linear combinations of the harmonicity equality to obtain

$$\sum\{w(vu)\Delta_h(\overrightarrow{vu}) \mid v \in F, u \in V - F, vu \in E\} = 0 \tag{78}$$

contradicting the fact that if $v \in F, u \in V - F$, then $h(u) \geq r > h(v)$. So for all $r \geq 0$, U_r consists only of a finite component containing V^0. We conclude

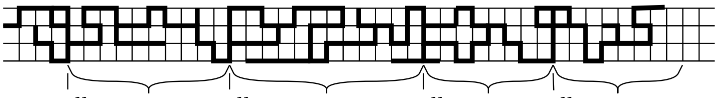

Fig. 7. Traversed edges forming walls and rooms in the square lattice graph on $\mathbb{Z} \times \{1, 2, 3, 4\}$

that $h(v) \to \infty$ if $v \to \infty$, and that h is a proper witness to the recurrence of non-reinforced random walks in G.

$\square$

Theorem 9. *Let $G = (V, E, w)$ be a strip-like weighted graph, and let w_{max}, w_{min}, k and $(V^i)_{i\in\mathbb{Z}}$ be as in 12. Then a once-reinforced random walk on G with large enough reinforcement factor δ is almost surely recurrent.*

Proof

Let G, w_{max}, w_{min}, k and $(V^i)_{i\in\mathbb{Z}}$ be as stated, let ρ_{max} denote the maximum degree of vertices in G and let h be the function of Lemma 13. We will use Lemma 3 to show that the condition of Lemma 11 is satisfied. Note that to show recurrence of the random walk, it suffices to show that at any time t_0, the probability of returning to V^0 is 1. Hence we may assume, without loss of generalization, that $v_{t_0} \in V^{\geq 0} = V^0 \cup V^1 \cup \ldots$, and that for purposes of this proof, $V^{-1} \cup V^{-2} \cup \ldots$ can be safely ignored. Unless stated otherwise, we will assume $i \geq 0$ whenever we write V^i.

We say that there is a *wall* at V^i, when all edges of E with both vertices in V^i are in E_t. The area between two successive walls we will call a *room*. Given that there are at most $k\rho_{max}$ edges involved, it can easily be seen that there is a $p_{wall} > 0$ such that the probability that a wall appears at V^i before a single vertex of V^{i+1} is visited is at least p_{wall} (irrespective of δ). It follows that, not taking into account rooms already present in any given initial situation, the expected average width of a room is at most p_{wall}^{-1}. Now pick $\epsilon > 0$ and $N > k/p_wall$ such that 94 will hold later. Let $d = N/w_{min}$, let $\eta = \epsilon/(d+1)$, and let c_dist, δ be as in Corollary 3.

Let $t > 0$, and fix the history up to time t. Let $\overrightarrow{v^*u^*}$ denote the next arc (at or after time t) that is added to A_t. Now first consider a room R with walls at V^i and V^j and width $\leq N/k$, at distance $\geq c_{dist}$ from the origin. It is easily seen that any non-self-intersecting E_t-path that does not extend beyond the room is of length at most N, and since walls connect all their vertices, paths that leave the room and return can be 'short-circuited'. Hence the tails of all arcs in the room extending A_t are connected by paths P with $\sum_{e\in P} 1/w(e) \leq d$,

$$P(\overrightarrow{v^*u^*} = \overrightarrow{vu} \,|\, \mathcal{F}_t)(1-\epsilon) \leq P(v^*u^* \text{ in } R \,|\, \mathcal{F}_t)\frac{w(vu)}{w_{ext}(R)} \leq P(\overrightarrow{v^*u^*} = \overrightarrow{vu} \,|\, \mathcal{F}_t)(1+\epsilon) \tag{79}$$

where $w_{ext}(R)$ denotes the total weight of all arcs in R that extend A_t. It follows that

$$P(\overrightarrow{v^*u^*} = \overrightarrow{vu} \mid \mathcal{F}_t)(\Delta_h(\overrightarrow{vu}) + \epsilon|\Delta_h(\overrightarrow{vu})|) \geq \frac{P(v^*u^* \text{ in } R \mid \mathcal{F}_t)}{w_{ext}(R)} w(vu)\Delta_h(\overrightarrow{vu}) \tag{80}$$

By taking linear combinations of the harmonicity equation, we can show that

$$\sum_{\substack{vu \text{ in } R \\ \overrightarrow{vu} \text{ extends } A_t}} w(vu)\Delta_h(\overrightarrow{vu}) = 0 \tag{81}$$

and hence

$$\sum_{\substack{vu \text{ in } R \\ \overrightarrow{vu} \text{ extends } A_t}} P(\overrightarrow{v^*u^*} = \overrightarrow{vu} \mid \mathcal{F}_t)\,(\Delta_h(\overrightarrow{vu}) + \epsilon|\Delta_h(\overrightarrow{vu})|) \geq 0 \tag{82}$$

Similarly, if we let i_t^{max} denote the largest index $i > 0$ such that a vertex of V^i has been visited at time t, then the area between $V^{i_t^{max}+1}$ and the wall V^{i_0} with the largest index i_0 can be considered to be an 'unfinished room' R_{last}. Here, taking linear combinations of the harmonicity equation yields

$$\sum_{\substack{vu \text{ in } R_{last} \\ \overrightarrow{vu} \text{ extends } A_t}} w(vu)\Delta_h(\overrightarrow{vu}) = 1 \tag{83}$$

and if R_{last} has width $\leq N/k$, we derive in the same manner as before

$$\sum_{\substack{vu \text{ in } R_{last} \\ \overrightarrow{vu} \text{ extends } A_t}} P(\overrightarrow{v^*u^*} = \overrightarrow{vu} \mid \mathcal{F}_t)\,(\Delta_h(\overrightarrow{vu}) + \epsilon|\Delta_h(\overrightarrow{vu})|) \geq \frac{P(v^*u^* \text{ in } R_{last} \mid \mathcal{F}_t)}{w_{ext}(R_{last})} \tag{84}$$

where $w_{ext}(R_{last})$ denotes the total weight of all arcs in R_{last} that extend A_t. Hence, if we pretend for the moment that we do not need to take into consideration edges within distance c_{dist} of the V^0, rooms of width $> N/k$, or the fact that to apply Lemma 11 we need to show that its condition holds *given* an arbitrary fixed history up to some time t_0, we obtain

$$E\left(\Delta_h(\overrightarrow{v^*u^*}) + \epsilon|\Delta_h(\overrightarrow{v^*u^*})| \,\middle|\, \mathcal{F}_t\right) \geq \frac{P(v^*u^* \text{ in } R_{last} \mid \mathcal{F}_t)}{w_{ext}(R_{last})} \tag{85}$$

Combined with the inequality

$$P(i_{t+1}^{max} = i_t^{max} + 1 \mid \mathcal{F}_t) \leq \frac{k\rho_{max}w_{max}}{w_{ext}(R_{last})} \cdot P(v^*u^* \text{ in } R_{last} \mid \mathcal{F}_t) \cdot \frac{1}{1-\epsilon} \tag{86}$$

this implies

$$E\left(\Delta_h(\overrightarrow{v^*u^*}) + \epsilon|\Delta_h(\overrightarrow{v^*u^*})| - (i_{t+1}^{max} - i_t^{max})\frac{1-\epsilon}{k\rho_{max}w_{max}} \,\middle|\, \mathcal{F}_t\right) \geq 0 \tag{87}$$

Now, if τ is a stopping time, then we can take the sum of the above equation over all $t < \tau$, to obtain

$$E\left(\sum_{\overrightarrow{vu}\in A_\tau} \Delta_h(\overrightarrow{vu}) + \epsilon \sum_{\overrightarrow{vu}\in A_\tau} |\Delta_h(\overrightarrow{vu})| - i_\tau^{max}\frac{1-\epsilon}{k\rho_{max}w_{max}}\right) \geq 0 \tag{88}$$

Now let us consider those aspects that we chose to ignore before. We will show that the number of edges involved in those aspects is relatively small. One of the aspects we ignored was the existence of rooms of width N/k or greater. But if a new room is 'growing', then the probability is less than $(1 - p_{wall})^{N/k}$ that the room will reach a width of N/k or greater, and even in that case the expected width of the room is at most $N/k + p_{wall}^{-1}$. Hence the expectation of the number of sets V^i contained in rooms of width $> N/k$ at any time τ is at most

$$E(i_\tau^{max})(1-p_{wall})^{N/k}(N/k + p_{wall}^{-1}) \tag{89}$$

Another aspect we ignored was that, in order to apply Lemma 11 we need to show that its condition holds *given* an arbitrary fixed history up to some time t_0. But any such initial situation is contained within the area defined by $V^0 \cup \ldots \cup V^{i_{t_0}^{max}}$. The same holds for edges within distance c_{dist} of the origin: all such edges are contained in $V^0 \cup \ldots \cup V^{c_{dist}}$. Edges in rooms extending from one of these areas may be affected, but the expected number of sets V^i involved in such an 'extension' is at most p_{wall}^{-1}. Hence the expected number of sets V^i for which the previous calculations do not apply is at most

$$\max(i_{t_0}^{max}, c_{dist}) + p_{wall}^{-1} + E(i_\tau^{max})(1-p_{wall})^{N/k}(N/k + p_{wall}^{-1}) \tag{90}$$

Each set V^i contains at most k vertices, each of which is adjacent to at most ρ_{max} edges. Each edge vu that we 'counted wrongly' before might have caused the expectation above to be higher than it should have been, but the difference will be at most $2|\Delta_h(\overrightarrow{vu})| \leq 2/w_{min}$. So the expectation above may be higher than it should have been, but not by more than

$$\frac{2k\rho_{max}}{w_{min}}\Big(\max(i_{t_0}^{max}, c_{dist}) + p_{wall}^{-1} + E(i_\tau^{max} \mid \mathcal{F}_{t_0})(1-p_{wall})^{N/k}(N/k + p_{wall}^{-1})\Big) \tag{91}$$

Combining everything, and taking into account that

$$\epsilon \sum_{\overrightarrow{vu}\in A_\tau} |\Delta_h(\overrightarrow{vu})| \leq \epsilon\frac{k\rho_{max}i_t^{max}}{w_{min}} \tag{92}$$

we derive that for any stopping time τ

$$E\left(\sum_{\overrightarrow{vu}\in A_\tau} \Delta_h(\overrightarrow{vu}) \,\middle|\, \mathcal{F}_{t_0}\right) \geq c \cdot E(i_\tau^{max} \mid \mathcal{F}_{t_0}) - c' \tag{93}$$

with

$$c = \frac{1-\epsilon}{k\rho_{max}w_{max}} - \frac{k\rho_{max}(2(1-p_{wall})^{N/k}(N/k + p_{wall}^{-1}) + \epsilon)}{w_{min}} > 0 \quad (94)$$

$$c' = \frac{2k\rho_{max}(\max(i_{t_0}^{max}, c_{dist}) + p_{wall}^{-1})}{w_{min}} \quad (95)$$

Note that we can and did choose ϵ and N such that $c > 0$. We conclude that the condition of Lemma 11 is satisfied.

□

Corollary 4. *Let $n \in N$. Then for δ large enough, the once-reinforced random walk with reinforcement factor δ on the square lattice graphs on $\mathbb{Z} \times \{1, \ldots, n\}$ and $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$ is almost surely recurrent.*

Remark 7. The main contributor to the size of the lower bound for δ for which this proof goes through is the requirement that by 94, $N/k > p_{wall}^{-1}$. Roughly estimated, p_{wall}^{-1} will be of order ρ_{max}^{k}, and to satisfy 94, ϵ and N should be of order $k^{-2}\rho_{max}^{-2}$ and $k\rho_{max}^{k}\log(k\rho)$. The proof of Lemma 12 can be refined (with more extensive use of the electrical-circuit paradigma) to show that δ and c_{dist} can be taken to be proportional to k/η. The resulting final lower bound for δ is of order $k^4\rho_{max}^{k+2}$.

Remark 8. The function of walls in the above proof is, as was stated before, to limit the occurrence of anomalous situations, where the edges that are close together in the graph of traversed edges do not form areas balanced by the harmonicity equations and vica versa. This approach fails when considering the random walk on the square lattice on $\mathbb{Z}^2$. Although it is possible to view $\mathbb{Z}^2$ as a row of vertexsets V_i, as i increases the size of the vertexsets would also increase, the probability of a wall forming would converge to 0, and the average width of a room would diverge to ∞.
Although presumably the reasoning above would hold up in a variant random walk, where the value of δ increases with the distance from some arbitrary origin, this is a marginal result at best. A better avenue of investigation for this problem is likely to find some other way of limiting the expected occurrence of anomalous situations.

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