Derived Category: Homework Set 1

Problem A

Let $\mathcal{A}$ be an abelian category. Let $f : x \to y$ be a morphism.

(1) Show that $\mathrm{Coker}(f) \cong \mathrm{Coker}\{\mathrm{Im}(f) \to y\}$.

(2) Show that $\mathrm{Ker}(f) \to x$ is injective and that $y \to \mathrm{Coker}(f)$ is surjective.

(3) Every monomorphism is a kernel of some morphism and every epimorphism is a cokernel of some morphism.

(4) Show that a morphism is an isomorphism if and only if it is both injective and surjective.

(5) Show that a complex

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3$$

is exact if and only if

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(N, M_1) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(N, M_2) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(N, M_3)$$

is exact as a complex of abelian groups.

Problem B

Let k be a field. A filtered k-vector space V consisits of vector subspaces V_i indexed by $i \in \mathbb{Z}$ such that $V_i \subseteq V_j$ for all $i < j$. A morphism $f : V \to W$ between filtered vector spaces is a k-linear map such that $f(V_i) \subseteq W_i$.

(1) Show that the category of filtered vector spaces is an addtive category where kernel and cokernel exist.

(2) Let $f : V \to W$ be a morphism of filtered vector spaces. Show that the induced map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism if and only if f is strict in the sense that $f(V_i) = f(V) \cap W_i$.

(3) Show that the category of filtered vector spaces is not an abelian category.

Problem C

Let $\mathcal{A}$ be an abelian category and let x be an object of $\mathcal{A}$. We define a point of x to be a pair (y, α), where $y \in \mathrm{Ob}(\mathcal{A})$ and $\alpha : y \to x$, modulo the equivalence relation

$$(y, \alpha) \sim (y', \alpha') \Leftrightarrow \exists\, z \text{ and epimorphisms } z \overset{p}{\twoheadrightarrow} y, z \overset{p'}{\twoheadrightarrow} y', \text{ such that } \alpha \circ p = \alpha' \circ p'.$$

Given a morphism $f : x \to x'$ and a point $u = (y, \alpha)$ of x, we define $f(u)$ to be the point of x' represented by $(y, f \circ \alpha)$. Note that the unique morphism $0 \to x$ defines a point of x, which will be called the zero point and denoted 0. Show the following diagram chasing principles.

1. $f : x_1 \to x_2$ is injective if and only if for any point u_1 of x_1, $f(u_1) = 0$ implies that $u_1 = 0$.
2. $f : x_1 \to x_2$ is surjective if and only if for any point u_2 of x_2, there exists a point u_1 of x_1 such that $f(u_1) = u_2$.
3. $f : x_1 \to x_2$ is the zero morphism if and only if $f(u_1) = 0$ for all points u_1 of x_1.
4. the complex $x \xrightarrow{f} y \xrightarrow{g} z$ is exact if and only if for every point v of y with $g(v) = 0$ there exists a point u of x such that $v = f(u)$.