Derived Category: Homework Set 2

Problem A

Let $\mathcal{A}$ be an abelian category. Let $i : x \hookrightarrow z$ and $j : y \hookrightarrow$ be two injective arrows and let $p : z \twoheadrightarrow z/x$ and $q : z \twoheadrightarrow z/y$ be the associated quotient objects. We define $x \cap y$ by the following Cartesian square

$$\begin{array}{ccc} x \cap y & \xrightarrow{j'} & x \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\ y & \xrightarrow{j} & z \end{array}$$

(1) Show that the arrows $j' : x \cap y \to x$ and $i' : x \cap y \to y$ are all injective.

(2) Show that

$$\begin{aligned} \operatorname{coker}(i - j : x \oplus y \longrightarrow z) &\cong \operatorname{coker}(-p \circ j : y \longrightarrow z/x) \\ &\cong \operatorname{coker}(q \circ i : x \longrightarrow z/y) \end{aligned}$$

(3) Show that

$$\begin{aligned} x \cap y &\cong \ker(i - j : x \oplus y \longrightarrow z) \\ &\cong \ker(p \circ j : y \longrightarrow z/x) \\ &\cong \ker(q \circ i : x \longrightarrow z/y) \end{aligned}$$

(4) Let $\langle x \cup y \rangle \hookrightarrow z$ be the image of $i - j : x \oplus y \to z$. Show that we have injective arrows $\bar{i} : x \hookrightarrow \langle x \cup y \rangle$ and $\bar{j} : y \hookrightarrow \langle x \cup y \rangle$.

(5) Show that the square

$$\begin{array}{ccc} x \cap y & \xrightarrow{j'} & x \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle \bar{i}} \\ y & \xrightarrow{\bar{j}} & \langle x \cup y \rangle \end{array}$$

is both cartesian and cocartesian.

Problem B

Let R be a commutative ring and let $\mathcal{A}$ be the category of R-modules. Let

$$0 \longrightarrow A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \longrightarrow 0$$

be a short exact sequence of complexes of R-modules.

(1) Assume that the above sequence is termwise split by the arrows $s^n : C^n \to B^n$ and $\pi^n : B^n \to A^n$. Show that $\delta^n = \pi^{n+1} \circ d_B^n \circ s^n : C^n \to A^{n+1}$ defines a morphism $\delta : C^\bullet \to A^\bullet[1]$ of complexes. As a morphism in $K(\mathcal{A})$, δ is independent of the choice of the splitting arrows $\{s^n\}$ and $\{\pi^n\}$.

(2) The induced map on cohomology $\delta : \mathrm{H}^n(C^\bullet) \to \mathrm{H}^{n+1}(A^\bullet)$ is the connecting morphism in the associated long exact sequence in cohomology

$$\cdots \xrightarrow{\delta} \mathrm{H}^n(A^\bullet) \xrightarrow{\alpha} \mathrm{H}^n(B^\bullet) \xrightarrow{\beta} \mathrm{H}^n(C^\bullet) \xrightarrow{\delta} \mathrm{H}^{n+1}(A^\bullet) \xrightarrow{\alpha} \cdots$$