Derived Category: Homework Set 3

Problem A

Let $\mathcal{A}$ be an abelian category. This exercise shows that, in general, there is no canonical delta-functor from $\mathcal{A}$ into $K(\mathcal{A})$.

Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{1}$$

be a short exact sequence in $\mathcal{A}$. Show that the only arrow $C \longrightarrow A[1]$ in $K(\mathcal{A})$ is the zero arrow. Assume that

$$A \longrightarrow B \longrightarrow C \xrightarrow{0} A[1]$$

is a distinguished triangle in $K(\mathcal{A})$. Show that the sequence (1) splits.

Problem B

Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left-exact additive functor. Let $\mathfrak{A}$ be a class of objects in $\mathcal{A}$ satisfying the following conditions.

(i) $\mathfrak{A}$ is closed under finite direct sums and if $0 \longrightarrow I \longrightarrow J \longrightarrow N \longrightarrow 0$ in $\mathcal{A}$ and both I and J are in $\mathfrak{A}$, then N is also in $\mathfrak{A}$.

(ii) If $0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$ is an exact sequence in $\mathcal{A}$ with $I \in \mathfrak{A}$, then $0 \longrightarrow F(I) \longrightarrow F(A) \longrightarrow F(B) \longrightarrow 0$ is an exact sequence in $\mathcal{B}$.

Let $I^\bullet$ (*resp.* $J^\bullet$) be a complex which is bounded below and $I^n \in \mathfrak{A}$ (*resp.* $J^n \in \mathfrak{A}$) for all all non-zero I^n (*resp.* J^n). Let $q : I^\bullet \to J^\bullet$ be a quasi-isomorphism. Show that $F(q) : F(I^\bullet) \to F(J^\bullet)$ is again a quasi-isomorphism.

Show that if we take $\mathfrak{A}$ to be the class of injective objects then both coditions (i) and (ii) are satisfied.