Symmetry & Quantum Information (1/4)

Mon: Entanglement intro
Tue: Pure state estimation (symmetric subspace)
Wed: Monogamy of entanglement (de Finetti, n-ext.)
Thu: Compression, tomography or E_{c} \big/{E_{D}} (Schur-Weyl)

1. Laws of Q. Mechanics

QMT system \rightarrow Hilbert space \quad H

\[ \dim < \infty \text{ for simplicity} \]

State: density operator \( g \in \mathcal{B}(H), \ g \geq 0, \ \text{tr}[g] = 1 \)

- pure state: \( \text{rk}[g] = 1 \ \iff \ g = |\psi\rangle \langle \psi| \quad \text{supposition} \)
  \[ \text{wave function} \quad \phi \]

- mixed state: otherwise, \( g = \sum \limits_{i} p_{i} \xi_{i} \text{ normal} \)

- classical w.r.t. ONS \{ |i\rangle \}\:

\[ g = \sum \limits_{i} p_{i} \langle i|\xi_{i} = (p_{i} \xi_{i}) \quad \overset{\text{pure iff}}{\longrightarrow} \text{deterministic} \]
Measurements: observable $X = X^+ \in B(H)$

$$X = \sum_{x \in \Omega} x \cdot P_x$$

- $\Omega$ is set of possible outcomes

- $P_x(\text{outcome } x) = \text{tr}[g \; P_x]$ \quad Ban's rule
  $$E(\text{outcome } x) = \text{tr}[g \; x]$$

- post-measurement state upon outcome $x$:
  $$s_x = \frac{P_x g P_x}{\text{tr}[P_x g]} \quad \text{"collapse of wave fn"}$$

\[\text{Tue:} \] naive general measurements

Ex: Can only discriminate $g, g$ perfectly if $\text{supp}(g) \perp \text{supp}(g')$. \[\text{Tue/Thu}\]

Ex: $[X, Y] \to 0$ $\Rightarrow$ probability of outcomes depends on order of measurements (in general)

"incompatible", joint meas. not sensible
Composite systems: \( A \) \( B \)

\[ H_{AB} = H_A \otimes H_B \]

Locally embed local observables as \( X_A \otimes I_B, I_A \otimes Y_B \)

\[ \text{tr}[\rho_{AB} (X_A \otimes I_B)] \]

\[ = \sum_{a,b} \langle a,b | \rho_{AB} (X_A \otimes I_B) | a,b \rangle \]

\[ = \sum_a \langle a | \sum_b \langle a | b \rangle \rho_{AB} (I_A \otimes (b)) | a \rangle X_A(a) \]

\[ = \text{tr}[\rho_A X_A] \]

\[ =: \text{tr}_B[\rho_{AB}] =: \rho_A \]

Partial trace

Def.: \( \rho_A = \text{tr}_B[\rho_{AB}] \) is called reduced density matrix, describes state of subsystem

- \( \rho_{AB} = \rho_A \otimes \rho_B \Rightarrow \rho_A = \rho_A \)

- Not every \( \rho_{AB} \) is of this form

\[ \rho_{AB} = \frac{1}{2} ( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes 1 + \mathbf{1} \otimes \sigma_z ) \]

boring!

Classical!
not even if $\rho_{AB}$ is pure

\[ \rho_{AB} \neq \rho_A \otimes \rho_B \]

(2) Entanglement

Def: $|\psi_{AB}\rangle$ is called entangled if

\[ |\psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \]

"maximally entangled state", "Bell pair"

Example: 

\[ |\Phi^{\pm}_{AB}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2 \]

\[ |\Psi^{\pm}_{AB}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \neq |\psi_A \otimes \psi_B\rangle \]

- Basic resource for QIT!
  - Non-local correlations
  - Computational speedup
  - Quantum communication

EX: $|\psi_{AB}\rangle$ product $\Rightarrow$ $\rho_A$ pure $\Rightarrow$ $\rho_B$ pure
Ex: Useful tool: **Schmidt decomposition (SVD)**:

\[ |\Psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \text{ & } p_i \geq 0: \]

\[ |\Psi_{AB}\rangle = \sum_i \sqrt{p_i} |\alpha_i\rangle \otimes |\beta_i\rangle \]

\[ L_o \quad g_A = \sum_i p_i |\alpha_i\rangle \langle \alpha_i| \text{ & } g_B = \sum_i p_i |\beta_i\rangle \langle \beta_i| \]

\[ L_o \quad \text{spec}(g_A) = \text{spec}(g_B) = \{ p_i \}: \text{**entanglement spectrum**} \]

L_o pure state ent. is "solved".

How about **mixed states**?

**Def:** \( g_{AB} \) is called **separable** if

\[ g_{AB} = \sum_i p_i g_A^i \otimes g_B^i \]

\[ \text{mixture of product states} \]

Otherwise entangled.

\[ \text{NP-hard} \]

How to decide if \( g_{AB} \) is entangled? \( \text{\textbf{wed}} \)

How to quantify entanglement? \( \text{\textbf{mark / thu}} \)
A first attempt:

\[ \text{SEP}(A:B) = \{ \sigma_{AB} \text{ separable is closed, convex} \} \]

\[ \forall \rho_{AB} \text{ entangled: } \exists X_{AB} \text{ observable:} \]

- \( t^* \rho_{AB} X_{AB} > 0 \)
- \( t^* \sigma_{AB} X_{AB} \leq 0 \) \((\forall \sigma_{AB} \in \text{SEP})\)

Entanglement witness (that detects \( \rho_{AB} \))

Ex: \( X_{AB} = (\Phi^+)^* (\Phi^+)_{AB} - \frac{1}{2} I_{AB} \) is an entanglement witness for \( \rho_{AB} = (\Phi^+)^* (\Phi^+)_{AB} \).

3. Quantum Operations

\[ A \rightarrow \tau \rightarrow C \]

Q. Operation: \( \tau: B(H_A) \rightarrow B(H_C) \) that is

- completely positive: \( \forall \rho_E, X_{AE} \geq 0: \text{ CPTP} \)

\[ (\tau \otimes \text{id}) X_{AE} \geq 0 \]
trace-preserving: \( tr [\mathcal{T}[X_A]] = tr [X_A] \) \( \forall X_A \)  
unitary time evolution  
\( \mathcal{T}[g] = U_g U_d^* \)  
adding auxiliary system  
\( \mathcal{T}[\mathcal{G}] = \mathcal{G} \otimes 1_{0 \times 01} \)  
forgetting system  
\( \mathcal{T}[\mathcal{G}_{AE}] = tr_E [\mathcal{G}_{AE}] \)

Shiozumi: Those suffice

Example: Measurement of \( X = \sum_x x \cdot p_x \) on \( \mathcal{H} \) be modeled by  
\( \mathcal{M} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{H}_x) \)  
\( \mathcal{M}[g] = \sum_x p_x g^x \otimes 1_x \chi_x \) 

\[ \text{True} \text{ more general measurements} \]

4 Quantifying Entanglement

How to compare \( S_{AB} \) vs \( AD \)?

Consider transformations that cannot cause entanglement...
**LOCC**: Class of q. operations composed of
- local q. operators
- classical communication of meas. outcomes

\[ \text{LOCC} \cdot \text{Sep} \subseteq \text{Sep} \]

**Def**: \( S_{AB} \geq_{\text{LOCC}} S_{AB} \) if \( S_{AB} \in \text{LOCC} \cdot S_{AB} \)

**L̅ distillable entanglement:**
\[ E_D = \text{Sep} \{ R: S_{AB} \otimes^n \geq \Phi^{+n}_{AB} \text{ for } n \text{ large} \} \]

**L̅̅ entanglement cost:**
\[ E_C = \text{Sep} \{ R: \Phi^{+n}_{AB} \geq S_{AB} \} \]