1. **Jointly typical sets** Consider sequences \((X^N, Y^N)\) of length \(N\) of IID random variables with distribution \(P(x, y)\). The jointly typical set is defined as

\[
J_{N, \epsilon}(P) = \{ (x^N, y^N) \text{ such that } x^N \in T_{N, \epsilon}(P_X), y^N \in T_{N, \epsilon}(P_Y), (x^N, y^N) \in T_{N, \epsilon}(P_{XY}) \}.
\]

(a) Show that if \(\tilde{X}^N\) and \(\tilde{Y}^N\) are both IID random variables distributed (independently!) according to \(P(x)\) and \(P(y)\) respectively, then

\[
Pr((\tilde{X}^N, \tilde{Y}^N) \in J_{N, \epsilon}(P)) \leq 2^{-N(I(X:Y) - 3\epsilon)}.
\]

*Hint: Use the properties of jointly typical sets that were proven in the lecture.*

(b) Show that, under the same assumptions for all \(\delta > 0\)

\[
Pr((\tilde{X}^N, \tilde{Y}^N) \in J_{N, \epsilon}(P)) \geq (1 - \delta)2^{-N(I(X:Y) + 3\epsilon)},
\]

for sufficiently large \(N\).

*Hint: First show that for sufficiently large \(N\)

\[
|J_{N, \epsilon}(P)| \geq (1 - \delta)2^{N(H(X,Y) - \epsilon)}.
\]

2. **Joint typicality for the binary symmetric channel** We consider the binary symmetric channel with error probability \(p\) and a uniform distribution on the source \(X\), that is

\[
P(X = 0) = P(x = 1) = \frac{1}{2}, \quad P(Y \neq X) = p.
\]

(a) Let \(Z = X \oplus Y\), where \(\oplus\) denotes addition modulo 2. Argue that \(Z\) is independent of \(X\).

(b) Show that \((x^N, y^N) \in J_{N, \epsilon}(P_{XY})\) if and only if \(x^N \in T_{N, \epsilon}(P_X)\) and \(z^N \in T_{N, \epsilon}(P_Z)\).