You do not have to hand in these exercises, they are for your practice only.

0. Exercises from MacKay: 9.20, 9.21

1. Jointly typical sets: In class you have seen some properties of jointly typical sets. In this exercise you can rederive these properties! Consider sequences $(X^N, Y^N)$ of length $N$ of IID random variables with distribution $P(x, y)$. The jointly typical set is defined as

$$J_{N,\epsilon}(P) = \{(x^N, y^N) \text{ such that } x^N \in T_{N,\epsilon}(P_X), y^N \in T_{N,\epsilon}(P_Y), (x^N, y^N) \in T_{N,\epsilon}(P_{XY})\}.$$ 

(a) Show that if $\tilde{X}^N$ and $\tilde{Y}^N$ are both IID random variables distributed (independently!) according to $P(x)$ and $P(y)$ respectively, then

$$\Pr((\tilde{X}^N, \tilde{Y}^N) \in J_{N,\epsilon}(P)) \leq 2^{-N(I(X;Y) - 3\epsilon)}.$$  

Hint: Use the properties of jointly typical sets that were proven in the lecture.

(b) Show that, under the same assumptions for all $\delta > 0$

$$\Pr((\tilde{X}^N, \tilde{Y}^N) \in J_{N,\epsilon}(P)) \geq (1 - \delta)2^{-N(I(X;Y) + 3\epsilon)}.$$  

for sufficiently large $N$.

Hint: First show that for sufficiently large $N$

$$|J_{N,\epsilon}(P)| \geq (1 - \delta)2^{N(H(X,Y) - \epsilon)}.$$ 

2. Joint typicality for the binary symmetric channel: We consider the binary symmetric channel with bit flip probability $f$ and a uniform distribution on the source $X$, that is

$$P(X = 0) = P(X = 1) = \frac{1}{2},$$

$$P(Y = 1|X = 0) = P(Y = 0|X = 1) = f.$$ 

(a) Let $Z = X \oplus Y$, where $\oplus$ denotes addition modulo 2. Argue that $Z$ is independent of $X$.

(b) Show that $(x^N, y^N) \in J_{N,\epsilon}(P_{XY})$ if and only if $x^N \in T_{N,\epsilon}(P_X)$ and $z^N \in T_{N,\epsilon}(P_Z)$. 