Introduction to Information Theory, Fall 2020

Practice problems for exercise class #9

You do **not** have to hand in these exercises, they are for your practice only.

0. **Exercises from MacKay:** 9.20, 9.21

1. **Jointly typical sets:** In class you have seen some properties of jointly typical sets. In this exercise you can rederive these properties! Consider sequences (X^N, Y^N) of length N of IID random variables with distribution $P(x, y)$. The jointly typical set is defined as

$$J_{N,\varepsilon}(P) = \left\{(x^N, y^N) \text{ such that } x^N \in T_{N,\varepsilon}(P_X),\ y^N \in T_{N,\varepsilon}(P_Y),\ (x^N, y^N) \in T_{N,\varepsilon}(P_{XY})\right\}.$$

   (a) Show that if $\tilde{X}^N$ and $\tilde{Y}^N$ are both IID random variables distributed (independently!) according to $P(x)$ and $P(y)$ respectively, then

$$\Pr((\tilde{X}^N, \tilde{Y}^N) \in J_{N,\varepsilon}(P)) \leqslant 2^{-N(I(X:Y)-3\varepsilon)}.$$

   Hint: Use the properties of jointly typical sets that were proven in the lecture.

   (b) Show that, under the same assumptions for all $\delta > 0$

$$\Pr((\tilde{X}^N, \tilde{Y}^N) \in J_{N,\varepsilon}(P)) \geqslant (1-\delta)2^{-N(I(X:Y)+3\varepsilon)}.$$

   for sufficiently large N.
   Hint: First show that for sufficiently large N

$$|J_{N,\varepsilon}(P)| \geqslant (1-\delta)2^{N(H(X,Y)-\varepsilon)}.$$

2. **Joint typicality for the binary symmetric channel:** We consider the binary symmetric channel with bit flip probability f and a uniform distribution on the source X, that is

$$P(X=0) = P(X=1) = \frac{1}{2},$$
$$P(Y=1|X=0) = P(Y=0|X=1) = f.$$

   (a) Let $Z = X \oplus Y$, where $\oplus$ denotes addition modulo 2. Argue that Z is independent of X.
   (b) Show that $(x^N, y^N) \in J_{N,\varepsilon}(P_{XY})$ if and only if $x^N \in T_{N,\varepsilon}(P_X)$ and $z^N \in T_{N,\varepsilon}(P_Z)$.