Problem 1 (Schur-Weyl duality).
Your goal in this exercise is to concretely identify irreducible representations of $U(2)$ and $S_n$ in the $n$-qubit Hilbert space. Let $j$ be such that $\frac{n}{2} - j$ is a nonnegative integer.

(a) Show that the subspace

$$\mathcal{H}_{n,j} := \left\{ |\phi\rangle \otimes |\psi^-\rangle^{\otimes \frac{n}{2} - j}, \ |\phi\rangle \in \text{Sym}^2(\mathbb{C}^2) \right\} \subseteq (\mathbb{C}^2)^{\otimes n}$$

is an irreducible $U(2)$-representation equivalent to $V_{n,j}$. Here, $|\psi^-\rangle = \frac{1}{\sqrt{2}} (|10\rangle - |01\rangle)$ is the singlet state. How can you obtain further $U(2)$-representations in $(\mathbb{C}^2)^{\otimes n}$ equivalent to $V_{n,j}$?

(b) Now construct an irreducible $S_n$-representation in $(\mathbb{C}^2)^{\otimes n}$ that is equivalent to $W_{n,j}$. How can you obtain further $S_n$-representations in $(\mathbb{C}^2)^{\otimes n}$ equivalent to $W_{n,j}$?

(c) Using part (b), confirm that the definition of $W_{n,j}$ and $W_{\tilde{n},j}$ via Schur-Weyl duality is equivalent to our original definition in lecture 3.

Problem 2 (PPT criterion).
In this exercise, you will study a simple, highly useful entanglement criterion. Given an operator $M_{AB}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, we define its partial transpose as the operator $M_{AB}^{TB}$ with matrix elements

$$\langle a, b|M_{AB}^{TB}|a', b'\rangle = \langle a, b|M_{AB}|a', b\rangle.$$ 

Note that this definition depends on the choice of basis for $\mathcal{H}_B$ (but not of the basis for $\mathcal{H}_A$).

(a) Show that $\text{tr} M_{AB}^{TB} = \text{tr} M_{AB}$.

(b) Observe that if $M_{AB} = X_A \otimes Y_B$ then $M_{AB}^{TB} = X_A \otimes Y_B^T$ and argue that this uniquely determines the partial transpose.

In particular, we can consider the partial transpose of a density operator $\rho_{AB}$.

(c) Show that if $\rho_{AB}$ is separable then $\rho_{AB}^{TB} \succeq 0$.

You thus obtain the so-called PPT criterion, short for positive partial transpose criterion: If the partial transpose $\rho_{AB}^{TB}$ is not positive semidefinite then $\rho_{AB}$ must be entangled.

(d) Verify using the PPT criterion that the ebit $|\Psi^+_2\rangle$ is entangled.

(e) Consider the family of isotropic two-qubit states,

$$\rho_{AB}(p) := p \tau_{\text{sym}} + (1 - p) \tau_{\text{anti}},$$

where $\tau_{\text{sym}}$ denotes the maximally mixed state on the symmetric subspace of two qubits and $\tau_{\text{anti}} = |\psi^-\rangle \langle \psi^-|$ the singlet state. For which values of $p \in [0, 1]$ does the PPT criterion establish entanglement?
In general, the PPT criterion is only a sufficient, but not a necessary criterion for entanglement. If \( \dim H_A \otimes H_B > 6 \), then there exist entangled states with a positive semidefinite partial transpose.

**Problem 3** (Dual representations).
This problem introduces the concept of a dual representation. To start, consider a representation \( \mathcal{H} \) of some group \( G \), with operators \( \{ R_g \} \). Let \( \mathcal{H}^* \) denote the dual Hilbert space, whose elements are “bras” \( \langle \phi \rangle \), and define operators \( R_g^* \) on \( \mathcal{H}^* \) by \( R_g^* \langle \phi \rangle := \langle \phi | R_g \rangle \).

(a) Verify that the operators \( \{ R_g^* \} \) turn \( \mathcal{H}^* \) into a representation of \( G \). This representation is called the dual representation of \( \mathcal{H} \).

(b) Show that if \( \mathcal{H} \) is irreducible then \( \mathcal{H}^* \) is irreducible.

A representation \( \mathcal{H} \) is called self-dual if \( \mathcal{H}^* \cong \mathcal{H} \).

(c) Show that the irreducible representations of SU(2), and hence all its representations, are self-dual.

(d) Show that any representation of \( S_3 \) is self-dual.

It is true more generally that any representation of \( S_n \) is self-dual.

**Problem 4** (Many copies of a bipartite pure state).
In this exercise, we will revisit the universal entanglement concentration protocol discussed in lecture 8. Let \( |\psi\rangle_{AB} \) be an arbitrary state of two qubits. Then \( |\psi\rangle_{AB}^\otimes_n \) is a vector in the Hilbert space

\[
(C^2)^\otimes_n \otimes (C^2)^\otimes_n \cong \left( \bigoplus_j V_{n,j}^A \otimes W_{n,j}^A \right) \otimes \left( \bigoplus_{j'} V_{n,j'}^B \otimes W_{n,j'}^B \right) \cong \bigoplus_{j,j'} V_{n,j}^A \otimes V_{n,j'}^B \otimes W_{n,j}^A \otimes W_{n,j'}^B.
\]

The superscripts \( A \) refer to the Schur-Weyl decomposition of the \( n A \)-systems, and likewise for \( B \). Now consider the representation of \( S_n \) on \( W_{n,j}^A \otimes W_{n,j'}^B \) given by the operators \( R_{\pi}^{(n,j)} \otimes R_{\pi}^{(n,j')} \). A vector in \( W_{n,j}^A \otimes W_{n,j'}^B \) is called an invariant vector if it is left unchanged by all these operators.

(a) Show that if \( j \neq j' \) then \( W_{n,j}^A \otimes W_{n,j'}^B \) contains no nonzero invariant vector for \( S_n \).

(b) Show that \( W_{n,j}^A \otimes W_{n,j'}^B \) contains a unique invariant vector (up to scalar multiples). Moreover, show that this vector is a maximally entangled state, which we denote by \( |\Phi^+\rangle_{W_{n,j}^A W_{n,j}^B} \).

**Hint:** Use problem 3 and Schur’s lemma.

(c) Conclude that \( |\psi\rangle_{AB}^\otimes_n \) can be written in the form

\[
|\psi\rangle_{AB}^\otimes_n = \sum_j \sqrt{p_j} |\Psi\rangle_{V_{n,j}^A V_{n,j}^B} \otimes |\Phi^+\rangle_{W_{n,j}^A W_{n,j}^B},
\]

where \( p_j = \text{tr} \{ P_j \rho_A^\otimes_n \} \) and where the \( |\Psi\rangle_{V_{n,j}^A V_{n,j}^B} \) are pure states in \( V_{n,j}^A \otimes V_{n,j}^B \).

(d) Use part (c) to analyze the universal entanglement concentration protocol discussed in class.

**Problem 5** (The controlled swap gate).
In this exercise, you will decompose the controlled swap (CSWAP) gate into a quantum circuit that consists of single-qubit and two-qubit gates only.
(a) Compute the three-qubit unitary that corresponds to the following quantum circuit:

Here, $V = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ is a square root of the $Z$-gate.

The unitary from part (a) is known as the Toffoli gate.

(b) Show that the controlled swap gate can be implemented by a sequence of Toffoli gates.