This handout summarizes the formalism of quantum information theory that we have developed in this course, starting from the axioms of quantum mechanics.

(A) **Systems**: To every quantum mechanical system, we associate a **Hilbert space** $\mathcal{H}$. For a joint system composed of two subsystems $A$ and $B$, with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, the Hilbert space is the tensor product $\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$.

(B) **States**: A **density operator** $\rho$ is an operator on $\mathcal{H}$ that satisfies (i) $\rho \geq 0$ and (ii) $\text{tr}[\rho] = 1$. Any density operator describes the state of a quantum mechanical system. If the rank of $\rho$ is one (i.e., of the form $\rho = \psi \psi^\dagger$ for some unit vector $|\psi\rangle \in \mathcal{H}$) then we say that $\rho$ is a **pure state**. Otherwise, $\rho$ is called a **mixed state**. An ensemble $\{p_i, \rho_i\}$ of quantum states can be described by the density operator $\rho = \sum_i p_i \rho_i$.

If $\rho_{AB}$ is the state of a joint system, the state of its subsystems can be described by the **reduced density matrices** $\rho_A = \text{tr}_B[\rho_{AB}]$ and $\rho_B = \text{tr}_A[\rho_{AB}]$. The latter states can be mixed even if $\rho_{AB}$ is pure. Conversely, any density operator $\rho_{AB}$ has a **purification** $\rho_{AB} = |\psi AB\rangle \langle \psi AB|$ (see Lectures 7 and 8).

(C) **Unitary dynamics**: Given a **unitary** operator $U$ on $\mathcal{H}$, the transformation $\rho \mapsto U \rho U^\dagger$ is in principle physical. In other words, the laws of quantum mechanics allow a way of evolving the quantum system for some finite time such that, when we start in an arbitrary initial state $\rho$, the final state is $U \rho U^\dagger$. If $\rho = |\psi\rangle \langle \psi|$ is a pure state, then this corresponds to $|\psi\rangle \mapsto U |\psi\rangle$.

(D) **Measurements**: A **POVM measurement** $\{Q_x\}_{x \in \Omega}$ with outcomes in some finite set $\Omega$ is a collection of operators on $\mathcal{H}$ that satisfies (i) $\sum x \in \Omega Q_x = I$ and (ii) $Q_x \geq 0$. Born’s rule asserts that the probability of outcome $x$ in state $\rho$ is given by the **Born rule**:

$$\text{Pr}_\rho(\text{outcome } x) = \text{tr}[\rho Q_x].$$

If $\rho = |\psi\rangle \langle \psi|$ is a pure state, then this can also be written as $\langle \psi| Q_x |\psi\rangle$. A POVM measurement that has precisely two outcomes is called a **binary POVM measurement**, and it has the form $\{Q, I - Q\}$, hence is specified by a single POVM element $0 \leq Q \leq I$. We can also consider POVMs with a continuum of possible outcomes (see Lecture 4).

We say that $\{P_x\}$ is a **projective measurement** if $\{P_x\}_{x \in \Omega}$ is a POVM where the $P_x$ are projections that are pairwise orthogonal (i.e., $Q_x Q_y = \delta_{x,y} Q_x$). If $\Omega \subseteq \mathbb{R}$, then the data $\{P_x\}_{x \in \Omega}$ is equivalent to specifying a Hermitian operator with spectral decomposition $O = \sum_x x P_x$, called an **observable**. If the outcome of a projective measurement is $x$ then the state of the system “collapses” into the **post-measurement state**

$$\rho' = \frac{P_x \rho P_x}{\text{tr}[P_x \rho]}$$

If $\rho = |\psi\rangle \langle \psi|$ is a pure state, then $\rho' = |\psi'\rangle \langle \psi'|$, where $|\psi'\rangle = P_x |\psi\rangle / \|P_x |\psi\rangle\|$.

Any POVM can be implemented using projective measurements on a larger system (see Lecture 2).
(E) **Operations on subsystems:** Consider a joint system with Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. If we want to perform a unitary $U_A$ on the subsystem modeled by $\mathcal{H}_A$, then the appropriate unitary on the joint system is $U_A \otimes I_B$. Similarly, if $\{Q_{A,x}\}_{x \in \Omega}$ is a POVM measurement on $\mathcal{H}_A$ then the appropriate POVM measurement on the joint system is $\{Q_{A,x} \otimes I_B\}_{x \in \Omega}$.

The standard formalism of quantum information theory includes two further notions that we did not discuss in this course: **Quantum channels** model general evolutions that can be obtained by composing unitary dynamics, adding ancillas, and taking partial traces. **Quantum instruments** can be thought of as implementations of POVM measurements that not only describe the statistics of outcomes but also model the post-measurement state.