Last time: Pure state estimation via $\text{Sym}^n(C^d)$.

\[ \Pi_n = \left( d+n-1 \right) \int d\gamma \quad \omega \otimes \omega =: \Pi_n \]

- Orthogonal proj. onto Sym. subspace
- Dimension of Sym. subspace
- prob. measure on pure states
  \( \{ \gamma \sim |\gamma \rangle \langle \gamma | \}, \text{ inv. under } \gamma \mapsto (U|\gamma \rangle \langle \gamma | U^\dagger = |U|\gamma \rangle \langle \gamma | U^\dagger \)

\[ \omega \otimes \Pi_n (U^\dagger) =: \Pi_n \]

Today: Will prove this formula using representation theory.
(Alternative: Calculate the integral by hand.)

Literature: Part I in Serre → Course homepage

**Group** $G$: Set with multiplication ("\$\cdot\$"), neutral element ("1"), inverses ("$g^{-1}$")

- $g,h \in G \implies g \cdot h \in G$
- $g \cdot 1 = 1 \cdot g = g$
- $g \cdot g^{-1} = g^{-1} \cdot g = 1$

Often omit "$\cdot$"!

**Examples:**

* Symmetric group $S_n := \{ \pi : \{1, \ldots, n\} \to \{1, \ldots, n\} \ |	ext{permutation} \}$
  - $\circ = \text{composition}$:
    \[ (\pi \circ \tau)(x) := \pi(\tau(x)) \]
  - $1 = \text{id}$ (identity map), $\tau^{-1} = \text{inverse function}$
* Unitary group \( U(d) \) := \{ U \text{ unitary } d \times d \text{ matrix} \}
  - \text{ matrix multiplication, } I = I = \text{identity matrix, } U^{-1} = U^t

* Special unitary group \( SU(d) \) := \{ U \text{ unitary } | \det(U) = 1 \}
  - \text{ subgroup of } U(d)

Other groups? \( D_8, \mathbb{Z}/n\mathbb{Z}, GL(d) \& SL(d) \); ...

Introduction to Representation Theory

**Unitary representation of group \( G \):**

* Hilbert space \( H \)

* Unitary operators \( \{ R_g : g \in G \} \) on \( H \) s.t.

\[
R_{gh} = R_g \cdot R_h \quad \& \quad R_i = I_H
\]

\( \text{NB: Always } \dim < \infty. \text{ Will say } \text{“the representation } H \text{”} \)

**Examples:**

* \( (C^d)^{\otimes n} \) is rep. of \( S_n \text{ and of } U(d) \)

\[
R_{\pi} (14_i \otimes \cdots \otimes 14_n) = 14_{\pi^{-1}(i)} \otimes \cdots \otimes 14_{\pi^{-1}(n)}
\]

\( T_U = U^{\otimes n} = U \otimes \cdots \otimes U \)

\[
[R_{\pi}, T_U] = 0
\]

\( \text{...So we can think of } \text{rep. of } S_n \times U(d) \)
* Representations of $S_3$: \[ \{\text{id}, 1 \mapsto 2, 1 \mapsto 3, 2 \mapsto 3, 1 \mapsto 2, 3 \mapsto 2 \mapsto 1\} \]

- **Trivial repr:** Exists for any group.
  \[ H = \mathbb{C}|0\rangle \quad R_{\pi}|0\rangle = |0\rangle \quad \forall \pi \]

- **Sign repr:**
  \[ H = \mathbb{C}|0\rangle \quad R_{\pi}|0\rangle = |0\rangle \quad \text{Sign}(\pi) \quad \text{for swaps } 1 \leftrightarrow 2 \text{ etc.} \]

- \[ H = \mathbb{C}^3 = \{\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle\} \]
  \[ R_{\pi} \text{ permutes coords, e.g. } R_{1 \leftrightarrow 2}\]

  \[ \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \rightarrow \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} \]

To understand a representation, want to decompose it into its smallest building blocks...

**Invariant subspace** ("subrepresentation"): \( \tilde{H} \subseteq H \text{ s.th. } \forall \phi \in \tilde{H}, R_g\phi \in \tilde{H} \) for all \( g \) in the group.

* \( H \) is called **irreducible** ("irrep") if \( \tilde{H} \) is trivial & \( H \) has only invariant subspaces

* If \( \tilde{H} \) is irrep. subspace so is \( \tilde{H}^+ \).

\[ H = \tilde{H} \oplus \tilde{H}^+ \]

\[ R_g = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \quad \text{"Smaller" representations} \]
If \( \tilde{H} \neq \{0\} \), then:

- "finest" decomposition: \( H = H_1 \oplus \ldots \oplus H_m \)
- Orthogonal & Irreducible

RT tells us how to decompose & what the reps are!

**Example:**

* Any 1-dim. repr. is an irrep.
* \( \otimes \) is not an irrep of \( S_3 \), because

\[
\tilde{H} = \{ \alpha |0\rangle + \beta |1\rangle + \gamma |2\rangle \mid \alpha + \beta + \gamma = 0 \} \subseteq \mathbb{C}^3
\]

is inv. subspace. \( \tilde{H} \) is irreducible. \( \rightarrow \) PSET

\( \tilde{H}^\perp = \mathbb{C} \langle |0\rangle + |1\rangle + |2\rangle \rangle \) is 1-dim

\( \rightarrow H = \tilde{H} \oplus \tilde{H}^\perp \) is decomposition into irreps

* Sym\(^3\)(\( \mathbb{C}^d \))?

For \( S_n \), invariant but not irreducible.

For \( U(d) \), invariant and **IRREDUCIBLE**!

\[ R_{\pi}(\Phi) = \langle \Phi | \rightarrow R_{\pi}(Tu(\Phi)) = Tu(R_{\pi}(\Phi)) = Tu(\Phi) \]

Proof below for \( d = 2 \).
Interlude: \( J : H \rightarrow H' \) s.t. \( JRg = R'gJ \) (4g)

If \( J \) is unitary: \( H, H' \) are called (unitarily) equivalent

\[
JRgJ^+ = R'g
\]

base change

"\( H \cong H' \)"

**Schur's Lemma**: Let \( J : H \rightarrow H' \) intertwiner.

1. If \( H, H' \) irreps: \( J \) invertible or \( J = 0 \).
2. If \( H = H' \) same irrep: \( J \cong \mathbb{I}_H \)

**Proof**: 1. \( \ker(J) \& \text{ran}(J) \) are inv. subspaces

2. Let \( \lambda \) be an eigenvalue. Then \( \ker((\lambda - J)) \neq 0 \), so \( \ker((\lambda - J)) = H \), so \( J = \lambda \mathbb{I}_H \).

Why do we care?!!

**Consequence**: \( \Pi_n = \Pi_n^i := (\mathbb{I} + d^{-1}) \int dt \ker e^{t \lambda} \mathbb{I} \)

Sketch: W.r.t. \( (C^d)^\otimes n = H \oplus H^\perp; H := \text{Sym}^n(C^d) \)

\[
\Pi_n = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix}, \quad \Pi_n^i = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad \Sigma_\alpha = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}
\]

\[
\Sigma_\alpha \Pi_n (\Sigma_\alpha)^\dagger \Pi_n^i (\Sigma_\alpha)^\dagger = \Pi_n^i \rightarrow J \text{ is intertwiner on irrep}
Schur's lemma

\[ J \alpha I \Rightarrow \Pi_n \cong \Pi_n \]

To see that \( \cong \), compare trace.

Next time: More details on this + proof that \( \text{Sym}^n(\mathbb{C}^2) \) is irreducible.